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DIAGONABILITY OF IDEMPOTENT MATRICES

ARTHUR STEGER

A ring $\mathscr{R}$ (commutative with identity) with the property that every idempotent matrix over $\mathscr{R}$ is diagonable (i.e., similar to a diagonal matrix) will be called an *ID*-ring. We show that, in an *ID*-ring $\mathscr{R}$, if the elements $a_1, a_2, \cdots, a_n \in \mathscr{R}$ generate the unit ideal then the vector $[a_1, a_2, \cdots, a_n]$ can be completed to an invertible matrix over $\mathscr{R}$. We establish a canonical form (unique with respect to similarity) for the idempotent matrices over an *ID*-ring. We prove that if $\mathscr{N}$ is the ideal of nilpotents in $\mathscr{R}$ then $\mathscr{R}$ is an *ID*-ring if and only if $\mathscr{R}/\mathscr{N}$ is an *ID*-ring. The following are then shown to be *ID*-rings: elementary divisor rings, a restricted class of Hermite rings, π-regular rings, quasi-semi-local rings, polynomial rings in one variable over a principal ideal ring (zero divisors permitted), and polynomial rings in two variables over a π-regular ring with finitely many idempotents.

In this paper, $\mathscr{R}$ will denote a commutative ring with identity, and $\mathscr{R}_n$ will denote the set of $n \times n$ matrices over $\mathscr{R}$. If $A, B \in \mathscr{R}_n$, then $A \cong B$ will mean that A is similar to B. We remark that if $\mathscr{R}$ is an *ID*-ring then every finitely generated projective R-module is the finite direct sum of cyclic modules, and that $\mathscr{R}$ is a directly indecomposable *ID*-ring if and only if every finitely generated projective $\mathscr{R}$-module is free. Most of the literature on this subject has been concerned with showing that a given ring $\mathscr{R}$ has the property that every finitely generated projective $\mathscr{R}$-module is free. This necessarily imposes the condition that $\mathscr{R}$ be indecomposable. In this paper, no such restriction is made.

2. Properties of *ID*-rings.

DEFINITION 1. $\mathscr{R}$ is said to be an *ID*-ring provided that for every $A = A^2 \in \mathscr{R}_n$, $n = 1, 2, \cdots$, there exists an invertible matrix $P \in \mathscr{R}_n$ such that PAP^{-1} is a diagonal matrix.

DEFINITION 2. The row vector $[a_1, a_2, \cdots, a_n]$ with components in $\mathscr{R}$ is said to be a basal provided that it can be completed to an invertible matrix over $\mathscr{R}$.

DEFINITION 3. The row vector X is said to be a characteristic vector of $A \in \mathscr{R}_n$ corresponding to $r \in \mathscr{R}$ provided (1) X is a basal vector and (2) $XA = rX$.

The following lemma, due to A. L. Foster, is an important tool in our development.

FOSTER'S LEMMA. *$\mathscr{R}$ is an ID-ring if and only if every idempotent matrix over $\mathscr{R}$ has a characteristic vector.*

From this lemma, which appears essentially as Theorem 10 in [2], one can quickly deduce that quasi-local rings and principal ideal domains are *ID*-rings. Then, known structure theorems suffice to show that principal ideal rings (see [7], p. 66), rings with descending chain condition, and Boolean rings are *ID*. These results will be extended in the next section.

THEOREM 1. *Let $A = A^2 \in \mathscr{R}_n$. If there exist invertible matrices $P, Q \in \mathscr{R}_n$ such that PAQ is a diagonal matrix then A is diagonable.*

Proof. Let $PAQ = B = \operatorname{diag}(b_1, b_2, \cdots, b_n)$ and let $U = Q^{-1}P^{-1} = (u_{ij})$. Then $(BU)^2 = BU$ and $BUB = B$. Hence $b_i = b_i^2 u_{ii}$, $b_i u_{ii}$ is idempotent, and by Lemma 2.1 of [9] $b_i \sim b_i u_{ii}$ for each i. Thus, we may assume that Q has been adjusted so that $b_i^2 = b_i$, $i = 1, 2, \cdots, n$. The equation $BUB = B$ now yields

(1) $b_i u_{ii} = b_i$, $i = 1, 2, \cdots, n$, and

(2) $b_i b_j u_{ij} = 0$, $i \neq j$, $i, j = 1, 2, \cdots, n$.

