

SIMPLIFYING INTERSECTIONS OF DISKS IN BING'S SIDE APPROXIMATION THEOREM

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In Bing's Side Approximation Theorem for 2-spheres in E^3 the disks on the approximating sphere and the disks on the given sphere may intersect in a very complicated manner. It is shown in this paper that the disks may be chosen so that there are the same number of disks on the approximating sphere as on the given sphere and the disks intersect in a one-to-one fashion. Furthermore, the approximating homeomorphism may be chosen so that it maps each disk on the given sphere onto the disk on the approximating sphere which it intersects.

Applications are given to a study of the preservation of tameness of subsets of the boundary of a crumpled cube under re-embeddings of the crumpled cube in E^3 .

An ε -mapping of a subset of E^3 into E^3 is a mapping which moves no point a distance as much as ε . An ε -set in E^3 is a set of diameter less than ε . David Gillman [6] and L. D. Loveland [10, 11] have used the following definition in connection with tame subsets of 2-spheres in E^3 .

Property $(*, F, V)$. If F is a closed subset of a 2-sphere S in E^3 and V is a component of $E^3 - S$, then $(*, F, V)$ is satisfied if for each $\varepsilon > 0$ there exist an ε -homeomorphism h of S into E^3 , a finite collection of mutually exclusive ε -disks $\{D_1, D_2, \dots, D_r\}$ on $h(s)$, and a finite collection of mutually exclusive ε -disks $\{E_1, E_2, \dots, E_n\}$ on S such that

- (i) $h(s)$ is polyhedral,
- (ii) $h(s) - \bigcup \text{Int } D_j \subset V$,
- (iii) $h(s) \cap s \subset \bigcup \text{Int } E_i$, and
- (iv) $(\bigcup E_i) \cap F = \emptyset$.

Bing's Side Approximation Theorem [3, Th. 16] may then be stated as follows.

THEOREM A (Bing). *If S is a 2-sphere in E^3 and V is a component of $E^3 - S$, then $(*, \emptyset, V)$ is satisfied.*

We show in § 6 that if $(*, F, V)$ is satisfied, then the homeomorphism h and the collections of disks may be chosen so that the following additional properties are satisfied.

- (v) $r = n$,

- (vi) $h(E_i) = D_i$, and
 (vii) $D_i \cap S = h(s) \cap E_i = (\text{Int } D_i) \cap (\text{Int } E_i)$.

We denote the property with conditions (i)—(vii) by $(**, F, V)$. Thus, as shown by Theorem 2, $(*, F, V)$ is equivalent to $(**, F, V)$. This strengthens Theorem A, that is, $(**, \emptyset, V)$ holds for any 2-sphere S and either complementary domain V in E^3 . Furthermore, it is a consequence of the Tietze Extension Theorem [8, page 82] and Dehn's Lemma [12] that if $(**, F, V)$ is satisfied and g is a homeomorphism of $S \cup V$ into E^3 , then $(**, g(F), g(V))$ is satisfied. The same invariance then also holds for $(*, F, V)$. See § 9. Some theorems on the invariance of tameness and half tameness (as defined below) of subsets of a 2-sphere S in E^3 under homeomorphisms of $S \cup V$ into E^3 follow from this result and are given in § 10.

Most of the definitions in this paper are like those used by Loveland in [10] and [11]. We say that a closed subset F of a 2-sphere in E^3 is half tame if it is a subset of some 2-sphere which is tame from one of its complementary domains and that F is tame if it is a subset of a tame 2-sphere.

To show that $(*, F, V)$ implies $(**, F, V)$ we begin with an approximation to S from V of the type given by $(*, F, V)$. Then by a series of steps involving a new selection of disks at each step we obtain a finite collection of mutually exclusive small disks $\{D_1, D_2, \dots, D_n\}$ on the approximating sphere $h(S)$ in one-to-one correspondence with a collection of n mutually exclusive small disks $\{E_1, E_2, \dots, E_n\}$ on S with the property that $h(S) - \bigcup \text{Int } D_i \subset V$, the closure of the component of $D_i - S$ containing $\text{Bd } D_i$ intersects S in the interior of the corresponding disk E_i , and $(\bigcup E_i) \cap F = \emptyset$. The proof is then completed with an application of the Tietze Extension Theorem, Dehn's Lemma, and a homeomorphism of the approximating sphere onto itself. Details are given in Sections 4, 5, and 6.

2. **Covering collections of disks.** This section includes a key step in obtaining the new collection of mutually exclusive small disks on the approximating sphere. This result is stated and proved as Lemma 3 below. The following lemma is a theorem of plane topology and is stated here without proof.

LEMMA 1. *If $\{K_1, K_2, \dots, K_n\}$ is a finite collection of mutually exclusive compact sets in E^2 such that $E^2 - \bigcup K_i$ is connected, then there is a collection of mutually exclusive disks $\{D_1, D_2, \dots, D_n\}$ in E^2 such that $K_i \subset \text{Int } D_i$.*

A special cellular decomposition of a 2-sphere S is a finite cellular decomposition of S in which each pair of intersecting cells intersect

in an arc. Each 2-sphere has a special cellular decomposition of arbitrarily small mesh [2]. The star $\text{st } C_p$ of a cell C_p of a cellular decomposition is the union of the cells which intersect C_p . If \mathcal{A} is a collection of sets we denote the union of the sets in \mathcal{A} by \mathcal{A}^* .

