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# MAXIMUM AND MONOTONICITY PROPERTIES OF INITIAL-BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

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**Various maximum and monotonicity properties of some initial boundary value problems for classes of linear second order hyperbolic partial differential operators in two independent variables are established. For example, let $M$ be such an operator in Cartesian coordinates $(x, y)$ and let $T$ be a domain bounded by a characteristic curve of $M$ with everywhere negative slope, and segments $OA$ and $OB$ of the positive $x$-axis and the positive $y$-axis, respectively; under certain restrictions on the coefficients of the operator $M$, if $Mu \leqq 0$ in $T$, $u = 0$ on $OA \cup OB$ and $\partial u/\partial y \leqq 0$ on $OA$ then $u(x, y) \leqq 0$ in $T$.**

**Such maximum and monotonicity properties also have applications to ordinary differential equations; the above mentioned maximum property yields a comparison theorem on the distance between zeros of solutions to some ordinary differential equations.**

The first maximum principles for a class of linear second order hyperbolic operators in two independent variables were formulated for problems in which conditions are imposed on the solution along characteristic curves [1; 3].

A maximum property of Cauchy's problem, in which the hypotheses on the solutions are imposed along noncharacteristic curves rather than characteristic curves, was first formulated by Weinberger [12] for a class of hyperbolic operators of the form

$$(1.1)\qquad Hu = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) + c\frac{\partial u}{\partial x} + d\frac{\partial u}{\partial y} \qquad a > 0\,,\ b > 0\,.$$

Namely, under certain restrictions on the coefficients of the operator $H$, if $\partial u/\partial y \leqq 0$ on the initial line $y = 0$ and if $Hu \geqq 0$ for $y > 0$ then $u$ attains its maximum on $y = 0$.

A generalized maximum property of Cauchy's problem was established by Protter [7] for essentially any smooth operator of the form (1.1). That is, the maximum of $u$ divided by an appropriate function of the form $e^{\gamma x}(1 - \beta e^{-\alpha y})$, over a sufficiently small strip $0 \leqq y \leqq y_0$, is attained on $y = 0$.

Recently, additional maximum properties and even some monotonicity properties of Cauchy and characteristic initial value problems have been obtained by Gloistehn [4] for some classes of linear and nonlinear hyperbolic operators in two independent variables. For example, under certain restrictions on the coefficients of the operator $H$ in (1.1), if $u \leqq 0$ and $\partial u/\partial y + \sqrt{a} \cdot \partial u/\partial x \leqq 0$ on $y = 0$, and if $Hu \geqq 0$ for $y > 0$ then $u \leqq 0$ and $\partial u/\partial y + \sqrt{a} \cdot \partial u/\partial x + \alpha u \leqq 0$ for $y \geqq 0$; here $\alpha(x, y)$ depends only on the coefficients of the operator $H$.

In the case of linear second order hyperbolic operators in more than two independent variables, Weinstein [14, 15], Weinberger [13] and the author [8; 9; 10] have established maximum properties of Cauchy's problem. A typical result for the wave operator

$$Wu = \frac{\partial^2 u}{\partial t^2} - \Delta u , \tag{1.2}$$

where $\Delta$ is the $n$-dimensional Laplace operator, is the following [10; 13; 14]. Let $N = ((n-2)/2)$ ($n$ even), $N = ((n-3)/2)$ ($n$ odd). If $\partial^k u/\partial t^k = 0$ ($k = 0, 1, \cdots, N$) and $\partial^{N+1}u/\partial t^{N+1} \leqq 0$ on the initial plane $t = 0$, and if $(\partial^N/\partial t^N)\cdot(Wu) \leqq 0$ for $t \geqq 0$ then $u \leqq 0$ for $t \geqq 0$. Here the $t$-derivatives of $u$ on the initial plane $t = 0$ are to be determined from the Cauchy data.

In this paper, we derive various maximum and monotonicity properties of some initial-boundary value problems for linear second order hyperbolic equations in two independent variables. These initial-boundary value problems, first considered by Hadamard [5; 6], may be formulated in the following way.

Let $L$ be a hyperbolic equation in characteristic coordinates (cf. [2]) of the form[1]

$$Lu = u_{\xi\eta} + au_\xi + bu_\eta + cu = F . \tag{1.3}$$

Let $C_l$, $C_0$ and $C_r$ be three curves with the following properties: (1) $C_l$, $C_0$ and $C_r$ may be represented as $\eta = F_l(\xi)$, $\eta = f(\xi)$ and $\eta = F_r(\xi)$, respectively, where $F_l, f$ and $F_r$ are continuously differentiable and $F'_l > 0$, $f' < 0$ and $F'_r > 0$, (2) $C_0$ and $C_l$ intersect at the point $O(0, 0)$, (3) $C_0$ and $C_r$ intersect at $D(\bar{\xi}_0, \bar{\eta}_0)$, where $\bar{\xi}_0 > 0$ and $\bar{\eta}_0 < 0$, and (4) $C_l$ and $C_r$ do not intersect. Let $C_l^+$ and $C_0^+$ be the parts of $C_l$ and $C_0$, respectively, where $\xi \geqq 0$. Let $C'_r$ and $C'_0$ be the parts of $C_r$ and $C_0$, respectively, where $\eta \geqq \bar{\eta}_0$.

In the initial-boundary value problem $I_l$, we assume that the coefficients of the operator $L$ are defined in the region "between" $C_0^+$ and $C_l^+$ and on the boundary $C_0^+ \cup C_l^+$, $u$ and $u_\xi$ (Cauchy data) are prescribed on $C_0^+$ and $u$ is prescribed on $C_l^+$.

In the initial-boundary value problem $I_r$, the operator $L$ is defined in the region "between" $C'_0$ and $C'_r$ and on the boundary $C'_0 \cup C'_r$, $u$ and $u_\eta$ (Cauchy data) are prescribed on $C'_0$ and $u$ is prescribed on $C'_r$.

