

## THE PSEUDO-RADICAL OF A COMMUTATIVE RING

ROBERT W. GILMER, JR.

If  $D$  is an integral domain with identity having quotient field  $K$ , the pseudo-radical of  $D$  is defined to be the intersection of all nonzero prime ideals of  $D$ . Consideration of the pseudo-radical arises naturally in examining the relation between the statements " $D$  has Jacobson radical zero" and " $D[u]$  has Jacobson radical zero, where  $u \in K$ ". Theorem 4 proves that the first statement implies the second. As a corollary it follows that if  $M$  is a prime ideal of the polynomial ring  $R[X]$  over a commutative ring  $R$  and if  $P = M \cap R$ , then  $M$  is an intersection of maximal ideals of  $R[X]$  if  $P$  is an intersection of maximal ideals of  $R$ . Consequently, if  $R$  is a Hilbert ring,  $R[X]$  is also a Hilbert ring. The remainder of the paper is devoted to a study of domains having nonzero pseudo-radical.

Goldman has defined in [6] the concept of a Hilbert ring: the commutative ring  $R$  with identity is a Hilbert ring if each proper prime ideal of  $R$  is an intersection of maximal ideals; here proper means an ideal different from  $R$ . The terminology is motivated by the observation that Hilbert's Nullstellensatz may be interpreted as asserting that each proper prime ideal of the polynomial domain  $K[X_1, \dots, X_n]$  for  $K$  a field, is an intersection of maximal ideals. In work done independently but at approximately the same time, Krull introduced in [10] the concept of a Jacobson ring; the reason for the terminology being obvious from the definition: the commutative ring  $R$  with identity is a Jacobson ring if for each proper ideal  $A$  of  $R$ , the radical of  $R/A$  and the Jacobson radical of  $R/A$  coincide. From these two definitions it is easily seen that  $R$  is a Hilbert ring if and only if  $R$  is a Jacobson ring. [10; p. 359]. In the remainder of this paper we shall use the term Hilbert ring for the notions described above.

In Theorem 3 of [6], Goldman shows  $R$  and the polynomial ring  $R[X]$  are simultaneously Hilbert rings. Also, (Theorem 5 of [6]),  $R[X]$  is a Hilbert ring if and only if for each maximal ideal  $M$  of  $R[X]$ ,  $M \cap R$  is maximal in  $R$ . We originally set out to consider the following "local" case of Goldman's Theorem 3:

*Suppose  $M$  is a proper prime ideal of  $R[X]$  and  $P = M \cap R$ . What is the relation between the following statements?*

- (1)  $M$  is an intersection of maximal ideals of  $R[X]$ .
- (2)  $P$  is an intersection of maximal ideals of  $R$ .

Corollary 2 shows that (2) implies (1). Goldman's Theorem 3

follows as a corollary. It can be seen that the above question may be reduced to considering the following problem.

*Suppose  $D$  is an integral domain with quotient field  $K$  and  $t \in K$ . What is the relation between the statements*

- (a) *the Jacobson radical of  $D$  is zero.*
- (b) *the Jacobson radical of  $D[t]$  is zero.*

Hence, as one might expect from the equivalence of the concepts Hilbert ring and Jacobson ring, we are led into a consideration of the Jacobson radical of a ring and its overrings. For integral domains it becomes natural in this connection to consider the *pseudo-radical*, defined as follows: If  $R$  is a commutative ring with identity, the pseudo-radical of  $R$  is the intersection of all nonzero prime ideals of  $R$ .

Section 2 of this paper lists some known results concerning relations between the ideal theory of  $R$  and that of the polynomial ring  $R[X]$ . Section 3 considers domains  $D$  with nonzero pseudo-radical and the effect of this hypothesis on the ideal theory of  $D[X]$ . In § 4 we consider the italicized questions previously listed.

Our notation and terminology will in general be that of Zariski-Samuel [19] [20]. In particular "ring" always means a commutative ring with identity and it is assumed that each subring contains the identity of the larger ring. We use the following terms which are not found in Zariski-Samuel. If  $D$  is an integral domain with quotient field  $K$  we say  $D$  has the *QR-property* if each domain between  $D$  and  $K$  is a quotient ring of  $D$  with respect to some multiplicative system in  $D$ . [5]. If for each prime ideal  $P$  of  $D$  the quotient ring  $D_P$  is a valuation ring, we say  $D$  is a *Prüfer domain*. For elementary properties of Prüfer domains see [7]. If each  $D_P$  is in fact a rank one discrete valuation ring, we say  $D$  is *almost Dedekind* [3].

