PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 2, 1967

APPLICATION OF INFINITARY LANGUAGES TO METRIC SPACES

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We attempt to lay the groundwork for applying the recently-developed theory of models for the infinitary languages $L^t_{\pi\varepsilon}$ to analysis. It will be shown that within one of these languages, axioms may be written whose class of models is precisely the metric spaces. We show that two complete separable metric spaces are elementarily equivalent in this language if and only if they are isomorphic and obtain an elimination of quantifiers for such spaces. A method is developed for transferring results on metric spaces to structures with metrics whose relations are closed under the metric topology. This class includes Banach Spaces.

When not otherwise indicated, definitions, notations, and model-theoretic results used in this paper may be found in [5].

The paper will be divided into three sections, as follows: I. Axiomatization of metric spaces, II. Theorems on metric systems, and III. Metric algebraic systems. Model-theoretic results will be introduced as needed.

I. Axiomatization of metric spaces.

Our method of axiomatization of metric spaces is chosen in order to make their definition possible with one sentence in $L^m_{\omega_1\omega}$ (for the definition of $L_{\pi\varepsilon}$, ω_1, and ω, see [5]). We begin with a definition of the rational numbers:

DEFINITION 1.1. $Q = \langle Q, +, \cdot, 0, 1, <\rangle$, a structure of type $\boldsymbol{q} = \langle 3, 3, 0, 0, 2\rangle$ will be called a rational number system if and only if:

(1) Q is an ordered field (see [9, pp. 77, 5]).

(2) $(\forall v_0)(0 < v_0 \rightarrow \bigvee_{i<\omega} \bigvee_{j<\omega}(v_0 \cdot \underbrace{(1 + \cdots + 1)}_{j \text{ factors}} = \underbrace{1 + \cdots + 1}_{i \text{ factors}}))$.

THEOREM 1.2. *Each rational number system is isomorphic to the rational numbers.*

Proof. Each ordered field contains an isomorphic copy of the rational numbers, and by axiom 2, each positive (thus each negative) element of our field is the ratio of two "integers".

DEFINITION 1.3. A structure $\mathfrak{A} = \langle A, M, Q, +, \cdot, 0, 1, <, d\rangle$ of type $\boldsymbol{m} = \langle 1, 1, 3, 3, 0, 0, 2, 3,\rangle$ is a metric system if and only if

$<Q, +, \cdot, 0, 1, <>$ is a rational number system, and $+, \cdot, <$ apply only to members of Q, and:

(1) $(\forall v_0)((M(v_0) \vee Q(v_0)) \wedge (\neg M(v_0) \vee \neg Q(v_0)))$
(2) $(\forall v/3)(d(v/3) \rightarrow M(v_0) \wedge M(v_1) \wedge 0 < v_2)$
(3) $(\forall v/4)(d(v/3) \wedge v_2 < v_3 \rightarrow d(v_0, v_1, v_3))$
(4) $(\forall v/3)(\exists v_3)(d(v/3) \rightarrow v_3 < v_2 \wedge d(v_0, v_1, v_3))$
(5) $(\forall v/2)(\exists v_2)(M(v_0) \wedge M(v_1) \rightarrow d(v/3))$
(6) $(\forall v/3)(d(v/3) \rightarrow d(v_1, v_0, v_2))$
(7) $(\forall v/5)(d(v/3) \wedge d(v_1, v_3, v_4) \rightarrow d(v_0, v_3, v_2 + v_4))$
(8) $(\forall v/2)(\exists v_2)(M(v_0) \wedge M(v_1) \wedge \neg(v_0 \equiv v_1) \rightarrow 0 < v_2 \wedge \neg d(v/3))$
(9) $(\forall v/3)(M(v_0) \wedge v_0 \equiv v_1 \wedge 0 < v_2 \rightarrow d(v/3))$.

The general idea of these axioms is to identify each real number with the set of rationals greater than it. Axioms 1 to 4 simply prepare the groundwork for the above interpretation. Axiom 5 says that to each pair of points there corresponds a distance. Axiom 6 gives the symmetry of distance, 7 gives the triangle inequality, 8 yields the fact that any pair of distinct points differ by a positive distance, and 9 says that the distance from a point to itself is 0. To formalize the above discussion, we have the following representation theorem for metric systems:

THEOREM 1.4. (a) *Let* $\mathfrak{A} = \langle A, M, Q, +, \cdot, 0, 1, <, d\rangle$ *be a metric system. Then* $\mathfrak{A}^* = \langle M, m\rangle$ *is a metric space, where* $m(x, y) = \inf\{q/\langle x, y, q\rangle \in d\}$ *(for each pair* $x, y \in M$*). This metric space is called the associated space of the metric system* $\mathfrak{A}$.