In the initial-boundary value problem $II_{lr}$, the operator $L$ is defined

[1] A subscript $\xi(\eta)$ denotes partial differentiation with respect to $\xi(\eta)$.

in the region "between" $C_l^+$, $C_r'$ and the segment $OD$ of the curve $C_0$ and also on the boundary $C_l^+ \cup OD \cup C_r'$, $u$ and either $u_\xi$ or $u_\eta$ are prescribed on $OD$ and $u$ is prescribed on $C_l^+ \cup C_r'$.

In § 2 and § 3, under certain conditions on the coefficients of the operator $L$, we establish some maximum properties of the initial-boundary value problems $I_l$, $I_r$ and $II_{lr}$. In § 4, the results of § 2 and § 3 are extended to an operator that is not expressed in terms of characteristic coordinates; namely, we consider a hyperbolic operator of the form

$$(1.4)\quad Mu = u_{yy} - h^2(x, y)u_{xx} + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u, \quad h > 0 .$$

In § 5, we obtain a sort of a monotonicity property, as well as another maximum property, of an initial-boundary value problem for an operator of the form (1.4); in § 6, an application of this maximum property yields a comparison theorem on the distance between zeros of solutions to some ordinary differential equations.

**2. Maximum properties of the initial-boundary value problems $I_l$ and $I_r$.** We consider a hyperbolic operator $L$ in characteristic coordinates of the form

$$(2.1)\qquad Lu = u_{\xi\eta} + au_\xi + bu_\eta + cu .$$

Let $A(\bar{\xi}_1, \bar{\eta}_1)$ and $B(\bar{\xi}_1, \bar{\eta}_2)$ be points on $C_0^+$ and $C_l^+$, respectively. Let $OA$ and $OB$ be the indicated segments of $C_0^+$ and $C_l^+$; the points $O$ and $A$ are assumed to belong to $OA$. Let $T_B$ denote the domain bounded by $OA$, $OB$ and the line $\xi = \bar{\xi}_1 > 0$ and let $\bar{T}_B$ denote the closure of $T_B$. We assume that the coefficients of $L$ are continuous in $\bar{T}_B$ and $b(\xi, \eta)$ has continuous first derivatives in $\bar{T}_B - OB$. We consider functions $u$ that are twice continuously differentiable in $\bar{T}_B - OB$ and continuous, together with their first derivatives, in $\bar{T}_B$.

We consider problem $I_l$; that is, $u$ and $u_\xi$ are prescribed on $C_0^+$ and $u$ is prescribed on $C_l^+$. In addition, suppose that

$$(2.2)\qquad u_\xi < 0 \quad \text{on} \quad OA - \{O\} .$$[2]

We have the following maximum property of problem $I_l$.

THEOREM 1. *Let the coefficients of $L$ satisfy the inequalities*

$$(2.3)\qquad b_\eta + ab - c \geqq 0$$

*and*

$$(2.4)\qquad c \geqq 0 ,$$

[2] The set $\{O\}$ contains only the point $O$.

*in* $T_B$, *and*

$$b \geqq 0 \quad on \quad OA . \tag{2.5}$$

*Let* $u$ *satisfy the inequality* (2.2) *and*

$$Lu \leqq 0 \quad in \quad T_B . \tag{2.6}$$

*Then if the maximum of* $u$ *in* $\bar{T}_B$ *is nonnegative it can only be attained on* $OA \cup OB$.

*Proof.* Let the maximum of $u$ in $\bar{T}_B$ occur at the point $Q$ and suppose that $Q$ does not lie on $OA \cup OB$. Then

$$u_\xi(Q) \geqq 0 . \tag{2.7}$$

Let $P$ denote the unique point of intersection of $OA$ and the characteristic $\Gamma(\xi = \text{constant})$ through $Q$.

The following fundamental identity is also used in the discussion of maximum principles for mixed elliptic-hyperbolic operators [1, p. 456]:

$$vLu = (vu_\xi)_\eta + (bvu)_\eta + [cv - (bv)_\eta]u , \tag{2.8}$$

where $v$ is a positive solution of the equation

$$v_\eta = av . \tag{2.9}$$

We integrate (2.8) along $\Gamma$ from $P$ to $Q$ and obtain

$$\begin{aligned} vu_\xi|_Q &= vu_\xi|_P + \int_P^Q vLud\eta - bvu|_P^Q + \int_P^Q vu(b_\eta + ab - c)d\eta \\ &= vu_\xi|_P + \int_P^Q vLud\eta + (bv)|_P[u(P) - u(Q)] - u(Q)\int_P^Q cvd\eta \\ &\quad + \int_P^Q v[u - u(Q)](b_\eta + ab - c)d\eta . \end{aligned} \tag{2.10}$$

Since $u(Q) \geqq 0$ and $u \leqq u(Q)$ in $\bar{T}_B$, the equation (2.10) and (2.2) through (2.7) imply a contradiction. This completes the proof of Theorem 1.

The conditions (2.3), (2.4) and (2.5) are "best possible" in the sense that one can give examples where the maximum property in Theorem 1 does not hold when these conditions fail to be satisfied (see Examples 1, 3 and 2, respectively, in § 4).

COROLLARY 1. *If* $c = 0$ *then the result of Theorem* 1 *holds without the requirement that the maximum of* $u$ *be nonnegative.*

COROLLARY 2. *If, in Corollary* 1, *we have* $u \leqq 0$ *on* $OA \cup OB$ *then* $u \leqq 0$ *in* $\bar{T}_B$ *holds without the requirement that the inequality* (2.2) *is strict.*

The proof of Corollary 2 consists of applying Corollary 1 to functions of the form $\omega = u - \varepsilon e^{\lambda(\xi+\eta)}$, with $\lambda$ chosen so large that $L\omega \leqq 0$, and then letting $\varepsilon \to 0$.

If we impose further restrictions on the data along $OA$ and $OB$ we can eliminate the restrictions (2.4) and (2.5) on the operator $L$.