2. The ideal theory of  $R$  and of  $R[X]$ . The relationship between the ideal theory of a ring  $R$  and its polynomial ring  $R[X]$  has been studied extensively; for the case when  $R$  is Noetherian principally by Krull [10], for  $R$  Prüfer by Seidenberg [15], for  $R$  Dedekind by Nagata [13], and for  $R$  arbitrary by numerous persons. We list now a few of the results which we shall need to refer to in § 3 and § 4.

RESULT A. If  $Q$  is a prime ideal of  $R$ , the set of all polynomials  $f(x)$  in  $R[X]$ , all of whose coefficients belong to  $Q$  is a prime ideal of  $R[X]$ , denoted by  $Q[X]$ . If  $M$  is prime in  $R[X]$  and if  $P = R \cap M$ , then  $R[X]/M \cong \bar{R}[\tau]$  where  $\bar{R} = R/P$  and  $\tau$  is the  $M$ -residue of  $X$ . We have  $P[X] \subseteq M$  and  $M = P[X]$  if and only if  $\tau$  is transcendental

over  $\bar{R}$ . [20; Appendix I].

RESULT B. If  $M_1, M_2, M_3$  are prime ideals of  $R[X]$  with  $M_1 \subset M_2 \subset M_3$ , then  $M_1 \cap R \subset M_3 \cap R$ . Also, if  $M_1 \cap R = M_2 \cap R$  then  $M_1 = [M_1 \cap R][X]$ . In particular if  $M_1 \neq (0)$  and  $M_1 \cap R = (0)$ ,  $M_1$  is a minimal prime of  $R[X]$ . [15; I; p. 505].

RESULT C. The element  $f(X) = a_0 + a_1X + \cdots + a_nX^n$  of  $R[X]$  is a unit of  $R[X]$  if and only if  $a_0$  is a unit of  $R$  and for  $i > 0$ ,  $a_i$  is nilpotent in  $R$ . As a consequence, it follows that the Jacobson radical  $R[X]$  is the set of nilpotent elements of  $R[X]$ . [16; p. 683].

3. The pseudo-radical of an integral domain. Suppose  $R$  is a ring. If  $R$  is not an integral domain then it is clear from the definition given in the introduction that the radical of  $R$  and the pseudo-radical of  $R$  coincide. Hence in considering the pseudo-radical in this section we restrict ourselves to the case when  $R$  is an integral domain. We note that for a one-dimensional domain  $D$  the pseudo-radical of  $D$  and the Jacobson radical of  $D$  are equal. Theorem 3 shows how the property that the pseudo-radical of the domain  $D$  is nonzero is reflected in the ideal theory of the polynomial domain  $D[X]$ . If  $D$  is a domain with quotient field  $K$  then by an *overring* of  $D$  we mean any domain between  $D$  and  $K$ .

LEMMA 1. Suppose  $D_1$  is a subring of the domain  $D_2$  and that each nonzero ideal of  $D_2$  intersects  $D_1$  in a nonzero ideal of  $D_1$ . If the pseudo-radical  $P_1^*$  of  $D_1$  is nonzero, then the pseudo-radical  $P_2^*$  of  $D_2$  is also nonzero.

The hypothesis clearly implies that  $P_1^* \subseteq P_2^* \cap D_1$ , from which the conclusion follows.

Examples are easily constructed for which  $P_2^*$  is nonzero but  $P_1^*$  is zero — see, for instance, Example 1.

The hypothesis of Lemma 1 is satisfied if and only if each nonzero prime ideal of  $D_2$  meets  $D_1$  in a nonzero ideal. [12; 105]. In particular the hypothesis of Lemma 1 holds if  $D_2$  is an overring of  $D_1$  or if  $D_2$  is integral over  $D_1$  [19; 259]. In the latter case we obtain a stronger conclusion than that of Lemma 1.