From (1),

$$BU = \begin{bmatrix} b_1 & b_1 u_{12} & \cdots & b_1 u_{1n} \\ b_2 u_{21} & b_2 & \cdots & b_2 u_{2n} \\ \vdots & \vdots & & \vdots \\ b_n u_{n1} & b_n u_{n2} & \cdots & b_n \end{bmatrix}.$$

If $X_k = [b_k u_{k1}, b_k u_{k2}, \cdots, b_k u_{kk-1}, 1, b_k u_{kk+1}, \cdots, b_k u_{kn}]$ then $X_k BU = b_k X_k$, $k = 1, 2, \cdots, n$. Now let

$$C = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

From (2), it follows that $|C| = 1$. Hence $(CP)A(CP)^{-1} = CBUC^{-1} = \operatorname{diag}(b_1, b_2, \cdots, b_n)$.

THEOREM 2. *Let $\mathscr{R}$ be an ID-ring. If $a_1, a_2, \cdots, a_n \in \mathscr{R}$ generate the unit ideal in $\mathscr{R}$ then the vector $[a_1, a_2, \cdots, a_n]$ is basal.*

Proof. Let $\sum_{i=1}^{n} x_i a_i = 1$ and let $B = (x_i a_j) \in \mathscr{R}_n$. Then $B^2 = B$ and $tr\, B = 1$. Since $\mathscr{R}$ is *ID*, $B \cong C = \operatorname{diag}(c_1, c_2, \cdots, c_n)$. If $X = [c_1, c_2, \cdots, c_n]$ then $XC = X$ and, since $\sum_{i=1}^{n} c_i = 1$,

$$\begin{vmatrix} c_1 & c_2 & c_3 \cdots c_n \\ -1 & 1 & 0 \cdots 0 \\ -1 & 0 & 1 \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ -1 & 0 & 0 \cdots 1 \end{vmatrix} = 1 .$$

Hence, B has a characteristic vector $Y = [y_1, y_2, \cdots, y_n]$ corresponding to 1. From $YB = Y$, we have $\sum_{i=1}^{n} y_i x_i a_j = y_j$, $j = 1, 2, \cdots, n$. Thus $(\sum_{i=1}^{n} y_i x_i)[a_1, a_2, \cdots, a_n] = [y_1, y_2, \cdots, y_n]$. Since Y is basal, so also is $[a_1, a_2, \cdots, a_n]$.

THEOREM 3. *If $\mathscr{R}$ is an ID-ring then every invertible ideal in $\mathscr{R}$ is principal.*

Proof. Let $\mathscr{K}$ be an invertible ideal in $\mathscr{R}$. Then there exist elements $a_1, a_2, \cdots, a_n \in \mathscr{K}$ and elements $x_1, x_2, \cdots, x_n$ in the full ring of quotients of $\mathscr{R}$ such that $x_i \mathscr{K} \subseteq \mathscr{R}$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} x_i a_i = 1$. It follows that $\mathscr{K} = (a_1, a_2, \cdots, a_n)$. Let $B = (x_i a_j) \in \mathscr{R}_n$. Then, as in Theorem 2, there exists a basal vector $Y = [y_1, y_2, \cdots, y_n]$ such that $y_j = \sum_{i=1}^{n} y_i x_i a_j$, $j = 1, 2, \cdots, n$. Now let $x_i = c_i/d$, $c_i, d \in \mathscr{R}$ and d not a zero divisor. If $p = \sum_{i=1}^{n} y_i c_i$ then $[pa_1, pa_2, \cdots, pa_n] = [dy_1, dy_2, \cdots, dy_n]$. Since Y is basal, $p\mathscr{K} = (d)$. Hence there is an $a \in \mathscr{K}$ such that $pa = d$. Thus, p is not a zero divisor. If $b \in \mathscr{K}$, then for some $r \in \mathscr{R}$, $pb = rd = pra$. Hence, $b = ra$ and $\mathscr{K} = (a)$.