THEOREM 2. *Let the coefficients of $L$ satisfy the inequality*

$$b_\eta + ab - c \geqq 0 \ \textit{in} \ T_B. \tag{2.3}$$

*Let $u$ satisfy the conditions*

$$u = 0 \quad \textit{and} \quad u_\xi \leqq 0 \,, \quad \textit{on} \quad OA \,, \tag{2.11}$$

$$u \leqq 0 \quad \textit{on} \quad OB \tag{2.12}$$

*and the differential inequality*

$$Lu \leqq 0 \quad \textit{in} \quad T_B \,. \tag{2.13}$$

*Then*

$$u \leqq 0 \quad \textit{in} \quad \bar{T}_B \,. \tag{2.14}$$

*Moreover, if the strict inequality in* (2.11) *holds on $OA - \{O\}$ then $u < 0$ in $T_B \cup AB$.*

*Proof.* We define the functions

$$u^\delta = e^{-\delta(\xi+\eta)} u \,, \qquad \delta > 0 \,.$$

Each function $u^\delta$ satisfies a differential inequality

$$L^\delta u^\delta \equiv u^\delta_{\xi\eta} + a^\delta u^\delta_\xi + b^\delta u^\delta_\eta + c^\delta u^\delta \leqq 0 \quad \text{in} \quad T_B \,, \tag{2.15}$$

where the coefficients of the hyperbolic operator $L^\delta$ are given by

$$a^\delta = a + \delta \,, \tag{2.16}$$

$$b^\delta = b + \delta \,, \tag{2.17}$$

$$c^\delta = c + \delta(a + b) + \delta^2 \,. \tag{2.18}$$

We note that for $\delta$ sufficiently large we have $b^\delta \geqq 0$ on $OA$ and $c^\delta \geqq 0$ in $\bar{T}_B$. Since the expression $b_\eta + ab - c$ is one of the two Laplace Invariants[3] under transformations of the dependent variable $u$ of the form $u = gU$, where $g$ is any positive function (cf. [1, p. 460]), we have

$$b^\delta_\eta + a^\delta b^\delta - c^\delta = b_\eta + ab - c \,. \tag{2.19}$$

[3] The other invariant is $a_\xi + ab - c$.

Suppose that the strict inequality in (2.11) holds on $OA - \{O\}$. Since

$$u_\xi^\delta = e^{-\delta(\xi+\eta)} u_\xi \quad \text{on} \quad OA \,, \tag{2.20}$$

Theorem 1 implies that $u^\delta < 0$ in $T_B \cup AB$. Therefore $u < 0$ in $T_B \cup AB$. This establishes the part of Theorem 2 when $u_\xi$ is negative on $OA - \{0\}$.

In order to complete the proof of Theorem 2, we introduce the class of functions

$$\omega = u - \varepsilon \phi e^{\lambda(\xi+\eta)} \,,$$

where $\phi$ is given by

$$\phi(\xi, \eta) = \eta - f(\xi) \qquad (\xi, \eta) \quad \text{in} \quad \bar{T}_B$$

and $\eta = f(\xi)$ is the equation of the curve $C_0$. We note that

$$\omega_\xi \,|_{OA} = u_\xi \,|_{OA} + \varepsilon f' e^{\lambda(\xi+\eta)} \,|_{OA} \,, \tag{2.21}$$

$$L\omega = Lu - \varepsilon e^{\lambda(\xi+\eta)}[\lambda(1 - f') - af' + b + \phi(\lambda^2 + \lambda(a + b) + c)] \,. \tag{2.22}$$

Since $f' < 0$ on $OA$ and $\phi \geqq 0$, we may choose $\lambda$ independently of $\varepsilon$ and so large that $L\omega \leqq Lu$ in $T_B$. It follows from (2.11) through (2.13) that $\omega$ satisfies the conditions of the first part of this proof and hence

$$u < \varepsilon \phi e^{\lambda(\xi+\eta)} \quad \text{in} \quad T_B \cup AB \,. \tag{2.23}$$

Finally, if we let $\varepsilon \to 0$ in (2.23), we obtain the desired result (2.14).

We remark that the condition (2.3) in Theorem 2 is "best possible" (see Example 1 in § 4). In addition we wish to emphasize that the condition (2.3) is invariant under a wide class of transformations of the dependent variable $u$ of the form $u = gU$ and also under transformations of the independent variables $\xi$ and $\eta$ which leave the form of the operator $L$ unchanged [1, p. 461].

Let $C(\xi_2, 0)$ be a point on $C_r'$. Take $A$ to be the point $D$ and let $DC$ be the indicated segment of $C_r'$. Let $T_C$ denote the domain bounded by $OD$, $DC$ and the line $\eta = 0$ and let $\bar{T}_C$ denote the closure of $T_C$. If we interchange $\xi$ and $\eta$, together with $a$ and $b$, in the above discussion we can establish, for example, the following maximum property of problem $I_r$ (see Theorem 2).

THEOREM 3. *Let the coefficients of $L$ satisfy the inequality*

$$a_\xi + ab - c \geqq 0 \quad in \quad T_C \,. \tag{2.24}$$

*Let $u$ satisfy the conditions*

$$u = 0 \quad and \quad u_\eta \leqq 0 \,, \quad on \quad OD \,, \tag{2.25}$$

(2.26) $$u \leqq 0 \quad on \quad DC$$

*and the differential inequality*

(2.27) $$Lu \leqq 0 \quad in \quad T_G\ .$$

*Then*

(2.28) $$u \leqq 0 \quad in \quad \bar{T}_G\ .$$

*Moreover, if the strict inequality in* (2.25) *holds on* $OD - \{D\}$ *then* $u < 0$ *in* $T_G \cup OC$.

The condition (2.24) is also "best possible" (see Example 1 in §4).