LEMMA 2. If the domain  $D_2$  is integral over its subring  $D_1$ , if  $P_i^*$  is the pseudo-radical of  $D_i$ , and if  $J_i$  is the Jacobson radical of  $D_i$ , then  $J_2 \cap D_1 = J_1$  and  $P_2^* \cap D_1 = P_1^*$ .

Both conclusions follow essentially from the “lying-over” theorem

of Krull [9; 749] and its consequences. For example, to show  $P_1^* = P_2^* \cap D_1$ , we have  $P_1^* \subseteq P_2^* \cap D_1$  by the proof of Lemma 1. And if  $t \in D_1 - P_1^*$ ,  $t \notin P$  for some nonzero prime  $P$  of  $D_1$ . By the lying over theorem, there is a nonzero prime ideal  $M$  of  $D_2$  such that  $P = M \cap D_1$ . Hence  $t \notin M$  and consequently  $t \notin P_2^* \cap D_1$ . Hence  $P_2^* \cap D_1 \subseteq P_1^*$  and equality holds.

LEMMA 3. *Let  $D$  be a domain with quotient field  $K$  and pseudo-radical  $P^*$ . The following statements are equivalent.*

(a)  $P^* \neq (0)$ .

(b)  $K$  is a simple ring extension of  $D$ .

(c)  $K$  is a finite ring extension of  $D$ .

(a)  $\rightarrow$  (b): If  $t$  is a nonzero element of  $P^*$ , then  $D[1/t] = D_{\{t^i\}_{i=1}^{\infty}}$  is a quotient ring of  $D$  such that each nonzero prime ideal of  $D$  meets  $\{t^i\}_{i=1}^{\infty}$ . Therefore (0) and  $D[1/t]$  are the only prime ideals of  $D[1/t]$ . [19; p. 224]. Consequently  $D[1/t]$  is the field  $K$  [19; p. 133] and (b) holds.

(b)  $\rightarrow$  (c): trivial

(c)  $\rightarrow$  (a): Let  $K = D[t_1, \dots, t_n]$  where  $t_i = a_i/b_i$ ;  $a_i, b_i \in D$ . Then clearly  $K = D[1/b] = D_{\{b^i\}_{i=1}^{\infty}}$  where  $b = b_1 b_2 \dots b_n$ . If  $M$  is a nonzero prime ideal of  $D$  then  $K = MK = MD_{\{b^i\}_{i=1}^{\infty}}$  so that  $M$  meets  $\{b^i\}_1^{\infty}$ ; hence  $b \in M$  and  $b \in P^* - \{0\}$ , showing that (a) holds.

REMARK 1. If  $D$  is a domain with pseudo-radical  $P^* \neq (0)$  we let  $d$  be a nonzero element of  $P^*$ . Then if  $M$  is any nonzero proper prime ideal of  $D$ ,  $d \in M$  so that  $M$  contains a minimal prime  $P$  of  $(d)$ . [11; p. 12]. But by choice of  $d$ ,  $P$  is a minimal prime of  $D$ . Hence if the pseudo-radical of  $D$  is nonzero, each nonzero proper prime of  $D$  contains a minimal prime of  $D$ .

THEOREM 1. *Suppose  $D$  is a domain with pseudo-radical  $P^*$ . If  $D$  is Noetherian,  $P^*$  is nonzero if and only if  $D$  is a one-dimensional semi-local domain. If  $D$  is a Krull domain,  $P^*$  is nonzero if and only if  $D$  is a semi-local principal ideal domain.*

If  $D$  is Noetherian, the conclusion that  $D$  is one-dimensional and semi-local is the content of Theorem 4 of [2]. And the converse is obvious.

If  $D$  is a Krull domain, let  $\{v_\alpha\}_{\alpha \in A}$  be a family of essential valuations for  $D$  (see [10; p. 82] for terminology) with  $v_\alpha$  associated with the minimal prime  $P_\alpha$ . If  $t$  is a nonzero element of  $P^*$ ,  $v_\alpha(t)$  is nonzero for only finitely many of the  $v_\alpha$ 's. But by choice of  $t$ ,  $v_\alpha(t)$  is positive for each  $\alpha \in A$ . Hence  $A$  is a finite set and  $D$  is an intersection of finitely many rank one discrete valuation rings. Hence  $D$  is a semi-

local principal ideal domain.<sup>1</sup> Again it is clear that the pseudo-radical of such a  $D$  is nonzero.