LEMMA 2. *If $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ is a special cellular decomposition of a 2-sphere S , \mathcal{D} is a finite collection of mutually exclusive disks each of which is a subset of the interior of a cell of \mathcal{C} , and $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m\}$ is a partition of \mathcal{D} (plus possibly some empty collections) such that $\mathcal{D}_p^* \subset \text{st } C_p$ ($p = 1, 2, \dots, m$), then there is a collection of mutually exclusive disks $\{D_1, D_2, \dots, D_m\}$ such that $\mathcal{D}_p^* \subset \text{Int } D_p \subset \text{st } C_p$.*

Proof. Without loss of generality we assume that for each p , \mathcal{D}_p is nonempty and there is a disk of \mathcal{D}_p in each cell of the star of C_p . Denote the subcollection of \mathcal{D}_q each disk of which lies in C_p by \mathcal{D}_{qp} . Now use Lemma 1 to cover each \mathcal{D}_{qp}^* by a disk D_{qp} which lies in C_p in such a way that the disks in the resulting collection are mutually exclusive. Then there is exactly one disk in $\text{Int } C_p$ associated with each cell in $\text{st } C_p$. A pattern like that in Figure 1 is obtained. For $q \neq p$ there is an arc A_{qp} from D_{qp} to C_q which lies except for end points in $(\text{Int } C_p - \bigcup D_{rp})$ such that the arcs in the resulting collection are mutually exclusive. Next there is an arc B_{qp} from D_{qp} to the end point of the arc A_{qp} such that B_{qp} lies, except for its intersection

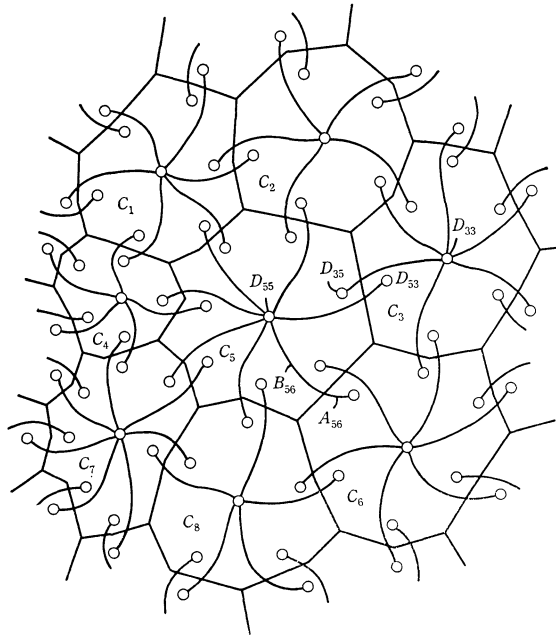


Figure 1.

with A_{qp} , in $(\text{Int } C_q - \bigcup D_{r_q} - \bigcup A_{s_q})$. The arcs in the resulting collection $\{A_{qp} \cup B_{qp}\}$ over all of S are mutually exclusive. The required disks are obtained (from the spider-like figures in Figure 1) by replacing each of these arcs by a thin disk.

LEMMA 3. *If $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ is a special cellular decomposition of a 2-sphere S , \mathcal{D} is a finite collection of mutually exclusive disks each of which is a subset of the interior of a cell of \mathcal{C} , and $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n\}$ is a partition of \mathcal{D} such that $\mathcal{D}_i^* \subset \text{st } C_p$ for some $C_p \in \mathcal{C}$, then there is a collection of mutually exclusive disks $\{D_1, D_2, \dots, D_n\}$ such that $\mathcal{D}_i^* \subset \text{Int } D_i \subset \text{st } C_q$ for some $C_q \in \mathcal{C}$.*

Proof. Assign each subcollection \mathcal{D}_i a cell $C_{p(i)}$ of \mathcal{C} such that $\mathcal{D}_i^* \subset \text{st } C_{p(i)}$. Let \mathcal{E}_p be the union of all \mathcal{D}_i 's assigned to C_p . Then $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m\}$ is a partition of \mathcal{D} (plus possibly some empty collections) satisfying the hypothesis of Lemma 2. Hence there is a collection of mutually exclusive disks $\{E_1, E_2, \dots, E_m\}$ such that $\mathcal{E}_p^* \subset \text{Int } E_p \subset \text{st } C_p$. Now use Lemma 1 to cover each \mathcal{D}_i^* in $\text{Int } E_p$ by a disk D_i in $\text{Int } E_p$ such that the disks in the resulting collection are mutually exclusive. Then $\{D_1, D_2, \dots, D_n\}$ is a collection of disks satisfying the conclusion of Lemma 3.

For a given collection \mathcal{D} of mutually exclusive disks on S to satisfy the hypothesis of Lemma 3 that each disk be contained in the interior of a cell of the cellular decomposition, it may be necessary to obtain a new cellular decomposition by a homeomorphism of S onto itself which carries the 1-skeleton of a given cellular decomposition off the disks of \mathcal{D} . The procedure is like that used by Bing in [2] and we simply state the following lemma for future use.

LEMMA 4. *If K is the 1-skeleton of a cellular decomposition of a 2-sphere S , $\varepsilon > 0$, and $\{D_1, D_2, \dots, D_n\}$ is a finite collection of mutually exclusive ε -disks on S , then there is an ε -homeomorphism h of S onto itself such that $(\bigcup D_i) \cap h(K) = \emptyset$.*

3. Intersection of a disk with a finite collection of disks.
Another important step in obtaining a new collection of disks on the approximating sphere is given in this section. The main result is Lemma 6, but the following lemma is needed first.