**3. A maximum property of the initial-boundary value problem** $II_{lr}$. Let $B(\bar{\xi}_0, \bar{\eta}_2)$ be the point of intersection of $C_l^+$ and the line $\xi = \bar{\xi}_0$ and let $T_B$ and $T_G$ be defined as in §2. Let $u$ satisfy the conditions

(3.1) $$u = 0 \quad \text{and either} \quad u_\xi < 0 \quad \text{or} \quad u_\eta < 0\ , \quad \text{on} \quad OD\ ,$$

(3.2) $$u \leqq 0 \quad \text{on} \quad OB \cup DC$$

and the differential inequality

(3.3) $$Lu \leqq 0 \quad \text{in} \quad T_B \cup T_G\ .$$

Since $f' < 0$ on $OD$, $u = 0$ and $u_\xi < 0 (u_\eta < 0)$, on $OD$, imply $u_\eta < 0 (u_\xi < 0)$ on $OD$. Hence, if the coefficients of $L$ satisfy the inequalities (2.3) and (2.24) then Theorem 2 and Theorem 3 imply

(3.4) $$u < 0 \quad \text{in} \quad T_B \cup T_G \cup DB \cup OC\ .$$

In this section, we determine a domain $\Sigma$ such that (1) $T_B \cup T_G \cup DB \cup OC \subset \Sigma$ and (2) under certain "invariant" conditions[4] on the coefficients of $L$, if (3.1) through (3.3) are satisfied then $u < 0$ in $\Sigma$.

Let $P(\xi_1, \eta_1)$ be any point such that $\bar{\xi}_0 < \xi_1 < \bar{\xi}_2$ and $0 < \eta_1 < \bar{\eta}_2$. Let $Q(\xi_1, \eta_0)$ denote the unique point of intersection of $DC$ and $\xi = \xi_1$ and let $R(\xi_0, \eta_0)$ denote the unique point of intersection of $OD$ and $\eta = \eta_0$. Hence, to each point $P(\xi_1, \eta_1)$ we may associate a unique point $S_P(\xi_0, \eta_1)$ and a characteristic rectangle with corners $P$, $Q$, $R$ and $S_P$ such that $Q$ and $R$ lie on $DC$ and $OD$, respectively; let $T$ denote the set of all points $P(\xi_1, \eta_1)$ such that $S_P$ is contained in $T_B$.[5] The set $T$ is a domain.

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[4] The conditions are stated in terms of Laplace Invariants (see footnote 3).

[5] In the definition of the set $T$ we may also use $OB$ instead of $DC$ so that $S_P$ lies on $OB$ and $T$ consists of all points $P$ such that $Q$ is contained in $T_G$.

Let $P(\xi_1, \eta_1)$ be any point in $T$ and let $Q$, $R$ and $S_P$ have coordinates as in the definition of the domain $T$. We integrate (2.8) along the characteristic from $P_1(\xi, \eta_0)$ to $P_2(\xi, \eta_1)$ and obtain[6]

$$(3.5)\quad \int_{P_1}^{P_2} vLud\eta = (vu)_\xi \big|_{P_1}^{P_2} + (bv - v_\xi)u \big|_{P_1}^{P_2} + \int_{P_1}^{P_2} uv(c - ab - b_\eta)d\eta \,.$$

We integrate (3.5) with respect to $\xi(\xi_0 \leqq \xi \leqq \xi_1)$ and obtain

$$(3.6)\quad (vu)(P) = (vu)(Q) + (vu)(S_P) - (vu)(R) + \int_R^Q (bv - v_\xi)ud\xi$$
$$- \int_{S_P}^P (bv - v_\xi)ud\xi + \iint v[Lu + u(b_\eta + ab - c)]d\xi d\eta \,,$$

where the double integral denotes integration over $\xi_0 \leqq \xi \leqq \xi_1$ and $\eta_0 \leqq \eta \leqq \eta_1$. Let $v^0$ be the particular solution of (2.9) given by

$$(3.7)\quad v^0 = \exp\left[\int_{\xi_0}^{\xi} b(\tau, \eta_0)d\tau + \int_{\eta_0}^{\eta} a(\xi, \rho)d\rho\right].$$

Then

$$(3.8)\quad (v^0)^{-1}(bv^0 - v^0_\xi) = 0 \quad \text{on} \quad \eta = \eta_0 \,,$$

$$(3.9)\quad (v^0)^{-1}(bv^0 - v^0_\xi) = b(\xi, \eta_1) - b(\xi, \eta_0) - \int_{\eta_0}^{\eta_1} a_\xi(\xi, \rho)d\rho$$
$$= \int_{\eta_0}^{\eta_1} [b_\eta(\xi, \rho) - a_\xi(\xi, \rho)]d\rho \quad \text{on} \quad \eta = \eta_1 \,.$$

It follows from (3.1) and (3.6) through (3.9) that

$$(3.10)\quad (v^0u)(P) = (v^0u)(Q) + (v^0u)(S_P) + \int_{\xi_0}^{\xi_1}\left[\int_{\eta_0}^{\eta_1}(a_\xi - b_\eta)d\rho\right](v^0u)(\xi, \eta_1)d\xi$$
$$+ \iint v^0[Lu + u(b_\eta + ab - c)]d\xi d\eta \,.$$

Let $\Sigma = T \cup T_B \cup T_C \cup DB \cup OC$. Suppose that there is a point $P$ in $\Sigma$ such that $u(P) = 0$. The inequality (3.4) implies that (1) $P$ is in $T$ and (2) we may assume without loss of generality that $u(P) = 0$ and $u \leqq 0$ in the characteristic rectangle with corners $P$, $Q$, $R$ and $S_P$. Let $\Sigma_B$ and $\Sigma_C$ denote the parts of $\Sigma$ where $\eta > 0$ and $\xi > \bar{\xi}_0$, respectively. Under the assumptions (2.24) and

$$(3.11)\quad b_\eta + ab - c \geqq 0 \quad \text{in} \quad \Sigma \,,$$

$$(3.12)\quad a_\xi \geqq b_\eta \quad \text{in} \quad \Sigma_B \,,$$

it follows from (3.2), (3.3) and (3.10) that $(v^0u)(S_P) \geqq 0$. Since $S_P$ is

[6] In this section, $u$ and the coefficients of $L$ are assumed to be sufficiently smooth in $T$ (see § 2).

in $T_B$, this is a contradiction. Hence $u < 0$ in $\Sigma$.

