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AN INVARIANT SUBSPACE THEOREM OF J. FELDMAN

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Theorem. Let t be a quasi-nilpotent bounded linear operator on a complex normed space X of dimension greater than one. Suppose further that there is a sequence $\{p_n(t)\}$ of polynomials in t and a nonzero compact operator s on X such that $p_n(t) \to s$ (in norm) as $n \to \infty$. Then t has a proper closed invariant subspace.

In [3], Feldman proves this theorem in the case when X is a Hilbert space. By adapting the proof given by Bonsall [2, Theorem (20.1)] of the Bernstein-Robinson invariant subspace theorem [1], the result can be shown to hold when X is a normed space, the necessary changes in the proof given in [2] being suggested by [3]. For the sake of completeness, the proof below repeats the relevant arguments in [2]. We need the following notation and simple results.

(i) If E is a nonempty subset of X and $x \in X$, *the distance from x to E*, $d(x, E)$, is defined by

$$d(x, E) = \inf \{\| x - y \| : y \in E\} .$$

(ii) Given a sequence $\{E_n\}$ of linear subspaces of X, define $\liminf E_n = \{x \in X : \exists \text{ a sequence } \{x_n\} \text{ with } x_n \in E_n \text{ and } x_n \to x\}$. It is clear that $\liminf E_n$ is a closed linear subspace of X and

$$\liminf E_n = \{x \in X : d(x, E_n) \to 0 \quad \text{as} \quad n \to \infty\} .$$

(iii) Given a finite dimensional subspace E of X and $x \in X$, $\exists\, u \in E$ such that $\| x - u \| = d(x, E)$. We call such a u *a nearest point of E to x*. Also, if F is a finite dimensional subspace of X such that $F \supset E$, $F \neq E$, $\exists\, v \in F$ such that $\| v \| = 1 = d(v, E)$.

Proof of theorem. Let $e \in X$, $\| e \| = 1$. Clearly we may assume that X has infinite dimension, and that $e, te, t^2e, \cdots$ are linearly independent. Let E_n be the linear span of $\{e, te, \cdots, t^{n-1}e\}$, and choose $e_n \in E_n$ such that

$$\| e_n \| = 1 = d(e_n, E_{n-1}) .$$

Since E_n is the linear span of $\{E_{n-1}, t^{n-1}e\}$, for each integer n there is a unique $\alpha_n \in \boldsymbol{C}$, $\alpha_n \neq 0$, such that

$$e_n - \alpha_n t^{n-1} e \in E_{n-1} . \tag{1}$$

Since $tE_{n-1} \subset E_n$, (1) gives

(2) $$t^r e_n - \alpha_n t^{n+r-1} e \in E_{n+r-1}$$

for $n \geqq 1$, $r \geqq 1$. Also, replacing n by $n + r$ in (1),

(3) $$e_{n+r} - \alpha_{n+r} t^{n+r-1} e \in E_{n+r-1} ,$$

and hence, by (2) and (3),

(4) $$t^r e_n - \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \in E_{n+r-1}$$

for $n \geqq 1$, $r \geqq 1$. We note that, since $d(e_n, E_{n-1}) = 1$, it follows from (4) that

$$d(t^r e_n, E_{n+r-1}) = \frac{|\alpha_n|}{|\alpha_{n+r}|} \qquad (n, r \geqq 1) .$$

We show that there is a subsequence $\{\alpha_{j(n)}/\alpha_{j(n)+1}\}$ of $\{\alpha_n/\alpha_{n+1}\}$ such that $\alpha_{j(n)}/\alpha_{j(n)+1} \longrightarrow 0$ as $n \longrightarrow \infty$. (This corresponds to the lemma in [3]). Suppose not. Then

$$\liminf_{n\to\infty} \left|\frac{\alpha_n}{\alpha_{n+1}}\right| = \lambda > 0 ,$$

and so there exists n_0 such that

$$\left|\frac{\alpha_n}{\alpha_{n+1}}\right| > \lambda/2 \quad \text{if} \quad n \geqq n_0 .$$

Since

$$\| t^r \| \geqq \| t^r e_n \| \geqq d(t^r e_n, E_{n+r-1}) = \left|\frac{\alpha_n}{\alpha_{n+r}}\right| ,$$

$$\| t^r \| \geqq \prod_{j=n}^{n+r-1} \left|\frac{\alpha_j}{\alpha_{j+1}}\right| .$$