LEMMA 5. *If S is a 2-sphere in E^3 , $\{E_1, E_2, \dots, E_n\}$ is a collection of mutually exclusive disks on S , V is a component of $E^3 - S$, D is a disk in E^3 with $\text{Bd } D$ contained in V , and G is the component of $D - S$ containing $\text{Bd } D$ such that $(\text{cl } G) \cap S = (\text{cl } G) \cap (\bigcup \text{Int } E_i)$ and*

$(\text{cl } G) \cap E_i \neq \emptyset$ ($i = 1, 2, \dots, n$) then the union of any collection of components of $D - G$ fails to separate $\text{Int } D$ and each component of $D - G$ intersects exactly one of the sets $(\text{cl } G) \cap E_i$ ($i = 1, 2, \dots, n$).

Proof. Let $K = D - G$. For each component K^α of K , $K^\alpha \cap (\text{cl } G - \text{Bd } D) \neq \emptyset$ and $G - \text{Bd } D$ is connected. It follows that if \mathcal{A} is an indexing set for the set of components of K and $\mathcal{B} \subset \mathcal{A}$ then $\text{Int } D - \bigcup_{\alpha \in \mathcal{B}} K^\alpha = (G - \text{Bd } D) \cup \bigcup_{\alpha \in \mathcal{A} - \mathcal{B}} K^\alpha$ is connected. It follows from the fact that spheres are unicoherent [13, page 60] that $K^\alpha \cap \text{cl}(\text{Int } D - K^\alpha)$ is connected for each $\alpha \in \mathcal{A}$ and therefore K^α may intersect at most one of the sets $(\text{cl } G) \cap E_i$. Since it must intersect at least one of these sets, K^α intersects exactly one.

LEMMA 6. *If the hypothesis of Lemma 5 is satisfied, then there is a collection of mutually exclusive disks $\{D_1, D_2, \dots, D_n\}$ in $\text{Int } D$ such that $\text{Bd } D_i \subset V$ and $(\text{cl } G_i) \cap S = (\text{cl } G_i) \cap \text{Int } E_i = (\text{cl } G) \cap \text{Int } E_i$ ($i = 1, 2, \dots, n$), where G_i is the component of $D_i - S$ containing $\text{Bd } D_i$.*

Proof. By Lemma 5 each component of $D - S$ intersects exactly one of the sets $(\text{cl } G) \cap E_i$. Let \mathcal{H}_i be the collection of all components of $D - G$ which intersect $(\text{cl } G) \cap E_i$. By Lemma 5, \mathcal{H}_i^* fails to separate $\text{Int } D$. Since the E_i 's are mutually exclusive, $\{\mathcal{H}_1^*, \mathcal{H}_2^*, \dots, \mathcal{H}_n^*\}$ is a collection of sets satisfying the hypothesis of Lemma 1. Hence there is a collection of mutually exclusive disks $\{D_1, D_2, \dots, D_n\}$ in $\text{Int } D$ such that $\mathcal{H}_i^* \subset \text{Int } D_i$. Note that $(\text{cl } G) \cap E_i = (\text{cl } G) \cap \mathcal{H}_i^*$ and $(\text{cl } G) \cap E_i$ fails to intersect E_k for $k \neq i$. Thus each D_i has the property that $\text{Bd } D_i \subset V$ and $(\text{cl } G_i) \cap S = (\text{cl } G_i) \cap \text{Int } E_i = (\text{cl } G) \cap \text{Int } E_i$. $D'_1, \{D_{11}, D_{12}\}, G_1, \{G_{11}, G_{12}\}$ illustrate this in Figure 2.

4. **Pairing disks in side approximations.** The preceding lemmas along with the Tietze Extension Theorem and Dehn's Lemma are used to prove the following theorem which shows that a side approximation to a sphere may be replaced by another such that the disks intersect in a one-to-one fashion. See Figure 2.

THEOREM 1. *If F is a closed subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$ such that $(*, F, V)$ is satisfied, and $\varepsilon > 0$, then there exist an ε -homeomorphism h of S into E^3 , a finite collection of mutually exclusive ε -disks $\{D_1, D_2, \dots, D_n\}$ on $h(S)$, and a collection of n mutually exclusive ε -disks $\{E_1, E_2, \dots, E_n\}$ on S such that*

- (i) $h(S)$ is polyhedral,
- (ii) $h(S) - \bigcup \text{Int } D_i \subset V$,
- (iii) $D_i \cap S = h(S) \cap E_i = (\text{Int } D_i) \cap (\text{Int } E_i)$, and

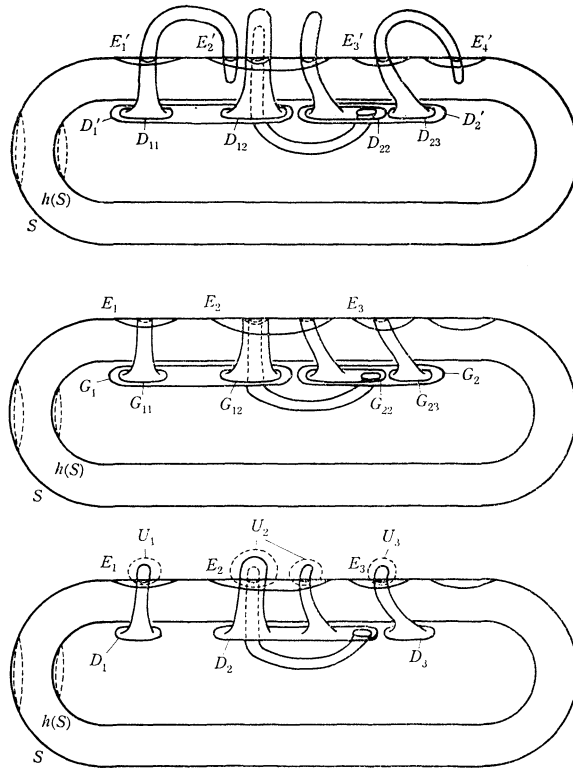


Figure 2.