If we interchange $\xi$ and $\eta$, together with $a$ and $b$, in the above discussion, the conditions (compare (3.11) and (3.12))

$$a_\xi + ab - c \geqq 0 \quad \text{in} \quad \Sigma \,, \tag{3.13}$$

$$b_\eta \geqq a_\xi \quad \text{in} \quad \Sigma_C \,, \tag{3.14}$$

also imply that $u < 0$ in $\Sigma$. We have established the following maximum property of problem $II_{lr}$.

THEOREM 4. *Let the coefficients of $L$ satisfy the inequalities*

$$\begin{aligned} a_\xi + ab - c &\geqq 0 \quad in \quad \Sigma \,, \\ b_\eta + ab - c &\geqq 0 \quad in \quad \Sigma \end{aligned} \tag{3.15}$$

*and either*

$$a_\xi + ab - c \geqq b_\eta + ab - c \quad in \quad \Sigma_B \tag{3.16}$$

*or*

$$b_\eta + ab - c \geqq a_\xi + ab - c \quad in \quad \Sigma_C \,. \tag{3.17}$$

*Let $u$ satisfy the conditions*[7]

$$u = 0 \quad and\ either \quad u_\xi < 0 \quad or \quad u_\eta < 0 \,, \quad on \quad OD \,, \tag{3.18}$$

$$u \leqq 0 \quad on \quad OB \cup DC \tag{3.19}$$

*and the differential inequality*

$$Lu \leqq 0 \quad in \quad \Sigma \,. \tag{3.20}$$

*Then*

$$u < 0 \quad in \quad \Sigma \,. \tag{3.21}$$

We remark that the domain $\Sigma$ is the "largest possible" in the sense that if we relax the strict inequalities in (3.18)—and hence also the strict inequality in (3.21)—then one can give examples where the maximum property $u \leqq 0$ holds only in the closure of $\Sigma$ (see Example 4 in § 4).

## 4. Maximum properties of the initial-boundary value problems $I_l'$ and $II_{lr}'$.

In this section we extend the results of § 2 and § 3 to a hyperbolic operator of the form

[7] We may replace the condition "either $u_\xi < 0$ or $u_\eta < 0$ on $OD$" by a condition involving the normal derivative of $u$ on $OD$ (cf. (4.15) in § 4).

(4.1) $Mu = u_{yy} - h^2(x, y)u_{xx} + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u, \quad h > 0$ .

For the sake of simplicity we consider only initial-boundary value problems for $M$ where $u$ and $u_y$ are prescribed on a portion of the $x$-axis and $u$ is prescribed on either the line $x = 0$ (problem $I_l'$) or the lines $x = 0$ and $x = d_0 > 0$ (problem $II_{lr}'$).

We recall that the characteristic curves of $M$ are the solutions of the ordinary differential equations

(4.2) $$\frac{dx}{dy} = h ,$$

(4.3) $$\frac{dx}{dy} = -h .$$

Let $A'(d, 0)$ and $D'(d_0, 0)$ be points on the positive $x$-axis. Let $B'(0, y_1)$ [respectively $C'(d_0, y_2)$] be the unique point of intersection of the line $x = 0[x = d_0]$ and the characteristic curve $\Gamma_-[\Gamma_+]$ with slope (4.3) [(4.2)] that passes through $A'(d, 0)$ [$O(0, 0)$]. Let $OA'$, $OD'$, $OB'$ and $D'C'$ be the indicated straight line segments. Let $T_{B'}$ and $T_{C'}$ be the domains bounded by $OB'$, $OA'$, $\Gamma_-$ and $D'C'$, $OD'$, $\Gamma_+$, respectively.[8]

We consider functions $u$ that are twice continuously differentiable in $\bar{T}_{B'} - OB'$ and continuous, together with their first derivatives, in $\bar{T}_{B'}$. We assume that the coefficients of $M$ are continuous in $\bar{T}_{B'}$, $\alpha$ and $\beta$ are continuously differentiable in $\bar{T}_{B'} - OB'$ and $h$ has continuous second derivatives in $\bar{T}_{B'} - OB'$. (We assume that analogous conditions hold when we consider the domain $T_{C'}$).

We define the operators

(4.4) $$\delta = \frac{\partial}{\partial y} + h\frac{\partial}{\partial x} ,$$

(4.5) $$D = \frac{\partial}{\partial y} - h\frac{\partial}{\partial x} .$$

The operators $\delta$ and $D$ are essentially the directional derivatives along the characteristic curves defined by (4.2) and (4.3), respectively.

In this section we assume also that $h$ is continuously differentiable and positive in $\bar{T}_{B'}$ (and $\bar{T}_{C'}$). If we introduce characteristic coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ as new independent variables (cf. [2]) then we can apply the results of § 2 to the transformed operator—an operator that is of the form (2.1). In terms of the operators $\delta$ and $D$ the conditions (2.3), (2.5) and (2.24) become

[8] The points $A'$ and $O$ do not belong to either $\Gamma_-$ or $\Gamma_+$.

$$(4.6)\quad 2E \equiv D\Big(\frac{D(h)-\alpha+\beta h}{h}\Big)$$
$$-\frac{1}{2h^2}(D(h)-\alpha+\beta h)(D(h)-\alpha-\beta h)-2\gamma \geqq 0 \quad \text{in} \quad T_{B'}\,,$$

$$(4.7)\qquad D(h)-\alpha+\beta h \geqq 0 \quad \text{on} \quad OA'$$

and (compare [1, p. 464, (5″)])

$$(4.8)\quad 2F \equiv \delta\Big(\frac{\delta(h)+\alpha+\beta h}{h}\Big)$$
$$-\frac{1}{2h^2}(\delta(h)+\alpha+\beta h)(\delta(h)+\alpha-\beta h)-2\gamma \geqq 0 \quad \text{in} \quad T_{C'}\,,$$

respectively. We have, for example, the following result.[9]

THEOREM 1′. *Let the coefficients of $M$ satisfy the inequalities* (4.6), (4.7) *and*

$$(4.9)\qquad \gamma \geqq 0 \quad in \quad T_{B'}\,.$$

*Let $u$ satisfy the condition*

$$(4.10)\qquad \delta(u) < 0 \quad on \quad OA' - \{O\}$$

*and the differential inequality*

$$(4.11)\qquad Mu \leqq 0 \quad in \quad T_{B'}\,.$$

*Then if the maximum of $u$ in $\bar{T}_{B'}$ is nonnegative it can only be attained on $OA' \cup OB'$.*

The following examples illustrate which conditions in the above theorems are "best possible".