Taking $n = n_0$, this gives

$$\| t^r \| \geqq (\lambda/2)^r \qquad (r \geqq 1) ,$$

and so

$$\lim_{r\to\infty} \| t^r \|^{1/r} \geqq \lambda/2 > 0 ,$$

contradicting the quasi-nilpotence of t. Therefore we can find a subsequence $\{j(n)\}$ such that

$$\frac{\alpha_{j(n)}}{\alpha_{j(n)+1}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty ,$$

i.e. such that

(5) $$d(te_{j(n)}, E_{j(n)}) \to 0 \quad \text{as} \quad n \to \infty .$$

Define linear mappings $t_n: E_n \to E_n$ $(n \geqq 1)$ by

$$t_n \mid E_{n-1} = t \mid E_{n-1} , \qquad t_n(e_n) = u_n ,$$

where u_n is a nearest point of E_n to te_n. We show that

(6) $$\| tx - t_n x \| \leqq d(te_n, E_n) \| x \| \qquad (x \in E_n, n \geqq 1) .$$

Let $x \in E_n$. Then $x = y + \lambda e_n$ for some $\lambda \in \boldsymbol{C}$, $y \in E_{n-1}$.

$$\| tx - t_n x \| = \| \lambda te_n - \lambda u_n \| = | \lambda | d(te_n, E_n) ,$$

and also

$$\| x \| \geqq d(x, E_{n-1}) = d(\lambda e_n, E_{n-1}) = | \lambda | d(e_n, E_{n-1}) = | \lambda | .$$

Therefore

$$\| tx - t_n x \| \leqq d(te_n, E_n) \| x \| \qquad (x \in E_n, n \geqq 1) .$$

From (5) and (6) we see that, if $\{x_n\}$ is a bounded sequence with $x_n \in E_{j(n)}$, then

(7) $$\| tx_n - t_{j(n)} x_n \| \to 0 \quad \text{as} \quad n \to \infty .$$

From (7) it follows that if $\{H_{n_k}\}$ is a sequence of subspaces with $H_{n_k} \subset E_{j(n_k)}$ and H_{n_k} invariant for $t_{j(n_k)}$, then $\liminf H_{n_k}$ is invariant for t.

We prove next, by induction on k, that for each integer k there is a constant A_k such that

(8) $$\| t^k x - t_n^k x \| \leqq A_k d(te_n, E_n) \| x \| \qquad (x \in E_n, n \geqq 1) .$$

The case when $k = 1$ is given by (6), $(A_1 = 1)$. Suppose that (8) holds for some k. Then, for $x \in E_n$,

$$\begin{aligned} \| t_n^k x \| &\leqq \| t^k x \| + A_k d(te_n, E_n) \| x \| \\ &\leqq (\| t^k \| + A_k d(te_n, E_n)) \| x \| \\ &\leqq (\| t^k \| + A_k \| t \|) \| x \| \\ &= B_k \| x \| , \quad \text{say} . \end{aligned}$$

Since $t_n^k E_n \subset E_n$, (6) gives

$$\begin{aligned} \| tt_n^k x - t_n^{k+1} x \| &\leqq d(te_n, E_n) \| t_n^k x \| \\ &\leqq B_k d(te_n, E_n) \| x \| . \end{aligned}$$

Therefore

$$\begin{aligned} \| t^{k+1} x - t_n^{k+1} x \| &\leqq \| t^{k+1} x - tt_n^k x \| + \| tt_n^k x - t_n^{k+1} x \| \\ &\leqq \| t \| \| t^k x - t_n^k x \| + \| tt_n^k x - t_n^{k+1} x \| \\ &\leqq (\| t \| A_k + B_k) d(te_n, E_n) \| x \| . \end{aligned}$$

Hence, by induction, (8) is proved.

It follows immediately from (8) that, given a polynomial $p(t)$ in t, there is a constant M such that

$$\| p(t)x - p(t_n)x \| \leqq Md(te_n, E_n) \| x \| \quad (x \in E_n, n \geqq 1) .$$

Hence we can find positive constants $\{M_r\}_{r \geqq 1}$ such that

$$\| p_r(t)x - p_r(t_n)x \| \leqq M_r d(te_n, E_n) \| x \| \tag{9}$$

for $x \in E_n$, $n \geqq 1$, $r \geqq 1$.