(iv) $(\bigcup E_i) \cap F = \emptyset$.

Proof. Let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be a special cellular decomposition of S of mesh less than $\epsilon/9$, and let

$$(1) \quad \delta = \min \{ 1/3 \min \{ \rho(C_p, C_q) : C_p \cap C_q = \emptyset ; C_p, C_q \in \mathcal{C} \} , \\ 1/2 \min \{ \epsilon/9 - \text{diam } C_p : C_p \in \mathcal{C} \} \} .$$

By $(*, F, V)$ there exist a $\delta/2$ -homeomorphism h_1 of S into E^3 , a finite collection of mutually exclusive $\delta/2$ -disks $\{D'_1, D'_2, \dots, D'_j\}$ on $h_1(S)$, and a finite collection of mutually exclusive $\delta/2$ -disks $\{E'_1, E'_2, \dots, E'_i\}$ on S such that

(2) $h_1(S)$ is polyhedral ,

(3) $h_1(S) - \bigcup \text{Int } D'_j \subset V$,

(4) $h_1(S) \cap S = (\bigcup \text{Int } D'_j) \cap S \subset \bigcup \text{Int } E'_i$, and

(5) $(\bigcup E'_i) \cap F = \emptyset$.

Note that $\text{Bd } D'_j \subset V$. If $(\bigcup D'_j) \cap S = \emptyset$ we are finished. Otherwise

let G_j denote the component of $D'_j - S$ containing $\text{Bd } D'_j$ and let $\{E_1, E_2, \dots, E_n\}$ be the subcollection of $\{E'_1, E'_2, \dots, E'_i\}$ such that

$$(6) \quad (\text{cl } G_j) \cap E_i \neq \emptyset \quad \text{for some } j (j = 1, 2, \dots, r) .$$

Then $\{E_1, E_2, \dots, E_n\}$ is the collection of disks on S in the conclusion of the theorem. Note that $(\bigcup E_i) \cap F = \emptyset$. The collection $\{D_1, D_2, \dots, D_n\}$ remains to be identified.

Let K be the 1-skeleton of \mathcal{C} . By Lemma 4 there is a $\delta/2$ -homeomorphism h_2 of $h_1(S)$ onto itself such that

$$(7) \quad (\bigcup D'_j) \cap h_2 h_1(K) = \emptyset .$$

It follows that for each $j (j = 1, 2, \dots, r)$

$$(8) \quad D'_j \subset \text{Int } h_2 h_1(C_p) \quad \text{for some } C_p \in \mathcal{C} .$$

Let $g = h_2 h_1$. Then g is a δ -homeomorphism. Suppose $x, y \in C_p$. Then $\rho(x, y) \leq \text{diam } C_p$ and it follows from (1) that

$$(9) \quad \begin{aligned} \rho(g(x), g(y)) &\leq \rho(g(x), x) + \rho(x, y) + \rho(g(y), y) \\ &< 1/2(\varepsilon/9 - \text{diam } C_p) + \text{diam } C_p \\ &\quad + 1/2(\varepsilon/9 - \text{diam } C_p) \\ &= \varepsilon/9 . \end{aligned}$$

Hence

$$(10) \quad \text{diam } g(C_p) < \varepsilon/9 \quad (p = 1, 2, \dots, m) .$$

Now suppose $g(C_p) \cap E_i \neq \emptyset$ and $g(C_q) \cap E_i \neq \emptyset$. Let $x \in g(C_p) \cap E_i$ and $y \in g(C_q) \cap E_i$. Note that g^{-1} is a δ -homeomorphism. Then by (1)

$$(11) \quad \begin{aligned} \rho(g^{-1}(x), g^{-1}(y)) &\leq \rho(g^{-1}(x), x) + \rho(x, y) + \rho(g^{-1}(y), y) \\ &< \delta + \delta/2 + \delta \\ &< 5/6 \min \{\rho(C_r, C_s) : C_r \cap C_s = \emptyset ; C_r, C_s \in \mathcal{C}\} . \end{aligned}$$

Hence $C_p \cap C_q \neq \emptyset$ and therefore

$$(12) \quad \begin{aligned} &\text{if } g(C_p) \text{ and } g(C_q) \text{ intersect the same } E_i, \text{ then} \\ &g(C_p) \text{ and } g(C_q) \text{ are identical or adjacent cells.} \end{aligned}$$

For each $j (j = 1, 2, \dots, r)$, let \mathcal{E}_j be the subcollection of E_i 's such that $(\text{cl } G_j) \cap E_i \neq \emptyset$. By (4) and (6),

$$(\text{cl } G_j) \cap S = (\text{cl } G_j) \cap (\bigcup \text{Int } E_i : E_i \in \mathcal{E}_j) .$$

Hence the hypotheses of Lemma 5 and Lemma 6 are satisfied and as a consequence there is a collection of mutually exclusive disks $\{D_{j,i} : i = j(1), j(2), \dots, n(j)\}$ in $\text{Int } D'_j$ such that $\text{Bd } D_{j,i} \subset V$ and $(\text{cl } G_{j,i}) \cap S =$

$(\text{cl } G_{ji}) \cap \text{Int } E_i = (\text{cl } G_j) \cap \text{Int } E_i$ where G_{ji} is the component of $D_{ji} - S$ containing $\text{Bd } D_{ji}$. Note that $h(S) - \bigcup \text{Int } D_{ji} \subset V$.