EXAMPLE 1. We consider an operator $M$ of the form $Mu = u_{yy} - u_{xx} + 3u$. Let $OA'$ and $OB'$ be the segments of the $x$-axis and the $y$-axis where $0 \leqq x \leqq 3\pi/4$ and $0 \leqq y \leqq 3\pi/4$, respectively. The domain $T_{B'}$ is given by $x + y < 3\pi/4$, $x > 0$ and $y > 0$. Since $h = 1$, $\gamma = 3$ and $\alpha = \beta = 0$, the conditions (4.7) and (4.9) are satisfied. However, the condition (4.6) becomes $\gamma \leqq 0$ which is not satisfied. Let $u(x, y) = -\sin 2y \cos(x - \pi/2)$. Then $Mu = 0$ in $T_{B'}$ and $\delta(u) = -2\cos(x - \pi/2) < 0$ when $y = 0$ and $0 < x \leqq 3\pi/4$. Since $u(r, (\pi + r)/2) = \sin^2 r > 0$ $(0 < r \leqq \pi/6)$ and $u = 0$ on $OA' \cup OB'$, the function $u$ does not attain its maximum on $OA' \cup OB'$. Therefore, the condition (4.6) in Theorem 1′ is "best possible". Moreover, if we set $\xi = y + x$ and $\eta = y - x$, this example shows that the

[9] The desired extension of Theorem 2 is contained in Theorem 5.

condition (2.3) in Theorem 1 and Theorem 2 is also "best possible".

EXAMPLE 2. Let $Mu = u_{yy} - u_{xx} - 2u_y$. Let $OA'$ and $OB'$ be the segments of the $x$-axis and the $y$-axis where $0 \leqq x \leqq \pi/3$ and $0 \leqq y \leqq \pi/3$, respectively. Then domain $T_{B'}$ is given by $x + y < \pi/3$, $x > 0$ and $y > 0$. Since $h = 1$, $\beta = -2$ and $\alpha = \gamma = 0$, the conditions (4.6) and (4.9) are satisfied but the condition (4.7) becomes $\beta \geqq 0$ which is not satisfied. Let $u(x, y) = (y - 1)e^y \cos(x - \pi/2)$. Then $Mu = 0$ in $T_{B'}$, $u \leqq 0$ on $OA' \cup OB'$ and $\delta(u) = \sin(x - \pi/2) < 0$ when $y = 0$ and $0 \leqq x \leqq \pi/3$. Since $u(r, 1 + r) = re^{1+r} \sin r > 0$ $(0 < r < 1/2(\pi/3 - 1))$, the condition (4.7) in Theorem 1′ is also "best possible".

EXAMPLE 3. Let $Mu = u_{yy} - u_{xx} - \gamma_0^2 u$, where $\gamma_0$ is a positive constant. Let $\beta_1$ be the first positive zero of $J_1(\rho)$, the Bessel function of order 1. Let $OA'$ and $OB'$ be the segments of the $x$-axis and the $y$-axis where $0 \leqq x \leqq d$ and $0 \leqq y \leqq d$ $(0 < d < \beta_1/\gamma_0)$, respectively. We note that condition (4.9) is not satisfied. Let $u(x, y) = J_0(\gamma_0\sqrt{x^2 - y^2})$, where $J_0(\rho)$ denotes the Bessel function of order 0. It is well known that $u$ has the properties (1) $Mu = 0$, (2) $u = 1$ on $y = x$ (and $y = -x$) and (3) $|u(x, y)| \leqq 1$ (cf. [2, p. 120] and [11]). Moreover, $\delta(u) = \gamma_0 J_0'(\gamma_0 x) = -\gamma_0 J_1(\gamma_0 x) < 0$ when $y = 0$ and $0 < x \leqq d$. Since $u$ attains its maximum on $y = x$, the condition (4.9) is also "best possible".

In order to extend Theorem 4 to the operator $M$ we first determine a domain $T'$ that plays the role of the domain $T$ in § 3. In the definition of the point $B'$, we take $A'$ to be the point $D'(d_0, 0)$. Let $\Gamma_{B'}$ and $\Gamma_{C'}$ be the characteristic curves given by (4.2) and (4.3), respectively, that pass through $B'$ and $C'$. Let $E$ be the characteristic quadrilateral bounded by $\Gamma_{B'}, \Gamma_{C'}, \Gamma_+$ and $\Gamma_-$. As in § 3, to each point $P'(x, y)$ in $E$, we may associate a unique point $S_{P'}$ and a characteristic quadrilateral with corners $P', Q', R'$ and $S_{P'}$ such that $Q'$ and $R'$ lie on $D'C'$ and $OD'$, respectively. Let $T'$ denote the domain that consists of all points $P'$ such that $S_{P'}$ is contained in $T_{B'}$. Moreover, as in § 3, let $\Sigma' = T' \cup T_{B'} \cup T_{C'} \cup \Gamma_- \cup \Gamma_+$ and let $\Sigma_{B'}$ and $\Sigma_{C'}$ be the parts of $\Sigma'$ "above $\Gamma_+$" and "above $\Gamma_-$", respectively.

We can now formulate the desired extension of Theorem 4. Since the Laplace Invariants $b_\eta + ab - c$ and $a_\xi + ab - c$ are given essentially by (4.6) and (4.8), respectively, we need only restate the conditions (3.15) through (3.17) in terms of the operators $\delta$ and $D$.