(b) *Let* $\mathscr{M} = \langle M, m\rangle$ *be a metric space. Then the associated system* $\mathscr{M}^* = \langle M \cup Q, M, Q, +, \cdot, 0, 1, <, d\rangle$ *of* $\mathscr{M}$ *is a metric system, where* Q *together with its relations and constants* $(+, \cdot, 0, 1, <)$ *is the rational numbers, our union* $M \cup Q$ *is disjoint, and*

$$d = \{\langle x, y, q\rangle / m(x, y) < q \in Q\} .$$

Proof. Consider the completion R of Q. If $x, y \in M$, by 5 $\{q/\langle x, y, q\rangle \in d\}$ is nonempty, and by 2, all its elements are positive (thus 0 is a lower bound). Thus $m(x, y)$ exists. It remains to be shown that m is a metric. To show the triangle inequality we note that axiom 7 gives us the fact that if $\langle x, y, q\rangle \in d$ and $\langle y, z, r\rangle \in d$, then $\langle x, z, q + r\rangle \in d$. Thus

$$\inf\{q/\langle x, y, q\rangle \in d\} + \inf\{r/\langle y, z, r\rangle \in d\} \geqq \inf\{s/\langle x, z, s\rangle \in d\} ,$$

in other words $m(x, z) \leqq m(x, y) + m(y, z)$. The other laws are checked with similar case. This shows (a), and under the assumptions of (b), axioms $1, \cdots, 9$ for metric systems can easily be shown.

LEMMA 1.5. (a) *If* $\mathfrak{A}, \mathfrak{B}$ *are metric spaces or metric systems and*

$\mathfrak{A} \cong \mathfrak{B}$, *then* $\mathfrak{A}^* \cong \mathfrak{B}^*$ (*where for metric spaces, isomorphism = isometry*).

(b) *If* $\mathfrak{A}$ *is a metric space or a metric system,* $\mathfrak{A} \cong (\mathfrak{A}^*)^*$.

(c) *For metric systems* $\mathfrak{A}, \mathfrak{B}, \mathfrak{A} \in \text{Iso Sub}(\mathfrak{B})$ *if and only if* $\mathfrak{A}^* \cong \mathfrak{C}$ *for some subspace* $\mathfrak{C}$ *of* $\mathfrak{B}^*$. (*We write* $\mathfrak{A} \in \text{Iso Sub}(\mathfrak{B})$ *if and only if* $\mathfrak{A} \cong \mathscr{D}$ *for some* $\mathscr{D} \subset \mathfrak{B}$, $\mathfrak{A} \in \text{Iso Sub}_\varepsilon(\mathfrak{B})$ *if and only if* $\mathfrak{A} \in \text{Iso Sub}(\mathfrak{B})$ *and* $\text{card}(\mathfrak{A}) < \varepsilon$.)

The proof is routine, though somewhat long, and left to the reader. We will now say that a metric system has property P if and only if its associated space has that property. We will also say that a metric system and a metric subsystem of it have property Q if and only if the associated space and subspace have that property (for example, the subspace may be dense). Often we will want to translate these properties into our first-order infinitary language. A proof similar to that of Theorem 17, [6] (with Q in place of $\mathfrak{A}$, and $\omega_1\omega$ in place of $\varepsilon\varepsilon$) shows that completeness cannot be characterized by a single sentence in $L^m_{\omega_1\omega}$. It is clear, however, that complete metric systems are precisely those which satisfy the following sentence in

$$L^m_{\omega_1\omega_1}\colon Ct = (\forall v/\omega)((\forall v_\omega)\, 0 < v_\omega \rightarrow \bigvee_{i<\omega} \bigwedge_{i<j<\omega} \bigwedge_{i<k<\omega} d(v_j, v_k, v_\omega)) \\ \rightarrow ((\exists v_{\omega+1})(\forall v_{\omega+2})\, 0 < v_{\omega+2} \rightarrow (\bigvee_{i<\omega} \bigwedge_{i<j<\omega} d(v_{\omega+1}, v_j, v_{w+2})))) ,$$

since this sentence simply says that each sequence (v/ω) which is cauchy (i.e., such that for each v there is an $i < \omega$ such that for each finite j, k such that j and k are greater than i, v_j and v_k are "closer together" than v_ω) there is a limit ($v_{\omega+1}$). It is also clear that separable metric systems are simply those satisfying Sep, where

$$\text{Sep} = (\exists v/\omega)(\forall v_\omega)(\forall v_{\omega+1})(M(v_\omega) \wedge 0 < v_{\omega+1} \rightarrow \bigvee_{i<\omega} d(v_\omega, v_i, v_{\omega+1})) .$$

Thus most of our future dealings with metric systems will take place in $L^m_{\omega_1\omega_1}$.

II. Theorems on metric systems.

THEOREM 2.1. *Let* $\mathfrak{A} \prec_{\omega_1, \omega_1} \mathfrak{B}$ *and let* $\mathfrak{A}$ *or* $\mathfrak{B}$ *be separable. Then* $\mathfrak{A} = \mathfrak{B}$. (For definition of $\prec_{\omega_1, \omega_1}$ see [5], Definition 2.2).

Proof. Assume $\mathfrak{A}$ is separable. Let $\mathfrak{C} = \langle \{n_0, n_1, \cdots\}, m' \rangle$ be a dense countable subspace of $\mathfrak{A}^*$, Then $\mathfrak{A} \models D[n_0, \cdots]$, where

$$D = (\forall v_\omega)(\forall v_{\omega+1})(M(v_\omega) \wedge 0 < v_{\omega+1} \rightarrow \bigvee_{i<\omega} d(v_\omega, v_i, v_{\omega+1})) .$$

Thus $\mathfrak{C}$ is dense in $\mathfrak{B}^*$, since by our assumption, $\mathfrak{B} \models D[n_0, \cdots]$.