Now for each fixed i ($i = 1, 2, \dots, n$), let \mathcal{D}_i be the subcollection of D_{jk} 's such that $(\text{cl } G_{jk}) \cap E_i \neq \emptyset$. By (12) each \mathcal{D}_i is contained in the star of a cell $g(C_p)$ for some $C_p \in \mathcal{C}$. Then $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n\}$ is a family of collections of disks satisfying the hypothesis of Lemma 3. Hence there is a collection of mutually exclusive disks $\{D''_1, D''_2, \dots, D''_n\}$ such that $\mathcal{D}_i^* \subset \text{Int } D''_i \subset \text{st } g(C_p)$. Note that $\text{Bd } D''_i \subset V$. We may assume without loss of generality that each disk D''_i has a polygonal boundary. Let G''_i be the component of $D''_i - S$ containing $\text{Bd } D''_i$. Each D''_i has the property that $\text{cl } G''_i$ intersects S in $\text{Int } E_i$. It remains to replace each D''_i by a disk D_i which intersects S in $\text{Int } E_i$ if it intersects S at all.

By the Tietze Extension Theorem extend the identity map on $(\text{cl } G''_i) \cap E_i$ to a map f_i on $D''_i - G''_i$ into a subdisk E''_i of $\text{Int } E_i$ which contains $(\text{cl } G''_i) \cap E_i$. Then extend f_i to D''_i by defining it to be the identity on G''_i . Since $D''_i \subset \text{st } g(C_p)$ and $\text{diam st } g(C_r) < \varepsilon/3$ for each $C_r \in \mathcal{C}$, it follows that $\text{diam } D''_i < \varepsilon/3$. Also $\text{diam } E_i < \varepsilon/36$. Therefore for each i ($i = 1, 2, \dots, n$),

$$(13) \quad \text{diam } f_i(D''_i) \leq \text{diam } D''_i + \text{Diam } E_i < 13\varepsilon/36 .$$

Let $\gamma_i = 1/2(13\varepsilon/36 - \text{diam } f_i(D''_i))$ and let U_i be a γ_i -neighborhood of the set of singularities of $f_i(D''_i)$ such that

$$(14) \quad U_i \cap S \subset \text{Int } E_i ,$$

$$(15) \quad (U_i \cup f_i(D''_i)) \cap (U_k \cup f_k(D''_k)) = \emptyset \quad \text{for } i \neq k, \text{ and}$$

$$(16) \quad (\bigcup U_i) \cap (g(S) - \bigcup \text{Int } D''_i) = \emptyset .$$

By Dehn's Lemma there is for each i ($i = 1, 2, \dots, n$), a polyhedral disk $D_i \subset f_i(D''_i) \cup U_i$ with $\text{Bd } D_i = \text{Bd } D''_i$. By the way U_i is defined, $(g(S) - \bigcup D''_i) \cup \bigcup D_i$ is a polyhedral 2-sphere, and for each i ,

$$(17) \quad \text{diam } D_i < 13\varepsilon/36 .$$

Now define $h = g$ on $g^{-1}(g(S) - \bigcup D''_i)$ and as follows on $g^{-1}(\bigcup D''_i)$. First extend the identity map on $\bigcup \text{Bd } D''_i = \bigcup \text{Bd } D_i$ to a homeomorphism h' of $\bigcup D''_i$ onto $\bigcup D_i$. Then for each $y \in \bigcup D''_i$,

$$(18) \quad \begin{aligned} \rho(h'(y), y) &\leq \max \{ \text{diam } D''_i + \text{diam } D_i : i = 1, 2, \dots, n \} \\ &< \varepsilon/3 + 13\varepsilon/36 = 25\varepsilon/36 . \end{aligned}$$

Since g is a δ -homeomorphism,

$$(19) \quad \rho(g(x), x) < \varepsilon/18 \quad \text{for each } x \in S .$$

Let $h = h'g$ on $g^{-1}(\bigcup D_i'')$. Then by (18) and (19),

$$(20) \quad \rho(h(x), x) < 25\varepsilon/36 + \varepsilon/18 = 3\varepsilon/4 \quad \text{on } g^{-1}(\bigcup D_i'').$$

Then h is the required homeomorphism and $\{D_1, D_2, \dots, D_n\}$ and $\{E_1, E_2, \dots, E_n\}$ are the required collections of disks.

5. Pairing disks with a homeomorphism. In Lemma 10 we prove the main lemma needed to show that the approximating homeomorphism may be chosen so that it maps each disk on the given sphere onto the corresponding disk of the approximating sphere. The proofs of Lemma 7 and Lemma 8 below are straight-forward and are not given.

LEMMA 7. *If D is a disk and $\{D_1, D_2, \dots, D_n\} \cup \{E_1, E_2, \dots, E_n\}$ is a collection of mutually exclusive disks in $\text{Int } D$, then there is a homeomorphism h of D onto itself which is the identity on $\text{Bd } D$ and carries E_i onto D_i ($i = 1, 2, \dots, n$).*

LEMMA 8. *If $\{D_1, D_2, \dots, D_r\}$ and $\{E_1, E_2, \dots, E_n\}$ are finite collections of mutually exclusive disks in the interior of a disk D , then there is a homeomorphism g of D onto itself which is the identity on $\text{Bd } D$ and is such that $(\bigcup g(E_i)) \cap (\bigcup D_j) = \emptyset$.*

LEMMA 9. *If \mathcal{C} is a special cellular decomposition of a 2-sphere S , $\{D_1, D_2, \dots, D_n\} \cup \{E_1, E_2, \dots, E_n\}$ is a collection of mutually exclusive disks with corresponding disks D_i and E_i in the interior of the same cell or in the interiors of adjacent cells of \mathcal{C} , then there is a homeomorphism h of S onto itself such that $h(E_i) = D_i$ and for each $x \in S$, x and $h(x)$ are either in the same cell or in adjacent cells of \mathcal{C} .*