THEOREM 4′. *Let the coefficients of $M$ satisfy the inequalities*

$$(4.12) \qquad \begin{aligned} E &\geqq 0 \quad in \quad \Sigma' \\ F &\geqq 0 \quad in \quad \Sigma' \end{aligned}$$

*and either*

$$F \geqq E \quad in \quad \Sigma_{B'} \tag{4.13}$$

*or*

$$E \geqq F \quad in \quad \Sigma_{C'} \,. \tag{4.14}$$

*Let* $u$ *satisfy the conditions*

$$u = 0 \quad and \quad u_y \leqq 0 \,, \quad on \quad OD' \,, \tag{4.15}$$

$$u \leqq 0 \quad on \quad OB' \cup D'C' \tag{4.16}$$

*and the differential inequality*

$$Mu \leqq 0 \quad in \quad \Sigma' \,. \tag{4.17}$$

*Then*

$$u \leqq 0 \quad in \quad \Sigma' \,. \tag{4.18}$$

*Moreover, if the strict inequality holds in* (4.15) *then the strict inequality holds also in* (4.18).

*Proof.* If the strict inequality holds in (4.15), Theorem 4 implies the desired result $u < 0$ in $\Sigma'$.

In order to complete the proof of Theorem 4′, we consider the functions

$$w = u - \varepsilon y e^{\lambda y} \qquad \varepsilon > 0 \,,$$

where $\lambda$ is chosen independently of $\varepsilon$ and so large that $Mw \leqq Mu$ in $\Sigma'$. Since (4.15) through (4.17) imply that $w$ satisfies the conditions of the first part of this proof, it follows that

$$u < \varepsilon y e^{\lambda y} \quad \text{in} \quad \Sigma' \,. \tag{4.19}$$

Hence, letting $\varepsilon \to 0$, we obtain (4.18).

The following example shows that the domain $\Sigma'$ in Theorem 4′ is the "largest possible".

EXAMPLE 4. Let $Mu = u_{yy} - u_{xx}$. Let $OD'$ and $OB'$ be the segments of the $x$-axis and the $y$-axis where $0 \leqq x \leqq \pi$ and $0 \leqq y \leqq \pi$, respectively, and let $D'C'$ be the segment of the line $x = \pi$ where $0 \leqq y \leqq \pi$. Then the domain $\Sigma'$ is given by $0 < x < \pi$ and $0 < y < \pi$. Let $u(x, y) = -\sin y \cos (x - \pi/2)$. Since $u \leqq 0$ in the closure of $\Sigma'$ but $u > 0$ when $0 < x < \pi$ and $y = \pi + \varepsilon$ $(0 < \varepsilon < \pi)$, the set $\Sigma'$ in Theorem 4′ is the "largest possible".

**5. A monotonicity property of the initial-boundary value problem $I_l'$.** In this section (the notation and the various smoothness assumptions are the same as in § 4) we consider the operator $M$ without introducing characteristic coordinates. In addition to an extension of Theorem 2 this more direct approach also yields a sort of a monotonicity property for $M$.

Our discussion is based upon the fundamental identity (see (2.8) and [1, p. 465]; compare also [4, p. 385, (1.2)])

$$D[v\delta(u)] = vMu + [D(v) - \beta v]D(u) - \gamma vu \,, \tag{5.1}$$

where $\delta$ and $D$ are the operators defined in (4.4) and (4.5) and $v$ is a positive solution of the equation

$$2hD(v) + v[D(h) - \alpha - \beta h] = 0 \,. \tag{5.2}$$ [10]

We rewrite (5.1) as

$$D[v(\delta(u) + \theta u)] = vMu + uvE \,, \tag{5.3}$$

where $E$ is defined in (4.6) and

$$\begin{aligned} \theta &= v^{-1}[\beta v - D(v)] \\ &= \frac{D(h) - \alpha + \beta h}{2h} \,. \end{aligned} \tag{5.4}$$

The following theorem is a consequence of (5.1) and (5.3).

THEOREM 5. *Let the coefficients of $M$ satisfy the inequality* (4.6). *Let $u$ satisfy the conditions*

$$u = 0 \quad and \quad u_y \leqq 0 \,, \quad on \quad OA' \,, \tag{5.5}$$

$$u \leqq 0 \quad on \quad OB' \tag{5.6}$$

*and the differential inequality*

$$Mu \leqq 0 \quad in \quad T_{B'} \,. \tag{5.7}$$

*Then*

$$u \leqq 0 \tag{5.8}$$

*and*

$$\delta(u) + \theta u \leqq 0 \,, \tag{5.9}$$

*in $T_{B'} \cup \Gamma_-$. Moreover, if the strict inequality in* (5.5) *holds on*

[10] On any characteristic curve given by $dx/dy = -h$, we see that $D(v) = dv/dy$ and, hence, the equation (5.2) becomes an ordinary differential equation.

*$OA' - \{O\}$ then the strict inequality holds also in* (5.8).

*Proof.* Suppose that the strict inequality in (5.5) holds on $OA' - \{O\}$. Since $D = d/dy$ on any characteristic curve $dx/dy = -h$, if we proceed as in the proof of Theorem 1 and Theorem 2—with the identity (5.1) playing the role of (2.8) and $u^\delta = e^{-\delta y}u$—we obtain $u < 0$ in $T_{B'} \cup \Gamma_-$. The remainder of the proof is a variation of a method used by Gloistehn [4] for the Cauchy problem. Assume that there is a point $Q'$ in $T_{B'} \cup \Gamma_-$ such that $[\delta(u) + \theta u]|_{Q'} = 0$. Let $\Gamma_{Q'}$ be the characteristic curve given by (4.3) that passes through $Q'$ and let $P$ denote the point of intersection of $\Gamma_{Q'}$ and $OA'$. Since $[\delta(u) + \theta u]|_P < 0$ by our hypotheses there is a point $Q$ on $\Gamma_{Q'}$ such that $[\delta(u) + \theta u]|_Q = 0$ and $\delta(u) + \theta u < 0$ on the arc of $\Gamma_{Q'}$ between $P$ and $Q$. Therefore, since $v > 0$ and $D$ is essentially differentiation along $\Gamma_{Q'}$, it follows that

$$D[v(\delta(u) + \theta u)]|_Q \geqq 0 \,. \tag{5.10}$$

The basic equation (5.3), together with $u(Q) < 0$, $Mu < 0$, (4.6) and (5.10), yields a contradiction. Thus $\delta(u) + \theta u$ is negative in $T_{B'} \cup \Gamma_-$ under the additional assumptions $u_y < 0$ on $OA' - \{O\}$ and $Mu < 0$ in $T_{B'} \cup \Gamma_-$.