Theorem 1 shows that a Noetherian domain or a Krull domain having nonzero pseudo-radical is one-dimensional and has only finitely many maximal ideals. That one-dimensionality is not a necessary condition for the pseudo-radical to be zero may be seen by considering a valuation ring  $D$  of rank  $> 1$  such that  $D$  has a minimal prime ideal. The following example shows that a domain with pseudo-radical nonzero may contain infinitely many maximal ideals.

EXAMPLE 1. Let  $A$  be the domain of all algebraic integers. The set  $\{M_\alpha\}$  of maximal ideals of  $A$  lying over a given maximal ideal  $(p)$  of  $Z$  is uncountable. [17; p. 31]. Moreover if  $N = A - (\bigcup_\alpha M_\alpha)$ ,  $\{M_\alpha A_N\}$  is the set of maximal ideals of  $A_N$ . Because  $A_N$  is one-dimensional, the pseudo-radical of  $A_N$  is  $\bigcap M_\alpha A_N = pA_N \neq (0)$ .

Example 1 is more complicated than it need be to illustrate the fact that a domain with infinitely many maximal ideals may have nonzero pseudo-radical. But Example 1 illustrates some of the difficulties involved in characterizing Prüfer domains having nonzero pseudo-radical. And other than a restatement of Lemma 3 or Theorem 3 for the special case when  $D$  is Prüfer we have, in fact, no such satisfactory characterization. The domain  $A_N$  of Example 1 is one-dimensional and has the property that its finitely generated ideals are principal; hence  $A_N$  is a domain with the  $QR$ -property [5; p. 99]. In particular  $A_N$  is Prüfer. Theorem 2 represents our best result concerning arbitrary Prüfer domains having nonzero pseudo-radical.

THEOREM 2. *Suppose  $D$  is a Prüfer domain with nonzero pseudo-radical  $P^*$ . If  $\{P_\alpha\}_{\alpha \in A}$  is the set of minimal prime ideals of  $D$ , the complete integral closure  $D'$  of  $D$  is  $\bigcap_\alpha D_{P_\alpha}$ ; therefore  $D'$  is completely integrally closed.*

We shall use in the proof the following result which appears in an unpublished paper of the author, *On the strong integral closure of an integral domain.*

*If  $D_2$  is an overring of the domain  $D_1$  and if the conductor of  $D_1$  in  $D_2$  is nonzero then  $D_1$  and  $D_2$  have the same complete integral closure.*

Hence if we show the conductor  $C$  of  $D$  in  $\bigcap D_{P_\alpha} = D^*$  is nonzero,

<sup>1</sup> This result should be known though we are unable to give an explicit reference for the proof. Its validity follows from the following: By Theorem 11.11 of [14],  $D$  is an almost Dedekind domain with only finitely many maximal ideals. By Theorem 3 of [3],  $D$  is a semi-local principal ideal domain.

then  $D$  and  $D^*$  have the same complete integral closure. But  $D^*$  is an intersection of rank one valuation rings, each of which is completely integrally closed [8; p.170], so  $D^*$  is completely integrally closed, and  $D^* = D'$  as asserted.

We show then that  $P^* \subseteq C$ . Thus let  $p \in P^*$  and let  $t \in D^*$ . We show  $pt \in D$ .

If  $\{M_\beta\}$  is the set of maximal ideals of  $D$  it suffices to show  $pt \in D_{M_\beta}$  for each  $\beta$ . [20; p.94]. Let  $P_\beta$  be the minimal prime of  $D$  contained in  $M_\beta$ . Since  $t \in D_{P_\beta}$ ,  $t = a/b$  where  $a \in D$ ,  $b \in D - P_\beta$ . If now  $b \notin M_\beta$ ,  $t \in D_{M_\beta}$  so that  $pt \in D_{M_\beta}$ . If  $b \in M_\beta$  then we have

$$pD_{M_\beta} \subseteq P_\beta D_{M_\beta} \subset bD_{M_\beta};$$

it follows that  $p/b \in D_{M_\beta}$  and  $(p/b)a = pt \in D_{M_\beta}$ . This completes our proof.