Proof. For each $C_p \in \mathcal{C}$ cover all D_i 's in C_p which are associated with E_i 's in the same cell C_q (which must either be C_p or a cell adjacent to C_p) by a disk F_{pq} in $\text{Int } C_p$ and cover all E_i 's in C_p which correspond to D_i 's in the same cell C_r by a disk H_{pr} in $\text{Int } C_p$ so that the disks in the total resulting collection are mutually exclusive. This may be done by Lemma 1. There is an arc A_{pq} from F_{pq} to H_{qp} such that A_{pq} lies except for end points in $(C_p \cup C_q) - (\bigcup F_{pr} \cup \bigcup H_{qs})$ and the arcs in the total resulting collection are mutually exclusive. See Figure 3. If these arcs are replaced by thin disks a collection of mutually exclusive disks $\{K_{pq}\}$ is obtained with $F_{pq} \cup H_{qp} \subset K_{pq}$. Now apply Lemma 7 to obtain a homeomorphism h of S onto itself such that $h(E_i) = D_i$ ($i = 1, 2, \dots, n$).

LEMMA 10. *If \mathcal{C} is a special cellular decomposition of a 2-sphere*

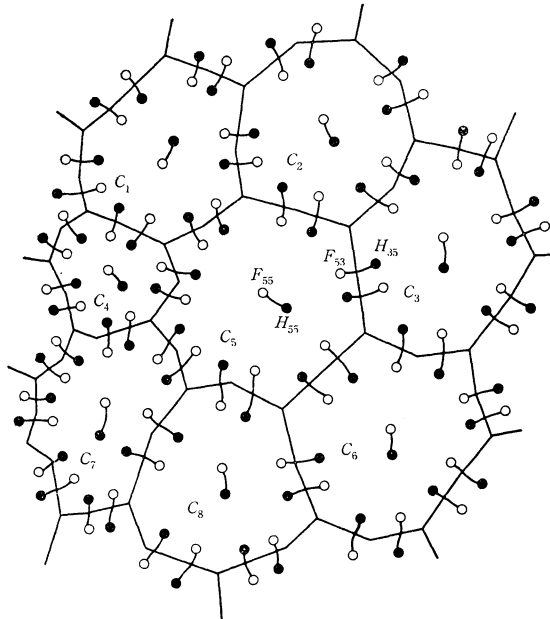


Figure 3.

$S, \{D_1, D_2, \dots, D_n\}$ is a collection of mutually exclusive disks on S , $\{E_1, E_2, \dots, E_n\}$ is a collection of mutually exclusive disks on S , and D_i and E_i are contained in the interior of the same cell or in the interiors of adjacent cells of \mathcal{C} , then there is a homeomorphism h of S onto itself such that $h(E_i) = D_i$ for each i , and for each $x \in S$, x and $h(x)$ are either in the same cell or in adjacent cells of \mathcal{C} .

Proof. By Lemma 8 there is a homeomorphism g of S onto itself which is the identity on the 1-skeleton of \mathcal{C} and sends each cell C_p onto itself such that $(\bigcup g(E_i)) \cap (\bigcup D_j) = \emptyset$. By Lemma 9 there is a homeomorphism f of S onto itself such that $fg(E_i) = D_i$ and for each $x \in S$, x and $f(x)$ are either in the same cell or in adjacent cells of \mathcal{C} . Then $h = fg$ is the required homeomorphism.

6. Equivalence of $(*, F, V)$ and $(**, F, V)$. The lemmas of § 5 are used to strengthen Theorem 1 by adding the condition that $h(E_i) = D_i$.

THEOREM 2. *If F is a closed subset of a 2-sphere S in E^3 and V is a component of $E^3 - S$, then $(*, F, V)$ is satisfied if and only if $(**, F, V)$ is satisfied.*

Proof. It is clear that $(**, F, V)$ implies $(*, F, V)$. To prove

the converse let $\varepsilon > 0$ and let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be a special cellular decomposition of S such that $\text{diam st } C_p < \varepsilon/3$ for each $C_p \in \mathcal{C}$. Let

$$(1) \quad \delta = 1/4 \min \{ \min \{ \varepsilon/3 - \text{diam st } C_p : C_p \in \mathcal{C} \} , \\ \min \{ \rho(C_p, S - \text{st } C_p) : C_p \in \mathcal{C} \} \} .$$

By Theorem 1 there is a $\delta/4$ -homeomorphism g of S into E^3 , a finite collection of mutually exclusive $\delta/4$ -disks $\{D_1, D_2, \dots, D_n\}$ on $g(S)$ and a finite collection of mutually exclusive $\delta/4$ -disks $\{E_1, E_2, \dots, E_n\}$ on S such that

$$(2) \quad g(S) \text{ is polyhedral ,}$$

$$(3) \quad g(S) - \bigcup \text{Int } D_i \subset V ,$$

$$(4) \quad D_i \cap S = h(S) \cap E_i = (\text{Int } D_i) \cap (\text{Int } E_i) , \quad \text{and}$$

$$(5) \quad (\bigcup E_i) \cap F = \emptyset .$$

Without loss of generality assume that $(\text{Int } D_i) \cap (\text{Int } E_i) \neq \emptyset$, ($i = 1, 2, \dots, n$). By Lemma 4 there is a $\delta/4$ -homeomorphism h_1 of S onto itself such that $(\bigcup E_i) \cap h_1(K) = \emptyset$ where K is the 1-skeleton of \mathcal{C} . Similarly there is a $\delta/4$ -homeomorphism h_2 of $g(S)$ onto itself such that $(\bigcup D_i) \cap h_2g(K) = \emptyset$. Note that