In order to complete the proof of Theorem 5, we consider again the functions

$$w = u - \varepsilon y e^{\lambda y} \qquad \varepsilon > 0 \,,$$

where $\lambda$ is chosen independently of $\varepsilon$ and so large that $Mw < Mu$ in $T_{B'}$. It follows from (5.5) through (5.7) and the first part of this proof that

$$u < \varepsilon y e^{\lambda y} \tag{5.11}$$

and

$$\delta(u) + \theta u < \varepsilon e^{\lambda y}(1 + \lambda y + \theta y) \,, \tag{5.12}$$

in $T_{B'} \cup \Gamma_-$. Therefore, letting $\varepsilon \to 0$, we obtain (5.8) and (5.9).

COROLLARY 3. *Let $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be two points in $T_{B'}$ that are joined by a characteristic curve $\Gamma$ of the family* (4.2) *and suppose that $y_1 \leqq y_2$. If* (4.6) *and* (5.5) *through* (5.7) *are satisfied then*

$$u(Q_2) \leqq u(Q_1) \exp\left[\int_{\Gamma\, Q_1}^{Q_2} \theta dy\right] . \tag{5.13}$$

The proof consists of multiplying (5.9) by $\exp\left[\int_{\Gamma\, y_1}^{y} \theta dy\right]$ and integrating along $\Gamma$ from $Q_1$ to $Q_2$.

**6. An application to ordinary differential equations.** In this section we establish a comparison theorem on the distance between zeros of solutions to some ordinary differential equations. Comparison theorems of this type have already been obtained by Weinberger [12] and Protter [7] as applications of some maximum properties of "pure" initial value problems. However, we show that in some cases a "stronger" result can be obtained by the use of a maximum property of an initial-boundary value problem.

We consider the ordinary differential equations[11]

$$(6.1)\quad (f_1(x)\phi'(x))' + g_1(x)\phi(x) = 0\,, \qquad f_1(x) > 0 \qquad c \leqq x \leqq d\,,$$

$$(6.2)\quad (f_2(y)\psi'(y))' + g_2(y)\psi(y) = 0\,, \qquad f_2(y) > 0 \qquad a \leqq y \leqq b\,.$$

Suppose that $\phi(x_1) = 0$ and $\phi(x) > 0$, $c \leqq x_1 < x \leqq x_2 \leqq d$. In addition, suppose that $\psi(y_1) = 0$ and $\psi'(y_1) < 0$, $a \leqq y_1 < b$. Let $M$ be the hyperbolic operator given by

$$(6.3)\quad Mu = u_{yy} - u_{xx} - f_1^{-1}f_1'u_x + f_2^{-1}f_2'u_y + (f_2^{-1}g_2 - f_1^{-1}g_1)u\,.$$

Then the function $u(x, y) = \phi(x)\psi(y)$ is such that

$$(6.4)\quad u = 0 \text{ and } u_y < 0\,, \text{ on } y = y_1 \text{ and } x_1 < x \leqq x_2\,,$$

$$(6.5)\quad u = 0 \text{ on } x = x_1 \text{ and } y_1 \leqq y \leqq b\,,$$

$$(6.6)\quad Mu = 0\,, \quad a \leqq y \leqq b \text{ and } c \leqq x \leqq d\,.$$

Hence, if the functions $\alpha = -f_1^{-1}f_1'$, $\beta = f_2^{-1}f_2'$ and $\gamma = f_2^{-1}g_2 - f_1^{-1}g_1$ are such that the operator $M$ satisfies the condition (4.6), Theorem 5 implies that $u < 0$ in the domain bounded by the lines $x = x_1$, $y = y_1$ and $x + y = x_2 + y_1$. Thus $\psi(y) < 0$ when $y_1 < y < y_1 + (x_2 - x_1)$. Since $\psi$ and $\psi'$ cannot vanish simultaneously and $x_1$, $x_2$ and $y_1$ were arbitrary, we have established the following comparison theorem (see [12, p. 512] and [7, pp. 123–125]).

THEOREM 6. *Let $m$ be the greatest lower bound of the distance between zeros of $\psi$ on the interval $a \leqq y \leqq b$ and let $m^*$ be the least upper bound of the distances between zeros of $\phi$ on the interval $c \leqq x \leqq d$. If*

$$(6.7)\quad 2f_2^{-1}f_2'' - (f_2^{-1}f_2')^2 - 4f_2^{-1}g_2 \geqq 2f_1^{-1}f_1'' - (f_1^{-1}f_1')^2 - 4f_1^{-1}g_1$$

*for $a \leqq y \leqq b$ and $c \leqq x \leqq d$, then*

$$(6.8)\quad m \geqq m^*\,.$$

[11] In this section, $v'$ denotes the derivative of the function $v$.

COROLLARY 4. *If, in Theorem* 6, *we have* $f_1(x) \equiv 1$, $g_1(x) \equiv \lambda^2$ *and*

$$2f_2f_2'' - (f_2')^2 + 4f_2(\lambda^2 f_2 - g_2) \geqq 0 \qquad a \leqq y \leqq b\,, \tag{6.9}$$

*then*

$$m \geqq \pi\lambda^{-1}\,. \tag{6.10}$$

We remark that, even under the conditions $\lambda^2 f_2(y) \geqq g_2(y)$ and $f_2(y)f_2''(y) \geqq (f_2'(y))^2$, the direct application of a maximum property for a "pure" initial value problem would yield only the "weaker" result $m \geqq \pi\lambda^{-1}/2$ [7, p. 124 Corollary 3].

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