Now let $b \in M'$, where $\mathfrak{B} = \langle B, M', \cdots \rangle$. Since $\mathfrak{C}$ is dense in $\mathfrak{B}^*$, there is a sequence $n_{i_0}, n_{i_1}, \cdots$ of the n_i's approaching b. Thus $\mathfrak{B} \models (\exists v_\omega)L[n_{i_0}, \cdots]$, where $L = (\forall v_{\omega+1})(0 < v_{\omega+1} \rightarrow \bigvee_{i<\omega} \bigwedge_{i<j<\omega} d(v_\omega, v_j, v_{\omega+1}))$. Thus $\mathfrak{A} \models (\exists v_\omega)L[n_{i_0}, \cdots]$, but since $\mathfrak{A}^*$ is a subspace of $\mathfrak{B}^*$ and limits are unique in metric spaces $\mathfrak{A}^* = \mathfrak{B}^*$. If $\mathfrak{A} = \langle A, M, \cdots \rangle$ this implies that $M = M'$, so $A = B$, thus $\mathfrak{A} = \mathfrak{B}$. This proves the theorem in the first case, but if $\mathfrak{B}$ is separable, then $\mathfrak{A}$ must also be, from which the theorem must follow in the second case.

It can be shown similarly that if $\mathfrak{A} \prec_{\omega_1, \omega_1} \mathfrak{B}$ and $\mathfrak{A}$ is dense in $\mathfrak{B}$, then $\mathfrak{A} = \mathfrak{B}$. The following definition and results may be found in Tarski.

Definition 2.2. Let $\mathfrak{A}$ be a structure of type $\boldsymbol{t}$, $\mathrm{Dom}(\boldsymbol{t}) = \varepsilon < \pi$, and let a be a well-ordering of A. Then

$$\mathrm{Des}_a(\mathfrak{A}) = (\bigwedge F) \wedge (\bigwedge F') \wedge (\bigwedge_{\boldsymbol{t}(p) \geqq 1}(\bigwedge F_p)) \wedge (\bigwedge_{\boldsymbol{t}(p) \geqq 1}(\bigwedge F'_p)) ,$$

where

$$F = \{v_i \equiv v_j / a_i = a_j\} \cup \{v_i \equiv c_j / a_i = c_j\} \cup \{c_i \equiv c_j / c_i = c_j\},$$
$$F' = \{\neg\, v_i \equiv v_j / a_i \neq a_j\} \cup \{\neg\, v_i \equiv c_j / a_i \neq c_j\} \cup \{\neg\, c_i \equiv c_j / c_i \neq c_j\},$$
$$F_p = \{\mathrm{R}_p(x{\cdot}j/\boldsymbol{t}(p)) / b{\cdot}j/\boldsymbol{t}(p) \in R_p,$$

where $b_{j_i} = a_k$ if $x_{j_i} = v_k$ and $b_{j_i} = c_k$ if $x_{j_i} = c_k\}$ and

$$F'_p = \{\neg\, \mathrm{R}_p(x{\cdot}j/\boldsymbol{t}(p))\ b{\cdot}j/\boldsymbol{t}(p) \notin R_p,\ b \text{ defined as for } F_p\} .$$

Corollary 2.3. *Let $\mathfrak{A}$ be a structure of type $\boldsymbol{t}$. Then* $\mathfrak{A} \in$ Iso Sub $(\mathfrak{B})$ *if and only if* $\mathfrak{B} \models (\exists X)\mathrm{Des}\,(\mathfrak{A})$, *where* $X = FV(\mathrm{Des}(\mathfrak{A}))$.

Note that if $c(A) < \pi$, then $\mathrm{Des}\,(\mathfrak{A})$ and $(\exists X)\mathrm{Des}\,(\mathfrak{A})$ are both formulas (the latter a sentence) of $\boldsymbol{L}^{\boldsymbol{t}}_{\pi\pi}$.

Definition 2.4. Let $\mathfrak{C}$ be a countable metric system, c a mapping from ω one-one onto C. Then let

$$Dc_c(\mathfrak{C}) = (\exists v/\omega)(\forall v_\omega)(\forall v_{\omega+1})(\mathrm{Des}_c(\mathfrak{C}) \wedge (M(v_\omega) \wedge 0 < v_{\omega+1} \rightarrow \bigvee_{i<\omega} d(v_i, v_\omega, v_{\omega+1}))) .$$

Note that $\mathfrak{A}$ is dense in $\mathfrak{B}$ for some $\mathfrak{A} \cong \mathfrak{C}$ if and only if

$$\mathfrak{B} \models Dc_c(\mathfrak{C}) .$$

Thus Dc_c should be viewed as a method of expressing within $\boldsymbol{L}^{\boldsymbol{m}}_{\omega_1\omega_1}$ the fact that a countable space is dense in another space. If it is unnecessary to distinguish our well-ordering c, we simply write $Dc(\mathfrak{C})$ in place of $Dc_c(\mathfrak{C})$.

LEMMA 2.5. *If $\mathfrak{A}$ is a metric system, $\mathfrak{A} \models$ Sep if and only if for some countable metric system $\mathfrak{C}$, $\mathfrak{A} = Dc(\mathfrak{C})$.*

The proof is left to the reader.