$$(6) \quad D_i \subset \text{Int } h_2g(C_q) \quad \text{for some } C_q \in \mathcal{C} ,$$

$$(7) \quad E_i \subset \text{Int } h_1(C_p) \quad \text{for some } C_p \in \mathcal{C} ,$$

$$(8) \quad \text{diam st } h_2g(C_q) < \text{diam st } C_q + \delta < \varepsilon/3 + \varepsilon/12 = 5\varepsilon/12 ,$$

$$(9) \quad h_2gh_1^{-1}(E_i) \subset \text{Int } h_2g(C_p) , \quad \text{and}$$

$$(10) \quad \rho(C_p, C_q) \leq \rho(h_1(C_p), C_p) + \rho(h_1(C_p), h_2g(C_q)) \\ + \rho(h_2g(C_q), C_q) \\ < \delta + 0 + \delta = 2\delta .$$

Suppose $C_p \cap C_q = \emptyset$. Then by (1) $\rho(C_p, C_q) > 4\delta$, a contradiction to (10). Hence $C_p \cap C_q \neq \emptyset$ and C_p and C_q are identical or adjacent cells. By Lemma 10 there is a homeomorphism h_3 of $h_2g(S)$ onto itself such that $h_3h_2gh_1^{-1}(E_i) = D_i$ and for each $x \in h_2g(S)$, x and $h_3(x)$ are either in the same cell $h_2g(C_r)$ or in adjacent cells. Thus by (8) h_3 is a $5\varepsilon/12$ -homeomorphism. Let $h = h_3h_2gh_1^{-1}$. Then h is the required homeomorphism.

THEOREM 3. *If S is a 2-sphere in E^3 and V is a component of $E^3 - S$, then $(**, \emptyset, V)$ is satisfied.*

Proof. This follows immediately from Theorem A and Theorem 2.

Theorem 3 is a strengthened form of Bing's Side Approximation Theorem, Theorem A.

7. **Generalization to a 2-manifold in a 3-manifold.** Bing has generalized Theorem A to connected 2-manifolds which separate a connected 3-manifold [5, Th. 1.1]. Using this and methods of the preceding sections we get the following form of this generalization. The proof is omitted since except for details it is like the proof of Theorem 1 followed by the proof of Theorem 2. A 2-manifold M^2 is tame in a 3-manifold M^3 if there is a triangulation of the 3-manifold such that M^2 is the closed union of elements of the triangulation of M^3 .

THEOREM 4. *If M^2 is a connected 2-manifold in a connected 3-manifold M^3 such that $M^3 - M^2 = V_1 \cup V_2$, V_1 and V_2 mutually separated, and f is a positive continuous real valued function on M^2 , then there is a homeomorphism h of M^2 into M^3 , and a locally finite collection of mutually exclusive disks $\{E_i\}$ on M^2 such that*

- (i) $\rho(h(x), x) < f(x)$ for each $x \in M^2$,
- (ii) $h(M^2)$ is tame,
- (iii) $h(M^2 - \bigcup \text{Int } E_i) = h(M^2) - \bigcup \text{Int } h(E_i) \subset V_1$,
- (iv) $h(E_i) \cap M^2 = E_i \cap h(M^2) = (\text{Int } h(E_i)) \cap (\text{Int } E_i)$,

and

- (v) $\text{diam } E_i < \text{minimum value of } f \text{ on } E_i$.

A closely related result, a variant of a theorem of Bing [3, Th. 18], also holds.

THEOREM 5. *If U is an open subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, and f is a positive continuous real valued function on U , then there is a homeomorphism h of S into E^3 and a locally finite collection of mutually exclusive disks $\{E_i\}$ in U such that*

- (i) $h(x) = x$ for each $x \in S - U$,
- (ii) $\rho(h(x), x) < f(x)$ for each $x \in U$,
- (iii) $h(U)$ is locally polyhedral,
- (iv) $h(U - \bigcup \text{Int } E_i) = h(U) - \bigcup \text{Int } h(E_i) \subset V$,
- (v) $U - \bigcup \text{Int } E_i \subset E^3 - h(S \cup V)$,
- (vi) $\text{diam } E_i < \text{minimum value of } f \text{ on } E_i$, and
- (vii) $h(E_i) \cap S = h(S) \cap E_i = (\text{Int } h(E_i)) \cap (\text{Int } E_i)$.

8. **Approximation of a 2-sphere by a singular 2-sphere.** To prove the invariance of $(**, F, V)$ under homeomorphisms of $S \cup V$ into E^3 it is convenient first to prove the following theorem.

THEOREM 6. *If F is a closed subset of a 2-sphere S in E^3 and V is a component of $E^3 - S$, then $(**, F, V)$ is satisfied if and only if for each $\varepsilon > 0$ there exist a finite collection of mutually exclusive ε -disks $\{E_1, E_2, \dots, E_n\}$ on S and an ε -mapping g of S into $(\bigcup \text{Int } E_i) \cup V$ such that*

- (i) *each singularity of $g(S)$ is contained in $(\bigcup \text{Int } E_i)$,*
- (ii) *$g(S - \bigcup \text{Int } E_i) \subset V$,*
- (iii) *$g(E_i) \cap S = g(S) \cap E_i = g(E_i) \cap \text{Int } E_i$, and*
- (iv) *$(\bigcup E_i) \cap F = \emptyset$.*

Proof. Suppose, $(**, V, F)$ is satisfied. Then apply the Tietze Extension Theorem as in the proof of Theorem 1 to obtain the mapping g required in the theorem. To prove the converse, follow a procedure as in the proof of Theorem 1 to apply Dehn's Lemma to each of the singular disks $g(E_i)$ so that an approximating sphere and then an approximating homeomorphism h of S onto this sphere are obtained. Here, of course, we need a form of Dehn's Lemma for nonpiecewise linear maps that can be obtained using one of Bing's approximation theorems [1, Th. 7]. Now $h(S)$ satisfies all of the requirements of property $(**, F, V)$ except the requirement that $h(S)$ be polyhedral. Bing's approximation theorem [1, Th. 1] can be used to complete the proof.