Recall that if $\mathscr{S}$ is the set of idempotents of $\mathscr{R}$ then $\langle \mathscr{S}, \cap, \cup, {}^* \rangle$ where $a \cap b = ab$, $a \cap b = a + b - ab$, and $a^* = 1 - a$, is a Boolean algebra (see [1]). It follows that if $a_1, a_2, \cdots, a_n \in \mathscr{S}$ and $a = \bigcup_{i=1}^{n} a_1$ then $a_1, a_2, \cdots, a_n$ generate the principal ideal (a) in $\mathscr{R}$.

THEOREM 4. (Canonical Form) *Let $\mathscr{R}$ be an ID-ring and let $A = A^2 \in \mathscr{R}_n$. Then $A \cong \operatorname{diag}(a_1, a_2, \cdots, a_n)$ where $a_i \mid a_{i+1}$, $i = 1, 2, \cdots, n-1$. Moreover, if $A \cong \operatorname{diag}(b_1, b_2, \cdots, b_n)$ with $b_i \mid b_{i+1}$, $i = 1, 2, \cdots, n-1$, then $a_i = b_i$, $i = 1, 2, \cdots, n$.*

Proof. Since $\mathscr{R}$ is *ID*, let $A \cong C = \operatorname{diag}(c_1, c_2, \cdots, c_n)$ and let $a_1 = \bigcup_{i=1}^{n} c_i$. Then there exist idempotents $x_1, x_2, \cdots, x_n$ such that $x_i a_1 = c_i$ for each i and $\bigcup_{i=1}^{n} x_i = 1$. Thus, $(x_1, x_2, \cdots, x_n) = 1$ and, by Theorem 2, $X = [x_1, x_2, \cdots, x_n]$ is basal. Since x_i is idempotent, $i = 1, 2, \cdots, n$, $XC = a_1 X$ and, as in the proof of Foster's Lemma,

$A \cong \operatorname{diag}(a_1, d_2, \cdots, d_n)$. By induction, $A \cong \operatorname{diag}(a_1, a_2, \cdots, a_n)$ where $a_i \mid a_{i+1}$, $i = 2, 3, \cdots, n-1$. Since a_1 divides each entry of C, $a_1 \mid a_2$. If also, $A \cong \operatorname{diag}(b_1, b_2, \cdots, b_n)$ with $b_i \mid b_{i+1}$, $i = 1, 2, \cdots, n-1$, then it is a consequence of Theorem 9.3 of [6] that $b_i = a_i$ for each i. This can also be seen directly as follows: since a_r divides each r-rowed minor of $\operatorname{diag}(a_1, a_2, \cdots, a_n)$, a_r divides $b_r = b_i b_2 \cdots b_r$. Similarly, b_r divides a_r and, since both a_r and b_r are idempotent, $a_r = b_r$, $r = 1, 2, \cdots, n$.

COROLLARY. *If $\mathscr{R}$ is ID and $A = A^2 \in \mathscr{R}_n$ then A has a characteristic vector corresponding to $|A|$.*

Proof. We need merely observe that if $A \cong \operatorname{diag}(a_1, a_2, \cdots, a_n)$ with $a_i \mid a_{i+1}$, $i = 1, 2, \cdots, n-1$, then $a_n = |A|$.

THEOREM 5. *Let $\mathscr{J}$ be the Jacobson radical of $\mathscr{R}$, let $\mathscr{N}$ be the ideal of nilpotents in $\mathscr{R}$, and let $\mathscr{K}$ be an arbitrary ideal in $\mathscr{R}$. If $\mathscr{K} \subseteqq \mathscr{J}$ and $\mathscr{R}/\mathscr{K}$ is an ID-ring then $\mathscr{R}$ is an ID-ring. If $\mathscr{K} \subseteqq \mathscr{N}$ then $\mathscr{R}$ is an ID-ring if and only if $\mathscr{R}/\mathscr{K}$ is an ID-ring.*

