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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS OF HIGHER ORDER

LU-SAN CHEN

It is known that the solution u of the heat equation $\partial u/\partial t = \Delta u$ under the boundary condition $u = 0$ decays as $e^{-\lambda t}u$ for some $\lambda > 0$ as $t \to \infty$. This gives us information about the asymptotic behavior of the solution u in time.

There arises the question whether such a theorem is valid for parabolic differential equations with variable coefficients.

In this note we shall treat this problem and prove that the theorem analogous to the above holds for parabolic differential inequalities of higher order under some additional restrictions.

Consider the unit sphere $\mathscr{D}$ in the n-dimensional Euclidean space E^n with boundary Γ and denote by $I(T)$ the interval $0 \leqq t \leqq T$ and by I the half-infinite interval $0 \leqq t < \infty$. The $(n+1)$-dimensional domain $\mathscr{D} \times I$ will be designated by R, while S will be the portion $\Gamma \times I$ of the boundary of R.

We are interested in the growth of functions $u(x, t)$ which satisfy the differential inequality of the form

$$(Lu)^2 \leqq c(t) \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2$$

in R and $D_x^\alpha u = 0$ $(|\alpha| \leqq s-1)$ on S. Here L is a parabolic differential operator of higher order written in the form

$$L \equiv (-1)^s \frac{\partial}{\partial t} - \sum_{|\alpha| \leqq 2s} a_\alpha D_x^\alpha, \tag{1}$$

where all the coefficients $a_\alpha = a_\alpha(x, t)$ are s-times continuously differentiable in (a neighborhood of) $R \cup S$ and

$$A \equiv \sum_{|\alpha| \leqq 2s} a_\alpha D_x^\alpha \tag{2}$$

$$\left(D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},\ \alpha = (\alpha_1, \cdots, \alpha_n),\quad |\alpha| = \alpha_1 + \cdots + \alpha_n \right)$$

is assumed to be uniformly elliptic throughout $R \cup S$, i.e., there is a constant k_0 depending only on A such that

$$\sum_{|\alpha| \leqq 2s} a_\alpha \xi^\alpha \geqq k_0 (\xi_1^2 + \cdots + \xi_n^2)^s$$

for any real vector $\xi = (\xi_1, \cdots, \xi_n)$.

Let $v(x, t)$ be a $2s$-times continuously differentiable function in

$R \cup S$ such that

$$D_x^\alpha v = 0 \qquad (|\alpha| \leqq s-1) \text{ on } S \tag{3}$$

and

$$v(x, t) = 0 \text{ in } (x, t) \in \mathscr{D} \times I(T_0) \tag{4}$$

for some T_0. The function v satisfying (3) and (4) is said to belong to the class V.

We shall prove the following theorem.

THEOREM. *Suppose that u satisfies the inequality*

$$(Lu)^2 \leqq c(t) \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2 \tag{5}$$

in $R \cup S$ and that $D_x^\alpha u$ ($|\alpha| \leqq s-1$) vanish on S and suppose that the condition

$$\lim_{t\to\infty} \int_{\mathscr{D}} e^{2\lambda t} u^2 \, dx = 0$$

holds for any $\lambda > 0$. If $c(t)$ is bounded and continuous in I and if $c(t) = 0(t^{-1})(t \to \infty)$ then u is identically equal to zero throughout R.

This is an analogue to Protter's theorem [2], where parabolic operators of second order are considered.

2. To prove the theorem, we prepare two inequalities deduced from the following lemmas whose proofs are found in [1].

LEMMA 1. *Assume that the differential operator A in (2) is uniformly elliptic in $R \cup S$. If v is in V, if $f = f(t)$ is in $C^1([0, \infty))$ and if $g = g(t)$ continuous in $[0, \infty)$ has no zero, then there exist two positive constants k_1 and k_2 depending only on A such that*

$$\begin{aligned} k_1 \iint_R f^2 \sum_{|\alpha| \leqq s} |D_x^\alpha v|^2 \, dxdt &\leqq \iint_R f^2 g^2 (Lv)^2 \, dxdt \\ &+ \iint_R (k_2 f^2 - 2ff' + f^2 g^{-2}) v^2 \, dxdt + \lim_{t\to\infty} \int_{\mathscr{D}} f^2 v^2 \, dx \, . \end{aligned}$$

LEMMA 2. *Suppose that v is in V and that $f = f(t)$ is in $C^\infty([0, \infty))$ and $g = g(t)$ continuous in $[0, \infty)$ have no zero, then for a given operator L in (1), there exists a constant k_3 depending only on A such that*

$$\iint_R ff''v^2dxdt \leqq \iint_R f^2(Lv)^2dxdt$$
$$+ k_3\iint_R (f^2g^2 + f'^2g^{-2}) \sum_{|\alpha|\leqq s} | D_x^\alpha v |^2 dxdt$$
$$+ \lim_{t\to\infty} \int_{\mathscr{D}} ff'v^2dx .$$