THEOREM 2.6. *Let $\mathfrak{A}$ be a complete separable metric system, $\mathfrak{B}$ a metric system. Then $\mathfrak{A} \equiv_{\omega_1, \omega_1} \mathfrak{B}$ if and only if $\mathfrak{A} \cong \mathfrak{B}$.* (A definition of $\equiv_{\pi, \varepsilon}$ is found in 2.1, [5].)

Proof. Let $\langle K, k\rangle$ be a dense countable subspace of $\mathfrak{A}^*$, where

$$\mathfrak{A} = \langle A, M, \cdots\rangle, \mathfrak{B} = \langle B, N, \cdots\rangle .$$

Then $\mathfrak{A} \models Dc(\langle K, k\rangle^*) \wedge Ct$, so $\mathfrak{B} \models Dc(\langle K, k\rangle^*) \wedge Ct$. By the remark following Definition 2.4, $\mathfrak{D}$ is dense in $\mathfrak{B}$ for some $\mathfrak{D} \cong \langle K, k\rangle$. Thus up to isomorphism, $\langle K, k\rangle$ is dense in $\mathfrak{B}^*$ and $\mathfrak{A}^*$, two complete metric spaces. It is well-known in the theory of metric spaces that if two complete spaces have the same dense subspace (up to isometry), then the spaces are isometric. Thus by Lemma 1.5, a, $\mathfrak{A} \cong \mathfrak{B}$.

We are going on to eliminate quantifiers for complete separable metric systems. First we make the notation $B(S)$ for all the boolean combinations of a set of formulas S (see Definition 5.1, [6]), and restate Theorem 5.2 of that paper.

THEOREM 2.7. *Let S be a consistent set of sentences (i.e., one with models) in $\boldsymbol{L}^t_{\pi\varepsilon}$, E a set of sentences in $\boldsymbol{L}^t_{\pi\varepsilon}$ such that there is a set $B'(E)$ with the property that for each $\phi \in B(E)$ there is a $\phi' \in B'(E)$ with $S \models \phi \leftrightarrow \phi'$. For any consistent set of sentences $T \subset \boldsymbol{L}^t_{\pi\varepsilon}$, $S \subset T$, assume there is an $A \subset E$ such that $T \subset T_{\pi\varepsilon}(S \cup A)$ (and $S \cup A$ consistent). Then for any sentence $\phi \in \boldsymbol{L}^t_{\pi\varepsilon}$ there is a $\theta \in B(E)$ such that $S \models \phi \leftrightarrow \theta$.*

LEMMA 2.8. *Let $S \subset \boldsymbol{L}^t_{\pi\varepsilon}$ be a consistent set of sentences, $E \subset \boldsymbol{L}^t_{\pi\varepsilon}$ a set of sentences such that* (1) *for any models $\mathfrak{A}, \mathfrak{B}$ of S, if $\mathfrak{A} \models \theta$, $\mathfrak{B} \models \theta$, $\theta \in E$, then $\mathfrak{A} \equiv_{\pi, \varepsilon} \mathfrak{B}$, and* (2) *for any model $\mathfrak{A}$ of S there is a $\theta \in E$ such that $\mathfrak{A} \models \theta$. Then* (a) *if $T \subset \boldsymbol{L}^t_{\pi\varepsilon}$ is a consistent set of sentences and $S \subset T$, then there is a $\theta \in E$ such that $T \subset T_{\pi\varepsilon}(S \cup \{\theta\})$ and $S \cup \{\theta\}$ is consistent,* (b) *if $\phi \in B(E)$, $S \models \phi \leftrightarrow \bigvee A$ for some $A \subset E$, and* (c) *for any sentence $\zeta \in \boldsymbol{L}^t_{\pi\varepsilon}$ there is a $\phi \in B(E)$ such that $S \models \zeta \leftrightarrow \phi$.*

Proof. (a) If $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of S, so there is a $\theta \in E$ such that $\mathfrak{A} \models \theta$. Now suppose $\mathfrak{B}$ is a model of S such that $\mathfrak{B} \models \theta$. Then $\mathfrak{A} \equiv_{\pi, \varepsilon} \mathfrak{B}$, so $\mathfrak{B} \models T$. Thus $T \subset T_{\pi\varepsilon}(S \cup \{\theta\})$. (b) $\phi = \bigvee_{i \in I} \bigwedge_{j \in J_i} \theta_{ij}$, $\theta_{ij} \in E$ or $\neg\, \theta_{ij} \in E$ for each i, j. Let $\phi_i = \bigwedge_{j \in J_i} \theta_{ij}$

be a disjunct. If for some $i, j, \theta_{ij} \in E$, we assert that either

$$S \models \phi_i \leftrightarrow \theta_{ij}$$

or $S \models \neg \phi_i$. To show this, assume not $S \models \phi_i \leftrightarrow \theta_{ij}$. Clearly, $S \models \phi_i \rightarrow \theta_{ij}$, thus not $S \models \theta_{ij} \rightarrow \phi_i$. But then by (1), $S \models \theta_{ij} \rightarrow \neg\phi_i$, therefore $S \models \neg \phi_i$. If for each j, $\neg \theta_{ij} \in E$, $\phi_i = \bigwedge_{j \in J_i} \neg (\neg \theta_{ij})$. Thus by (2) $S \models \phi_i \leftrightarrow \bigvee E - \{\theta_{ij}/j \in J_i\}$. Thus in any of the above cases $S \models \phi \leftrightarrow \bigvee_{k \in K_j} \theta_k$, with each $\theta_i \in E$. Thus