9. Invariance of $(*, F, V)$ under homeomorphisms of $S \cup V$ into E^3 . It is shown in this section that $(**, F, V)$ is invariant under homeomorphisms of $S \cup V$ into E^3 . It follows then from Theorem 2 that $(*, F, V)$ is invariant.

THEOREM 7. *If F is a closed subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, $(**, F, V)$ is satisfied, and h is a homeomorphism of $S \cup V$ into E^3 , then $(**, h(F), h(V))$ is satisfied.*

Proof. Consider first the case in which $V = \text{Int } S$. Let $\varepsilon > 0$. There is a $\delta > 0$ such that the image $h(X)$ of each δ -subset X of $S \cup \text{Int } S$ is an $\varepsilon/3$ -subset of $h(S \cup \text{Int } S)$. By Theorem 6 there exist a finite collection of δ -disks $\{E_1, E_2, \dots, E_n\}$ on S and a δ -mapping f of S into $(\bigcup \text{Int } E_i) \cup \text{Int } S$ such that

- (1) each singularity of $f(S)$ is contained in $(\bigcup \text{Int } E_i)$,
- (2) $f(S - \bigcup \text{Int } E_i) \subset \text{Int } S$,
- (3) $f(E_i) \cap S = f(S) \cap E_i = f(E_i) \cap (\text{Int } E_i)$, and
- (4) $(\bigcup E_i) \cap F = \emptyset$.

Consider any $x \in h(S)$. Then $\rho(h^{-1}(x), fh^{-1}(x)) < \delta$. Hence

$$\rho(x, hfh^{-1}(x)) < \varepsilon/3 .$$

Also $\text{diam } h(E_i) < \varepsilon/3$ and it follows that $\text{diam } hfh^{-1}(h(E_i)) < \varepsilon$. Now for the sphere $h(S)$, the closed subset $h(F)$, and the collection of disks $\{h(E_1), h(E_2), \dots, h(E_n)\}$, the mapping hfh^{-1} satisfies the requirements of the mapping g in Theorem 6. Hence it follows from Theorem 6 that $(**, h(F), h(\text{Int } S))$ is satisfied. A similar proof can be used for the case in which $V = \text{Ext } S$ by enclosing S in the interior of a large 3-cell K , so that h would be uniformly continuous on $K - \text{Int } S$.

10. Embeddings of crumpled cubes in E^3 . A crumpled cube is a space which is homeomorphic to a 2-sphere plus its interior in E^3 . In this section we give some theorems on the preservation of tameness and half tameness of subsets of the boundary of a crumpled cube under re-embeddings of the crumpled cube in E^3 .

Loveland has shown that if $(*, F, V)$ is satisfied for a closed subset F of a 2-sphere S in E^3 with complementary domain V , then F is half tame [11] and if $(*, F, \text{Int } S)$ and $(*, F, \text{Ext } S)$ are both satisfied, then F is tame [10]. It is clear that if a 2-sphere S is tame from V , then $(*, F, V)$ is satisfied for each closed subset F of S . The next two theorems follow from these results and Theorem 7.

THEOREM 8. *If F is a closed subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, $(*, F, V)$ is satisfied, and h is a homeomorphism of $S \cup V$ into E^3 , then $h(F)$ is half tame.*

THEOREM 9. *If F is a closed subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, $(*, F, V)$ is satisfied, and h is a homeomorphism of $S \cup V$ into E^3 such that $h(S)$ is tame from $E^3 - h(S \cup V)$, then $h(F)$ is tame.*

Hosay [7] and Lininger [9] have independently shown that if S is a 2-sphere in E^3 , V is a component of $E^3 - S$, and $\varepsilon > 0$, then there is an ε -homeomorphism h of $S \cup V$ into E^3 such that $h(S)$ is tame from $E^3 - h(S \cup V)$. The next theorem follows from this and Theorem 9.

THEOREM 10. *If F is a closed subset of a 2-sphere S in E^3 , V is a component of $E^3 - S$, $(*, F, V)$ is satisfied, and $\varepsilon > 0$, then there is an ε -homeomorphism h of $S \cup V$ into E^3 such that $h(F)$ is tame.*

As pointed out by Loveland in [10] it is known from results of Bing [4] and methods of Gillman [6] that if F is a finite union of

tame finite graphs and tame Sierpiński curves on a 2-sphere S in E^3 , then $(*, F, \text{Int } S)$ and $(*, F, \text{Ext } S)$ are satisfied. Theorem 9 and this result imply the following theorem.

THEOREM 11. *If F is a finite union of tame finite graphs and tame Sierpiński curves on a 2-sphere S in E^3 , V is a component of $E^3 - S$, and h is a homeomorphism of $S \cup V$ into E^3 such that $h(S)$ is tame from $E^3 - h(S \cup V)$, then $h(F)$ is tame.*

Question. Can Theorem 11 be generalized to tame closed subsets of a 2-sphere with no degenerate components? If it were known that $(*, F, V)$ holds for such subsets of a 2-sphere, then the theorem would generalize immediately.

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