Finally, if  $D$  is a domain having nonzero pseudo-radical  $P^*$ , we observe how this fact is reflected in the ideal theory of  $D[X]$ .

**THEOREM 3.** *Suppose  $D$  is a domain with pseudo-radical  $P^*$ ,  $X$  is an indeterminate over  $D$ , and  $K$  is the quotient field of  $D$ . The following statements are equivalent.*

- (a)  $P^* \neq (0)$ .
- (b) *there is a linear polynomial  $f(X) \in D[X]$  such that  $(f(X))$  is a maximal and minimal prime ideal in  $D[X]$ .*
- (c) *there exists a prime ideal  $M$  of  $D[X]$  such that  $M$  is both maximal and minimal in  $D[X]$ .*
- (d) *there exists a maximal ideal of  $D[X]$  lying over  $(0)$  in  $D$ .*

(a)  $\rightarrow$  (b): By Lemma 3, (a) implies  $K = D[1/t]$  for some nonzero element  $t$  of  $D$ . If then  $f(X) = tX - 1$ , we have  $D[X]/(tX - 1) \cong D[1/t] = K$ . Hence  $P = (tX - 1)$  is a maximal ideal of  $D[X]$ .

If  $P_1$  is a prime ideal properly contained in  $P$ ,  $P_1 = AP$  for some ideal  $A$  of  $D[X]$  since  $P$  is principal. Since  $P \supset P_1$  this implies  $A \subseteq P_1 = AP \subseteq A$ . Thus  $P_1 = P_1P = P_1P^2 = \dots$  and  $P_1 \subseteq \bigcap_{n=1}^{\infty} P^n = \bigcap_{n=1}^{\infty} (f^n(X))$ , which is clearly zero. Hence  $P_1 = (0)$ ,  $P$  is minimal, and (b) holds.

(b)  $\rightarrow$  (c): trivial.

(c)  $\rightarrow$  (d): Suppose  $M$  is both maximal and minimal in  $D[X]$  and let  $P = M \cap D$ .  $P[X] \subseteq M$  and  $P[X]$  is not maximal in  $D[X]$  by Results C and A. Hence  $M \supset P[X] \supseteq (0)$ . By the minimality of  $M$  it follows that  $P[X] = (0)$  and hence that  $P = (0)$  and (d) holds.

(d)  $\rightarrow$  (a): We let  $M$  be a maximal ideal of  $D[X]$  such that  $M \cap D = (0)$ . Then  $D[X]/M \cong D/D \cap M[\tau] \cong D[\tau]$  is a field;  $\tau$  the  $M$ -residue of  $X$ . This implies  $\tau$  is algebraic over  $D$  so that  $d\tau$  is integral over  $D$  for some nonzero  $d \in D$ . [18; 78]. Then  $D[d\tau]$  has

quotient field  $D[\tau] = D[d\tau][\tau]$ . By Lemma 3, the pseudo-radical of  $D[d\tau]$  is nonzero. By Lemma 2,  $P^* \neq (0)$  also.

To prove (d)  $\rightarrow$  (a) we could have appealed to a theorem of Nagata [13; I; p. 85], proved using different methods, which shows that (d) implies condition (b) of Lemma 3.

**4. A generalization of a theorem of Goldman.** Let  $R$  be a ring and let  $M$  be a prime ideal of the polynomial ring  $R[X]$ . If  $P = M \cap R$  we seek a relation between (\*) and (\*\*):

(\*)  $P$  is an intersection of maximal ideals of  $R$ .

(\*\*)  $M$  is an intersection of maximal ideals of  $R[X]$ .

Since  $R[X]/M \cong \bar{R}[\tau]$  where  $\bar{R} = R/P$  and  $\tau$  is the  $M$ -residue of  $X$ , we see that (\*\*) holds if and only if the Jacobson radical of  $\bar{R}[\tau]$  is zero. And (\*) is equivalent to the condition that the Jacobson radical of  $\bar{R}$  is zero. By Result A,  $\tau$  is transcendental over  $\bar{R}$  if and only if  $M = P[X]$ . And by Result C,  $\tau$  transcendental over  $\bar{R}$  implies the Jacobson radical of  $\bar{R}[\tau]$  is zero. Hence (\*\*) does not imply (\*) in general. In fact, Theorem 3 shows that if the pseudo-radical of the domain  $D$  is nonzero, then there is a maximal ideal  $M$  of  $D[X]$  such that (\*) is not valid for  $M \cap D = (0)$ .