$$S \models \phi \leftrightarrow \bigvee_{i \in I} \bigvee_{k \in K_i} \theta_k ,$$

and eliminating repetitions, $S \models \phi \leftrightarrow \bigvee_A$, which is condition (b) (note that while $\bigvee A$ may not be in $L^t_{\pi\varepsilon}$, it will be in $L^t_{\pi'\varepsilon}$ for some $\pi' \geqq \pi$, and all our statements have meaning in that language). (c) This is immediate for (a) and (b) and Theorem 2.7, using

$$B'(E) = \{\bigvee A/A \subset E\} .$$

THEOREM 2.9. *Let $\phi \in L^m_{\omega_1\omega_1}$ be a sentence. Then if Σ is the conjunction of the axioms for a metric system together with* Sep *and Ct, (i.e., is an axiom for complete separable metric systems) then $\Sigma \models \phi \leftrightarrow \theta$ for some $\theta \in B(E)$, where $E = \{Dc(\mathfrak{C})/\mathfrak{C}$ a countable metric system$\}$.*

Proof. Let $S = \{\Sigma\}$. Then S is a consistent set of sentences in $L^m_{\omega_1\omega_1}$. E is also a set of sentences in $L^m_{\omega_1\omega_1}$, and S, E satisfy conditions (1) (by the proof of Theorem 2.6) and (2) (by Lemma 2.5) of Lemma 2.8. Thus our conclusion follows from (c) of that lemma.

To eliminate quantifiers for all formulas in $L^m_{\omega_1\omega_1}$ for the theory of complete separable metric spaces, we shall need the following:

DEFINITION 2.10. Let t be a type, i an ordinal. Then $t * i$ is the type defined by $t = t * i/\mathrm{Dom}(t)$ and if $j < i$, then $t * i(\mathrm{Dom}(t) + j) = 0$.

DEFINITION 2.11. Let $\mathfrak{C} = \langle C, K, \cdots \rangle$ be a countable metric system,

$$\mathfrak{C}' = \langle C, K, \cdots, e_j \rangle_{j<\omega}, \quad \{e_j/j < \omega\} = D \subset K .$$

Then if c is a mapping of ω one-one onto C, $DC_c(\mathfrak{C}, D)$ is the sentence of $L^{m*\omega}_{\omega_1\omega_1}$ defined as follows:

$$\begin{aligned} DC_c(\mathfrak{C}, D) = (\exists v/\omega)(\forall v_\omega)(\forall v_{\omega+1})(\mathrm{Des}_c(\mathfrak{C}') \\ \wedge (M(v_\omega) \wedge 0 < v_{\omega+1} \rightarrow \bigvee_{i<\omega} d(v_i, v_\omega, v_{\omega+1}))) . \end{aligned}$$

If distinguishing c is unnecessary, we again leave it out and write $DC(\mathfrak{C}, D)$ in place of $DC_c(\mathfrak{C}, D)$. $DC(\mathfrak{C}, D)$ should be considered as $Dc(\mathfrak{C})$ with some constants added.

THEOREM 2.12. *Let $\mathfrak{A}, \mathfrak{B}$ be $\boldsymbol{m} * \omega$-structures such that $\mathfrak{A}/\boldsymbol{m}, \mathfrak{B}/\boldsymbol{m}$ are complete separable metric systems. If $\mathfrak{A} \models DC(\mathfrak{C}, D)$ and $\mathfrak{B} \models DC(\mathfrak{C}, D)$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. Let $C = \{c_0, c_1, \cdots\}$, where $\mathfrak{C}$ is dense in $\mathfrak{A}/\boldsymbol{m}$. Let

$$\theta = DC_c(\mathfrak{C}, D) ,$$

where $c = \langle c_0, c_1, \cdots \rangle$ Suppose $\mathfrak{B} \models \theta$. Then there is a sequence $\{b_n/n < \omega\}$ satisfying θ without the existential quantifier. The map $f: \mathfrak{C} \rightarrow \mathfrak{C}'$ such that $f(c_n) = b_n$ is an isomorphism because of Des $(\mathfrak{C})$. In particular, it is an isomorphism from $\mathfrak{C}$ to $\mathfrak{C}'$ preserving constants. Since $\mathfrak{C}$ is dense in $\mathfrak{A}/\boldsymbol{m}$, $\mathfrak{C}'$ is dense in $\mathfrak{B}/\boldsymbol{m}$, thus this isomorphism can be extended to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

COROLLARY 2.13. *Let $\theta \in \boldsymbol{L}^{\boldsymbol{m}*\omega}_{\omega_1\omega_1}$ be a sentence. Then $\Sigma \models \theta \leftrightarrow \phi$, where $\phi \in B(H)$, $H = \{DC_c(\mathfrak{C}, D)/\mathfrak{C}$ countable, $D \subset C$, and c a map from ω one-one onto $C\}$.*

Proof. Clearly $S = \{\Sigma\}$ is a consistent set of sentences in $\boldsymbol{L}^{\boldsymbol{m}*\omega}_{\omega_1\omega_1}$, and H is also a set of sentences in $\boldsymbol{L}^{\boldsymbol{m}*\omega}_{\omega_1\omega_1}$. By Theorem 2.12, condition (1) of Lemma 2.8 is satisfied. Condition (2) follows from the fact that any separable metric space with a countable number of constants has a countable dense subsystem containing those constants. If D is that set of constants, and $\mathfrak{C}$ is that system, then $\mathfrak{C} \models DC_c(\mathfrak{C}, D)$ for any mapping of ω one-one onto C. Thus our Corollary follows from (c) of that lemma.