Proof. Let $\mathscr{K} \subseteqq \mathscr{J}$ and assume that $\mathscr{R}/\mathscr{K}$ is *ID*. Let $A = A^2 = (A_{ij}) \in \mathscr{R}_n$ and $A^* = (a_{ij} + \mathscr{K})$. Then $(A^*)^2 = A^*$ and if $d = |A|$ then $d + \mathscr{K} = |A^*|$. By the corollary to Theorem 4, we may let $X^* = [x_1 + \mathscr{K}, x_2 + \mathscr{K}, \cdots, x_n + \mathscr{K}]$ be a characteristic vector of A^* corresponding to $d + \mathscr{K}$. Then, if $X = [x_1, x_2, \cdots, x_n]$, $XA = dX + Y$ where the components of Y are in $\mathscr{K}$. Since $A^2 = A$ and $d^2 = d$, $XA = dXA + YA$, $YA = (1-d)XA = (1-d)Y$, and $(X + (2d-1)Y)A = dX + dY = d(X + (2d-1)Y)$. Since $\mathscr{K} \subseteqq \mathscr{J}$, $u + \mathscr{K}$ is a unit of $\mathscr{R}/\mathscr{K}$ if and only if u is a unit of $\mathscr{R}$. It follows, therefore, that since X^* is basal so also is $X + (2d-1)Y$. By Foster's Lemma, $\mathscr{R}$ is *ID*. Now let $\mathscr{K} \subseteqq \mathscr{N}$. Since $\mathscr{N} \subseteqq \mathscr{J}$, we need only prove that if $\mathscr{R}$ is *ID* then $\mathscr{R}/\mathscr{K}$ is *ID*. Hence, assume that $\mathscr{R}$ is *ID* and $A^* = (A^*)^2 = (a_{ij} + \mathscr{K}) \in (\mathscr{R}/\mathscr{K})_n$. It will suffice to show that there exists an idempotent matrix $F = (f_{ij}) \in \mathscr{R}_n$ such that $f_{ij} + \mathscr{K} = a_{ij} + \mathscr{K}$, $i, j = 1, 2, \cdots, n$. If $A = (a_{ij})$ then $A^2 = A + B$ where the components of B are in $\mathscr{K}$. Thus B is nilpotent. Let k be the least natural number such that $B^k = Z =$ zero matrix. If $k = 1$, there is nothing left to prove. Hence, assume that $k > 1$ and let $C = A + (I - 2A)B$. Then the components of $C - A$ are in K and, since $AB = BA$,

$$C^2 = A^2 + 2A(I - 2A)B + (I - 2A)^2 B^2 .$$

Therefore, $C^2 - C = B + (I - 2A)^2(B^2 - B)$. Since $(I - 2A)^2 = I + 4B$,

$C^2 = C + B^2(4B - 3I)$. If we let $D = B^2(4B - 3I)$, we have $C^2 = C + D$ where the components of D are in $\mathscr{K}$ and, for some natural number $l < k$, $D^l = Z$. Repeating this process, we arrive in a finite number of steps at the required matrix F.

COROLLARY. *Let $\mathscr{N}$ be the ideal of nilpotents in $\mathscr{R}$ and let $x_1, x_2, \cdots, x_k$ be indeterminates. Then $\mathscr{R}[x_1, x_2, \cdots, x_k]$ is ID if and only if $(\mathscr{R}/\mathscr{N})[x_1, x_2, \cdots, x_k]$ is ID.*

Proof. The corollary follows by observing that $\mathscr{N}[x_1, x_2, \cdots, x_k]$ is the ideal of nilpotents in $\mathscr{R}[x_1, x_2, \cdots, x_k]$ and that

$$\mathscr{R}[x_1, x_2, \cdots, x_k]/\mathscr{N}[x_1, x_2, \cdots, x_k] \approx (\mathscr{R}/\mathscr{N})[x_1, x_2, \cdots, x_k] .$$

3. **Classes of ID-rings.** As an immediate consequence of Theorem 1, we have:

THEOREM 6. *An elementary divisor ring is an ID-ring.*

THEOREM 7. *Let $\mathscr{R}$ be a Hermite ring with Jacobson radical $\mathscr{J}$. If $\mathscr{R}$ has the property that $ab = 0$ implies either $(a) = (a^2)$ or $a \in \mathscr{J}$ or $b \in \mathscr{J}$ then $\mathscr{R}$ is an ID-ring.*

Proof. Let $A = A^2 = (a_{ij}) \in \mathscr{R}_n$ and let Q be an invertible matrix such that $QA = B = (b_{ij})$ is triangular; i.e., $b_{ij} = 0$ if $i < j$. Let $Q^{-1} = (p_{ij})$. Then $X = [b_{11}p_{11}, b_{11}p_{12}, \cdots, b_{11}p_{1n}]$ is the first row of QAQ^{-1}. If $(b_{11}) = (b_{11}^2)$ then there is an idempotent e such that $b_{11} \sim e$. By Theorem 3.9 of [6], there are vectors $X_2, X_3, \cdots, X_n$ such that $\begin{vmatrix} X \\ X_2 \\ \vdots \\ X_n \end{vmatrix} = e$. If $C = \begin{bmatrix} X \\ eX_1 \\ \vdots \\ eX_2 \end{bmatrix}$ then $|C + (1 - e)I| = 1$. Thus, the vector $Y = [b_{11}p_{11} + 1 - e, b_{11}p_{12}, \cdots, b_{11}p_{1n}]$ is basal and $Y(QAQ^{-1}) = X = eX = eY$; i.e, Y is a characteristic vector of QAQ^{-1} corresponding to e. If $b_{11} \in \mathscr{J}$ then $1 - b_{11}p_{11}$ is a unit of $\mathscr{R}$ and

$$[1 - b_{11}p_{11}, - b_{11}p_{12}, \cdots, - b_{11}p_{1n}]$$

is a characteristic vector of QAQ^{-1} corresponding to 0. Suppose now that neither of these assumptions on b_{11} is true. From the equation, $BA = QA^2 = QA = B$, we obtain $b_{11}(1 - a_{11}) = 0$. By the hypothesis on $\mathscr{R}$, $1 - a_{11} \in \mathscr{J}$, a_{11} is a unit of $\mathscr{R}$, and $[a_{11}, a_{12}, \cdots, a_{1n}]$ is a characteristic vector of A corresponding to 1. In any event, A has a characteristic vector and Foster's Lemma completes the proof.

THEOREM 8. *A π-regular ring is an ID-ring.*

Proof. Let $\mathscr{R}$ be π-regular with Jacobson radical $\mathscr{J}$. Then $\mathscr{R}/\mathscr{J}$ is regular and, therefore an elementary divisor ring (see [3], p. 365). The conclusion follows from Theorems 5 and 6.

THEOREM 9. *A quasi-semi-local ring is an ID-ring.*

Proof. Let $\mathscr{R}$ be quasi-semi-local with Jacobson radical $\mathscr{J}$. Since, by definition, $\mathscr{R}$ has only a finite number of maximal ideals, $\mathscr{R}/\mathscr{J}$ is a finite direct sum of fields. Theorem 5 completes the proof.

THEOREM 10. *Let $\mathscr{R}$ be an ID-ring and let $\mathscr{S}$ be a subring of $R[[x]]$ which contains $\mathscr{R}$. If $\mathscr{S}$ has the property that $u \in \mathscr{S}$ and u is a unit of $\mathscr{R}[[x]]$ imply that u is a unit of $\mathscr{S}$ then $\mathscr{S}$ is an ID-ring.*