We turn to the case when  $\bar{R}[\tau]$  is algebraic over  $\bar{R}$ —that is, the case where  $M \supset P[X]$ . Here  $d\tau$  is integral over  $\bar{R}$  for some nonzero  $d \in \bar{R}$ . Further  $\bar{R}$  has Jacobson radical zero if and only if  $\bar{R}[d\tau]$  has Jacobson radical zero. Noting that  $\tau$  is in the quotient field of  $\bar{R}[d\tau]$  and that  $\bar{R}[d\tau][\tau] = \bar{R}[\tau]$  it is then apparent that our earlier question concerning (\*) and (\*\*) is equivalent to a consideration of this question: Let  $D$  be a domain and  $t$  be an element of the quotient field of  $D$  such that  $D$  is integrally closed in  $D[t]$ . What is the relation between these statements?

(#) The Jacobson radical of  $D$  is zero.

(##) The Jacobson radical of  $D[t]$  is zero. Our earlier observations concerning (\*) and (\*\*) show that (##) does not imply (#) in general. We shall show, however, that (#) implies (##). From this it will follow that (\*) implies (\*\*); then Goldman's Theorem 3 of [6] follows as a corollary. We first prove one preliminary lemma.

**LEMMA 4.**<sup>2</sup> *Suppose  $R$  is a commutative ring with Jacobson radical  $(0)$ . If  $A = \{a_1, a_2, \dots, a_n\}$  is a finite set of regular elements of  $R$ , the intersection of all maximal ideals of  $R$  not meeting  $A$  contains no nonzero regular element. If "Jacobson radical" is replaced by "pseudo-radical" and "maximal" is replaced by "nonzero prime" the*

<sup>2</sup> For the proof of Lemma 4 as well as for several other helpful suggestions concerning the preparation of this paper, the author is grateful to W. J. Heinzer.

*statement obtained is also valid.*

We prove the first statement of the lemma. Let  $r$  be a nonzero regular element of  $R$ . By hypothesis  $x = ra_1 \cdots a_n$  is nonzero so there is a maximal ideal  $M$  of  $R$  such that  $x \notin M$ . Hence  $r \notin M$  and no  $a_i$  is in  $M$ ; therefore  $r$  does not belong to the intersection of the maximal ideals of  $R$  not meeting  $A$ .

**THEOREM 4.** *If  $D$  is a domain with quotient field  $K$  and if the Jacobson radical of  $D$  is  $(0)$  then for  $t \in K$ ,  $D[t]$  has Jacobson radical zero.*

We let  $t = a/b$ ;  $a, b \in D$ . If  $D_1 = D[t]$  and if  $D_2 = D[1/b] = D_{\{b^i\}_{i=1}^\infty}$ , then  $D \subseteq D_1 \subseteq D_2$ . If  $\{M_\alpha\}$  is the set of maximal ideals of  $D$  not containing  $b$ , Lemma 4 shows  $\bigcap M_\alpha = (0)$ . For each  $\alpha$  we have  $M_\alpha = M_\alpha D_2 \cap D \supseteq M_\alpha D_1 \cap D \supseteq M_\alpha$ ; hence  $M_\alpha = M_\alpha D_1 \cap D$  for each  $\alpha$ . Consequently  $M_\alpha D_1 \subset D_1$  so that  $M_\alpha D_1$  is contained in a maximal ideal  $P_\alpha$  of  $D_1$ . By the maximality of  $M_\alpha$  in  $D$ , we must have  $P_\alpha \cap D = M_\alpha$  for each  $\alpha$ . Thus  $(\bigcap P_\alpha) \cap D = \bigcap (P_\alpha \cap D) = \bigcap M_\alpha = (0)$ . Because  $D_1$  is an overring of  $D$ , this implies  $\bigcap P_\alpha = (0)$ ; in particular the intersection of all maximal ideals of  $D_1$ —that is, the Jacobson radical of  $D_1$ —is zero.

In view of our remarks preceding Lemma 4, the next two corollaries follow immediately from Theorem 4.