We now need the concept of substitution for free occurrences of variable and for constants in a formula. Intuitively, by

$$Sb\{y_i/i \in I\}, \{x_i/i \in I\}\theta$$

we mean the formula obtained by substituting y_i for x_i at each free occurrence of x_i or constant x_i, and leaving unchanged other terms in θ. Somewhat more formally, we consider the function $f: V \cup C \rightarrow V \cup C$ (see Definition 1.1, [5]) such that for each $i \in I, f(x_i) = y_i$, and if $x_i \in V \cup C, i \notin I, f(x_i) = x_i$, and:

DEFINITION 2.14. $Sb_f\theta$ is the formula defined inductively as follows:

(1) $Sb_f(x_i \equiv x_j) = f(x_i) \equiv f(x_j)$,

(2) If $j \in (V \cup C)^{t(p)}$, then $Sb_f(P_p(x \circ j)) = P_p(f \circ x \circ j)$,

(3) $Sb_f(\neg\, \theta) = \neg\, Sb_f\theta$,

(4) $Sb_f(\bigvee X) = \bigvee \{Sb_f\theta/\theta \in X\}$,

(5) $Sb_f(\bigwedge X) = \bigwedge \{Sb_f\theta/\theta \in X\}$,

(6) $Sb_f(\forall\, W)\theta = (\forall\, W)Sb_{f'}\theta$, where $f'(x) = f(x)$ for $x \in V \cup C - W$, $f'(x) = x$ if $x \in W$,

(7) $Sb_f(\exists\, W)\theta = (\exists\, W)Sb_{f'}\theta$.

We also define $Sb(\{y_i/i \in I\}, \{x_i/i \in I\})\theta = Sb_f\theta$ with f as constructed in the last sentence before our definition.

An immediate consequence of the definition is the fact that if $\mathfrak{A} \models Sb(\{y_i/i \in I\}, \{x_i/i \in I\})(\theta \rightarrow \phi)$ and none of the x_i occur in θ, then $\mathfrak{A} \models \theta \rightarrow Sb(\{y_i/i \in I\}, \{x_i/i \in I\})\phi$. Also, if a sentence $\theta \models \phi$ and none of the x_i occur in θ, then $\theta \models Sb(\{y_i/i \in I\}, \{x_i/i \in I\})\phi$.

THEOREM 2.15, *If $F \in \boldsymbol{L}^{m}_{\omega_1\omega_1}$ is a formula, then $\Sigma \models F \leftrightarrow \theta$ for some $\theta \in B(E')$, where $\phi \in E'$ if and only if*

$$\phi = Sb(\{v_i/i \in \omega\}, \{e_i/i \in \omega\})\, DC(\mathfrak{C}, D),$$

with $\mathfrak{C}, D$ as in Definition 2.11, $D = \{e_i/i \in \omega\}$.

Proof. We can always put F in an equivalent form F' such that $\{v_i/i \in \omega\}$ is the set of free variables of F'. Thus we lose no generality in assuming $\{v_i/i \in \omega\}$ is the set of free variables of F. Now consider the sentence $\phi = Sb(\{e_i/i \in \omega\}, \{v_i/i \in \omega\})\, F \in \boldsymbol{L}^{m*\omega}_{\omega_1\omega_1}$. For some $\theta \in B(H)$, $\Sigma \models \phi \leftrightarrow \theta$. Since Σ contains no reference to the e_i, we also have $\Sigma \models Sb(\{v_i/i \in \omega\}, \{e_i/i \in \omega\})(\phi \leftrightarrow \theta)$, thus $\Sigma \models F \leftrightarrow Sb(\{v_i/i \in \omega\},\{e_i/i \in \omega\})\theta$. But by (3), (4), and (5) of Definition 2.14, $Sb(\{v_i/i \in \omega\}, \{e_i/ \in \omega\})\theta \in B(E')$.

Thus we have an elimination of quantifiers in the theory of complete separable metric systems for all formulas of $\boldsymbol{L}^{m}_{\omega_1\omega_1}$. This elimination does not always take place within that language. We shall proceed to universal equivalence for complete metric systems. Using Corollary 2.3, Tarski has shown that two structures are universally equivalent in $\boldsymbol{L}^{t}_{\pi\pi}$ (in symbols $\mathfrak{A} \equiv_{U^t_{\pi\pi}} \mathfrak{B}$) if and only if Iso $\mathrm{Sub}_\pi(\mathfrak{C}) =$ Iso $\mathrm{Sub}_\pi(\mathfrak{B})$. Thus if we allow $\boldsymbol{U} = \boldsymbol{U}^{m}_{\omega_1\omega_1}$, we have the fact that $\mathfrak{A} \equiv_{\boldsymbol{U}} \mathfrak{B}$ if and only if Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{A}) =$ Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{B})$ for any two metric systems.