Proof. Let $A = A^2 \in \mathscr{S}_n$ and let A' be the matrix in $\mathscr{R}_n$ obtained from A by suppressing all positive powers of x. If $A' = Z =$ zero matrix and $A \neq Z$, let k be the highest power of x which divides (in $R[[x]]$) each entry in A. Then we may write $A = x^kB$; and some entry in B is not divisible by x. Since A is idempotent $x^kB = x^{2k}B^2$. Thus, $B = x^kB^2$ and, since $k > 0$, we have arrived at a contradiction. Again, let $A = A^2 \in \mathscr{S}_n$. Then $(A')^2 = A'$ and, since $\mathscr{R}$ is ID, it follows from Theorem 4 that the entries of A' generate in $\mathscr{R}$ a principal ideal (e) where e is idempotent. Then $(1-e)A$ is idempotent and $((1-e)A)' = Z$. Thus, $(1-e)A = Z$. Let P be an invertible matrix in $\mathscr{R}_n$ such that $PA'P^{-1} = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ where $a_i \mid a_{i+1}$, $i = 1, 2, \cdots, n-1$. Therefore, $a_1 = e$ and $PAP^{-1} = B = (b_{ij})$ with $b_{11} = e + r_1x + r_2x^2 + \cdots$. Then, if $Y = [1 - e + b_{11}, b_{12}, \cdots, b_{1n}]$, $(1-e)B = Z$ implies $YB = eY$. Since $1 - e + b_{11}$ is a unit in $R[[x]]$, by the hypothesis on $\mathscr{S}$, Y is a characteristic vector corresponding to e. The theorem follows from Foster's lemma.

Theorem 10 shows for example that the domain of complex valued functions of a complex variable which are analytic at some point z_0 in the complex plane is an ID-ring, or that the domain of real valued functions of a real variable analytic at some real number r_0 is ID. It is also true that the domain of entire functions is ID. This has, however, nothing to with Theorem 10; but it is rather a consequence of Theorem 7 in conjunction with a theorem proved in [4] to the effect that in the domain of entire functions every finitely generated ideal is principal.

The problem of determining, given a ring $\mathscr{R}$, whether or not $\mathscr{R}[x]$ is *ID* is a difficult one. An important result in this area is due to Seshadri who proved in [8] that if $\mathscr{R}$ is a principal ideal domain then $\mathscr{R}[x]$ is *ID*. In particular, $\mathscr{K}[x, y]$, where $\mathscr{K}$ is a field, is $\mathscr{I}D$. The character of $\mathscr{K}[x, y, z]$ is open. Horrocks showed ([5]), p. 718) that if $\mathscr{R}$ is a regular local ring of dimension 2 with a field of coefficients then $\mathscr{R}[x]$ is *ID*. Chase, on the other hand, has constructed an example (unpublished) of a complete local domain $\mathscr{R}$ such that $\mathscr{R}[x]$ is not *ID*. The ring in Chase's example has dimension 1, is not a regular local ring, and in fact is not integrally closed.

THEOREM 11. *Let $\mathscr{R}$ be a ring with $\mathscr{N}$ its ideal of nilpotents.* (1) *If $\mathscr{R}/\mathscr{N}$ is a principal ideal ring then $\mathscr{R}[x]$ is ID*; (2) *if $\mathscr{R}/\mathscr{N}$ is a Boolean ring then $\mathscr{R}[x, y]$ is ID*; *and* (3) *if $\mathscr{R}$ is a π-regular ring with finitely many idempotents then $\mathscr{R}[x, y]$ is ID.*

Proof. The assertions of this theorem are a consequence of applying the Corollary to Theorem 5 to Seshadri's result. First, assume that $\mathscr{R}/\mathscr{N}$ is a principal ideal ring. It is a consequence of the result on page 66 of [7] that $\mathscr{R}/\mathscr{N}$ is a finite direct sum of principal ideal domains. Thus (1) has been established. Now assume that $\mathscr{R}/\mathscr{N}$ is a Boolean ring and let $A = A^2 \in ((\mathscr{R}/\mathscr{N})[x, y])_n$. Then the set of coefficients of the entries in A together with 1 generate a finite Boolean subring $\mathscr{S}$ of $\mathscr{R}/\mathscr{N}$ whose unit element is the unit element of $\mathscr{R}/\mathscr{N}$. Since $\mathscr{S}$ is the finite direct sum of fields, A is diagonable and (2) has been proved. Finally, assume that $\mathscr{R}$ is a π-regular ring with finitely many idempotents. Then $\mathscr{R}/\mathscr{N}$ is the finite direct sum of fields. This completes the proof of (3).

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UNIVERSITY OF NEW MEXICO