**COROLLARY 1.** *If  $\tau$  is algebraic over the domain  $D$ , if  $D[\tau]$  is an integral domain, and if the Jacobson radical of  $D$  is  $(0)$ , then the Jacobson radical of  $D[\tau]$  is  $(0)$ .*

**COROLLARY 2.** *Let  $M$  be a prime ideal of the polynomial domain  $D[X]$  over the domain  $D$ , and let  $M \cap D = P$ . If  $M = P[X]$ ,  $M$  is an intersection of maximal ideals of  $D[X]$ . If  $M \supset P[X]$  and if  $P$  is an intersection of maximal ideals of  $D$ ,  $M$  is an intersection of maximal ideals of  $D[X]$ .*

**COROLLARY 3.** (Goldman) *Let  $X$  be an indeterminate over the ring  $R$ .  $R[X]$  is a Hilbert ring if and only if  $R$  is a Hilbert ring.*

From the definition of a Hilbert ring, homomorphic images of Hilbert rings are again Hilbert. Thus if  $R[X]$  is a Hilbert ring, so is  $R[X]/(X) \cong R$ . The converse follows from Corollary 2.

It follows from Corollary 1 and from Result C that if the domain



$D$  has Jacobson radical zero then  $D_1$  has Jacobson radical zero where  $D_1$  is any domain which is a finite ring extension of  $D$ . The hypothesis that  $D_1$  be a domain is necessary as may be seen by taking  $D = Z$  and  $D_1 = Z[X]/(X^2)$ . Also, it is clear that Corollary 3, as well as most of our other results could be stated for polynomial rings in finitely many indeterminates.

In conclusion we consider what can be said about the ideal structure of a domain  $D$  such that for some element  $t$  of the quotient field of  $D$ ,  $D[t]$  has Jacobson radical  $(0)$ . Up to this point what we have observed is that we cannot conclude in general that  $D$  has Jacobson radical zero. We now note that by Lemma 1 it is clear that if  $D[t]$  is not the quotient field of  $D$  then  $D$  has pseudo-radical zero. If  $D$  is finite-dimensional we have a partial converse of this result.

**THEOREM 5.** *If the finite-dimensional domain  $D$  has Jacobson radical nonzero and pseudo-radical zero, there exists an overring  $D[t]$  of  $D$  such that  $D[t]$  has Jacobson radical zero.*

We first observe that if  $d$  is a nonzero element of  $D$ , the intersection of all nonzero primes of  $D$  not containing  $d$  is  $(0)$ . This follows from Lemma 4. Hence if  $z_1$  is a nonzero element of the Jacobson radical of  $D$ , what we have just noted implies  $D_1 = D[1/z_1]$  is a domain with pseudo-radical  $(0)$ . Also  $D_1$  has dimension less than that of  $D$  since  $z_1$  belongs to each maximal ideal of  $D$ . If the Jacobson radical  $J_1$  of  $D_1$  is zero we have proved the theorem. If  $J_1 \neq (0)$  then  $J_1 \cap D \neq (0)$  also since  $D_1$  is an overring of  $D$ . Hence if  $z_2$  is a nonzero element of  $J_1 \cap D$  and if  $D_2 = D_1[1/z_2]$ ,  $D_2$  has pseudo-radical zero, the dimension of  $D_2$  is less than that of  $D_1$ , and  $D_2 = D[1/z_1 z_2]$  is a simple ring extension of  $D$ . If the Jacobson radical  $J_2$  of  $D_2$  is zero our proof is complete; if not, we continue the process. The process stops after finitely many steps since  $D$  is finite-dimensional and since for a one-dimensional domain the Jacobson radical and the pseudo-radical coincide.

**COROLLARY 4.** *Suppose  $D$  is a finite-dimensional domain with nonzero Jacobson radical. In order that each simple overring  $D[t]$  of  $D$  which is not a field have nonzero Jacobson radical it is necessary and sufficient that the pseudo-radical of  $D$  be nonzero.*

Corollary 4 follows at once from Theorem 5 and the remarks preceding Theorem 5. The hypothesis of finite-dimensionality for  $D$  cannot be dropped; if  $D$  is a valuation ring having no minimal prime ideal, then each simple overring  $D[t]$  of  $D$  has nonzero Jacobson radical, but the pseudo-radical of  $D$  is  $(0)$ .

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