DEFINITION 2.16. Let K be a class of complete metric systems. Then $SC(K) = \{\mathfrak{B}/$for some $\mathfrak{A} \in K, \mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{B}$ is a seperable complete metric system$\}$. $ISC(K) = \{\mathfrak{D}/\mathfrak{D} \cong \mathfrak{B}$ for some $\mathfrak{B} \in SC(K)\}$.

$$ISC(\{\mathfrak{A}\}) = ISC(\mathfrak{A})\,.$$

THEOREM 2.17. *Let $\mathfrak{A}, \mathfrak{B}$ be complete metric systems. Then* $\mathfrak{A} \equiv_{\boldsymbol{U}} \mathfrak{B}$

if and only if $ISC(\mathfrak{A}) = ISC(\mathfrak{B})$.

Proof. Assume first that Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{A})$ = Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{B})$ (i.e., that $\mathfrak{A} \equiv_U \mathfrak{B}$). Let $\mathfrak{C} \in ISC(\mathfrak{A})$. Since $\mathfrak{C}$ is separable and complete there is a countable $\mathfrak{D} \subset \mathfrak{A}$ such that $\mathfrak{D}$ is dense in an isomorphic copy of $\mathfrak{C}$. By our assumption, there is a countable $\mathfrak{C} \subset \mathfrak{B}$ such that $\mathfrak{C} \cong \mathfrak{D}$. Since $\mathfrak{B}$ is complete, the completion of $\mathfrak{C}$ is contained in $\mathfrak{B}$ (by previous remarks, this is isomorphic to $\mathfrak{C}$). Thus $ISC(\mathfrak{A}) \subset ISC(\mathfrak{B})$. Similarly, $ISC(\mathfrak{B}) \subset ISC(\mathfrak{A})$.

Conversely, assume $ISC(\mathfrak{A}) = ISC(\mathfrak{B})$. Let $\mathfrak{C} \in$ Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{A})$. Since $\mathfrak{A}$ is complete, the completion of an isomorphic copy of $\mathfrak{C}$ is contained in $\mathfrak{A}$, thus another isomorphic copy of its completion is contained in $\mathfrak{B}$. Thus an isomorphic copy of $\mathfrak{C}$ is contained in $\mathfrak{B}$. This shows that Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{A}) \subset$ Iso $\mathrm{Sub}_{\omega_1}(\mathfrak{B})$, and the reverse inclusion is shown similarly.

COROLLARY 2.18. *Let $\mathfrak{A}$, $\mathfrak{B}$ be complete separable metric systems. Then $\mathfrak{A} \equiv_U \mathfrak{B}$ if and only if for some $\mathfrak{C} \subset \mathfrak{B}$, $\mathfrak{A} \cong \mathfrak{C}$ and for some $\mathfrak{D} \subset \mathfrak{A}$, $\mathfrak{B} \cong \mathfrak{D}$.*

Universal equivalence, even for complete separable spaces, is not as strong as elementary equivalence. The following example, when formalized, provides a counterexample. Let

$$N = \{f\colon \omega \to \boldsymbol{R} / \Sigma_{i=1}^{\infty}(f(i))^2 < \infty\} ,$$

where $\boldsymbol{R}$ is the real numbers. Define $m(f, g) = \sqrt{\Sigma_{i=1}^{\infty}(f(i) - g(i))^2}$. Then m can be shown to be a metric on N. Let

$$S = \{f \in N / 0 \leqq f(1) \leqq 1\} .$$

Then $S \subset N$ and the following isometry takes N into S. Let $F\colon N \to S$ by $F(f)(i) = f(i - 1)$, $i > 1$ and $F(f)(1) = 0$.

If $\mathfrak{B}$ is the metric system associated with N, $\mathfrak{A}$ that associated with S, then $\mathfrak{A}, \mathfrak{B}$ are complete and separable, $\mathfrak{A} \subset \mathfrak{B}$, and $\mathfrak{B} \cong \mathfrak{C} \subset \mathfrak{A}$, but not $\mathfrak{A} \cong \mathfrak{B}$. If we take $S' = \{f \in N / f(0) \in Q\}$ (Q the set of rationals), we have an example (using S', N) of a sepable complete metric system universally equivalent to a separable incomplete system. There are also examples of separable systems universally equivalent to inseparable systems.

III. Metric algebraic systems. The theory of Banach Algebras (see Naimark) has been one of the most important in analysis during the past twenty years. The theories of metric groups (see Montgomery-Zippin), Banach Spaces (see Dunford-Schwartz), normed spaces (see Day), and many others have been essential to functional analysis.

Measure theory can, in many respects, be reduced to the theory of certain metric Boolean algebras, c.f. [4, pp. 165–174]. All these theories contain a common feature: they are theories of algebraic structures with metric topologies. Most of them share a second feature: they are defined in terms of a norm rather than a metric. A metric, however, can easily be defined by allowing $m(a, b) = \| a - b \|$. In this section we will extend the results obtained for metric spaces to algebraic structures with metric topologies.