First, if $v \in V$, then by putting $f = e^{\lambda t}$ and $g = 1$, Lemma 1 implies that there exists two positive constants k_1 and k_2 depending only on A such that

(6)
$$k_1\iint_R e^{2\lambda t} \sum_{|\alpha|\leqq s} | D_x^\alpha v |^2dxdt$$
$$\leqq \iint_R e^{2\lambda t}(Lv)^2dxdt + \iint_R e^{2\lambda t}(k_2 + 1)v^2dxdt.$$

Next, if $v \in V$, then by putting $f = e^{\lambda t}$ and $g = \sqrt{\lambda}$ it follows from Lemma 2 that there exists a constant k_3 depending only on A such that

(7)
$$\iint_R e^{2\lambda t}\lambda^2v^2dxdt$$
$$\leqq \iint_R e^{2\lambda t}(Lv)^2dxdt + 2k_3\iint_R e^{2\lambda t}\lambda \sum_{|\alpha|\leqq s} | D_x^\alpha v |^2 dxdt.$$

These are analogues to Protter's estimates, Lemma 3 and Lemma 4 in [2].

From (6) and (7), we get

$$\left(k_1 - \frac{2(k_2 + 1)k_3}{\lambda}\right)\iint_R e^{2\lambda t} \sum_{|\alpha|\leqq s} | D_x^\alpha v |^2 dxdt$$
$$\leqq \left(1 + \frac{k_2 + 1}{\lambda^2}\right)\iint_R e^{2\lambda t}(Lv)^2dxdt, \quad v \in V .$$

So, if λ is sufficiently large, for instance, if $\lambda \geqq \lambda_0$, we have

(8)
$$\frac{k_1}{2}\iint_R e^{2\lambda t} \sum_{|\alpha|\leqq s} | D_x^\alpha v |^2dxdt$$
$$\leqq 2\iint_R e^{2\lambda t}(Lv)^2dxdt$$

for $v \in V$.

3. Now we give the proof of theorem. Let $\varphi = \varphi(t)$ be an infinitely many times differentiable function of t such that

$$\varphi(t) = \begin{cases} 0 & , \ 0 \leqq t \leqq T_1 \\ 0 < \varphi < 1, & T_1 \leqq t \leqq T_2 \\ 1 & , \ T_2 \leqq t < \infty \end{cases}$$

for some T_1 and $T_2 (T_1 < T_2)$.

The function $v(x, t) = \varphi(t) . u(x, t)$ is in the class V and the inequality (8) is valid for v. We put

$$R(T_2 - T_1) = \mathscr{D} \times (I(T_2) - I(T_1)) \quad \text{and} \quad R(T_2) = \mathscr{D} \times (I - I(T_2)).$$

The inequality (8) implies that

$$\frac{k_1}{2} \iint_{R(T_2)} e^{2\lambda t} \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2 \, dx dt$$
$$\leqq 2 \iint_{R(T_2 - T_1)} e^{2\lambda t} (Lv)^2 dx dt + 2 \iint_{R(T_2)} e^{2\lambda t} (Lu)^2 dx dt \, .$$

We substitute (5) into the last integral on the right and get

$$\iint_{R(T_2)} \left[\frac{k_1}{2} - 2c(t) \right] e^{2\lambda t} \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2 \, dx dt$$
$$\leqq 2 \iint_{R(T_2 - T_1)} e^{2\lambda t} (Lv)^2 dx dt \, .$$

Since $c(t) = 0(t^{-1}) (t \rightarrow \infty)$ by the assumption, we see that there exists a positive constant δ such that $k_1/2 - 2c(t) > \delta$ if $t \geqq T_3$ for some sufficiently large $T_3 (> T_2)$. It holds that

$$\iint_{R(T_3)} \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2 \, dx dt \leqq \frac{2}{\delta} e^{2\lambda (T_2 - T_3)} \iint_{R(T_2 - T_1)} (Lv)^2 \, dx dt \, .$$

Since $\lambda (\geqq \lambda_0)$ is arbitrary, letting $\lambda \longrightarrow \infty$, we see at once that $u \equiv 0$ in $R(T_3)$.

As $c(t)$ is bounded in I, we can apply the theorem in [1] for this function u and we can conclude that u vanishes throughout R.

References

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Department of Mathematics
Taiwan Provincial Cheng-Kung University, Tainan
and
Mathematical Institute
Nagoya University, Nagoya