DEFINITION 3.1. A metric algebraic system is a structure

$$\mathfrak{A} = \langle A, M, Q, +, 0, 1, <, d, R_p \rangle_{p<o}$$

with $o \leqq \omega$ and:

(a1) $\mathfrak{A}/m$ is a metric system.

(a2) If $o = t(p)$, we have: $M(c_p)$. If $0 < t(p)$, we have:

$$(\forall\, v/t(p))(R_p(v/t(p)) \rightarrow \bigwedge\nolimits_{i<t(p)} M(v_i)) .$$

(a3)
$$\begin{aligned}&(\forall\, v/t(p))(\exists v_{2t(p)})(\forall\, t(p)/v/2t(p))(R_p(v/t(p))\\ &\quad \vee \neg\, [0 < v_{2t(p)} \wedge (R_p(t(p)/v/2t(p))\\ &\quad \wedge (\bigwedge\nolimits_{i<t(p)} d(v_i,\, v_{t(p)+i},\, v_{2t(p)})))]) .\end{aligned}$$

(For notation used here see [5], discussion following Def. 1.1.)

In the above axioms, (a2) simply restricts the relations (and constants) to the metric space, while (a3) closes all the relations (for the definition of a closed relation extend Dunford-Schwartz, p. 57, Def. 3 in the obvious manner). Since continuous operations are closed as relations, it is clear that metric topological groups, etc. can be considered by use of the above axioms. Banach Spaces can also be considered, despite the fact that they are defined by use of two metric spaces, rather than one. We simply work with the disjoint union of the universes of the metric spaces involved, and define a metric on that set by setting it equal to the appropriate of the two existing metrics for two elements of the same set, and setting it equal to 1 otherwise. Both sets now become closed and we proceed with the axiomatization in the obvious manner. The remaining systems mentioned at the beginning of this section can similarly be axiomatized as metric algebraic systems.

DEFINITION 3.2. Let $\mathfrak{A}$ be a metric algebraic system, and let W be a predicate defined in terms of the relations of $\mathfrak{A}/\boldsymbol{m}$. Then $\mathfrak{A}$ is said to have the property W if and only if $\mathfrak{A}/\boldsymbol{m}$ satisfies W.

In the above definition, W could define the property of being separable, complete, etc. Note here that any substructure of a metric algebraic system is itself a metric algebraic system, provided it remains a metric system.

DEFINITION 3.3. Let $A' \subset R_p$. Then A' is dense in R_p if and only if for any $\langle a_0, \cdots, a_{t(p)-1}\rangle \in R_p$ and any $q > 0$, there is an

$$\langle a'_0, \cdots, a'_{t(p)-1}\rangle \in A'$$

such that for each $j < t(p)$, $\langle a_j, a'_j, q\rangle \in d$. A metric algebraic subsystem $\mathfrak{C}$ of a metric algebraic system $\mathfrak{A}$ is called dense in $\mathfrak{A}$ if and only if $\mathfrak{C}/\boldsymbol{m}$ is dense in $\mathfrak{A}/\boldsymbol{m}$ and the restriction of each R_p to C is dense in R_p.

Thus a subset of a relation is called dense here if and only if it is dense in the product topology. The above definition is the "correct" one for density of subsystems, as shown by the following fact: we can now make Definition, Theorem, Lemma, or Corollary 3.x(x $\geqq 4$) from the corresponding Definition, Theorem, Lemma, or Corrollary 2.x by making the following alterations (where necessary) in their texts: change "metric system" to "metric algebraic system of type $\boldsymbol{t}$" and change "$\boldsymbol{m}$" (the type of metric systems) to "$\boldsymbol{t}$". The proofs must be altered somewhat, although only one (done below) creates any problem in the new setting. Finally, the meaning of the words has been changed somewhat. Our new $DC_c(\mathfrak{C}, D)$, for example, now refers to all the relations in our system, rather then only the metric relations. A separable Banach Space, for example, is determined by the relation of vector sum (as well as norm, i.e., our metric) and scalar product on a dense countable subset.

LEMMA 3.5. (*Corresponding to* Lemma 2.5). *If* $\mathfrak{A}$ *is a metric algebraic system of type* $\boldsymbol{t}$, $\mathfrak{A} \models$ Sep *if and only if for some countable metric algebraic system of type* $\boldsymbol{t}$, $\mathfrak{C}$, $\mathfrak{A} \models Dc(\mathfrak{C})$.

Proof. First note that if N_p is any countable subset of $A^{t(p)}$, then there is a countable $C_p \subset A$ such that $N_p \subset C_p{}^{t(p)}$ (for example, let C_p be the collection of all points which are any coordinate for any point of N_p). Since the domain o of $\boldsymbol{t}$ is countable, we can take $C = \bigcup_{p<o} C_p \cup Q$, and let $\mathfrak{C}$ be C together with the restrictions of all the relations of $\mathfrak{A}$. Then $\mathfrak{C}$ is clearly countable and $\mathfrak{A} \models Dc(\mathfrak{C})$.

A remark left implicit in § II is that two separable complete systems are isomorphic if and only if they contain isomorphic dense subsystems. This remains true here only because our relations are required to be closed.

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Received April 20, 1966. This research was supported in part by National Science Foundation Grants GP-4361 and GP-5710.

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