Since $st = ts$ and $s \neq 0$, we may assume that $s^{-1}(0) = (0)$, for otherwise $s^{-1}(0)$ is a proper closed invariant subspace for t. Therefore $se \neq 0$, and we can choose α with $0 < \alpha < 1$ and $\alpha \| s \| < \| se \|$. Choose sequences $\{E_n^i\}_{i=0}^{j(n)}$ of subspaces of $E_{j(n)}$ such that

$$(0) = E_n^0 \subset E_n^1 \subset \cdots \subset E_n^{j(n)} = E_{j(n)} ,$$

where $\dim E_n^i = i$ and E_n^i is invariant for $t_{j(n)}$. Since $d(e, E_n^0) = \| e \| = 1$ and $d(e, E_n^{j(n)}) = 0$, for each n there is a greatest i, i_n say, such that $d(e, E_n^{i_n}) > \alpha$. Put $F_n = E_n^{i_n}$, $G_n = E_n^{i_n+1}$. Then

$$d(e, F_n) > \alpha , \quad d(e, G_n) \leqq \alpha \qquad (n \geqq 1) ,$$

and so

$$e \notin \liminf F_{n_k} \tag{10}$$

for any subsequence $\{n_k\}$. Let y_n, z_n be nearest points of G_n to e, se respectively, and let $v_n \in G_n$ with $\| v_n \| = 1 = d(v_n, F_n)$. We can write

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ z_n &= x'_n + \beta'_n v_n , \end{aligned}$$

where x_n, $x'_n \in F_n$ and β_n, $\beta'_n \in \boldsymbol{C}$. We have

$$\begin{aligned} | \beta_n | &= d(\beta_n v_n, F_n) = d(y_n, F_n) \leqq \| y_n \| \\ &\leqq \| y_n - e \| + \| e \| = d(e, G_n) + \| e \| \leqq 2 \| e \| . \end{aligned}$$

Similarly

$$| \beta'_n | \leqq 2 \| se \| .$$

Also, for $n \geqq 1$,

$$\begin{aligned} \| sy_n \| &\geqq \| se \| - \| sy_n - se \| \geqq \| se \| - \| s \| \| y_n - e \| \\ &= \| se \| - \| s \| d(e, G_n) \geqq \| se \| - \alpha \| s \| > 0 . \end{aligned} \tag{11}$$

By the compactness of s and the boundedness of $\{\| y_n \|\}$, $\{| \beta_n |\}$, $\{| \beta'_n |\}$, we can find a subsequence $\{n_k\}$ such that

$$\beta_{n_k} \to \beta , \qquad \beta'_{n_k} \to \beta' , \qquad sy_{n_k} \to y \quad \text{as} \quad k \to \infty .$$

We show that $y \in \liminf G_{n_k}$. Let $\varepsilon > 0$. $\exists\ n_0$ such that

$$\| s - p_{n_0}(t) \| < \frac{\varepsilon}{4\| e \|} .$$

By (5), $\exists\ k_0$ such that

$$d(te_{j(n_k)}, E_{j(n_k)}) < \frac{\varepsilon}{4M_{n_0}\| e \|} \quad \text{if} \quad k \geqq k_0 .$$

Since $\| y_n \| \leqq 2\| e \|$ $(n \geqq 1)$, by (9)

$$\| p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \leqq M_{n_0} d(te_{j(n_k)}, E_{j(n_k)}) \cdot 2\| e \|$$

for $k \geqq 1$. Therefore $k \geqq k_0$ implies that

$$\begin{aligned} \| sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| &\leqq \| sy_{n_k} - p_{n_0}(t)y_{n_k} \| \\ &\quad + \| p_{n_0}(t)y_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \\ &\leqq \| s - p_{n_0}(t) \| \| y_{n_k} \| \\ &\quad + 2M_{n_0}\| e \| d(te_{j(n_k)}, E_{j(n_k)}) \\ &\leqq \frac{\varepsilon}{4\| e \|} \cdot 2\| e \| + 2M_{n_0}\| e \| \cdot \frac{\varepsilon}{4M_{n_0}\| e \|} = \varepsilon . \end{aligned}$$

Since $sy_{n_k} \to y$, $\exists\ k_1 \geqq k_0$ such that $\| sy_{n_k} - y \| < \varepsilon$ if $k \geqq k_1$. Thus if $k \geqq k_1$,

$$\begin{aligned} \| y - p_{n_0}(t_{j(n_k)})y_{n_k} \| &\leqq \| y - sy_{n_k} \| + \| sy_{n_k} - p_{n_0}(t_{j(n_k)})y_{n_k} \| \\ &< \varepsilon + \varepsilon = 2\varepsilon . \end{aligned}$$

But $p_{n_0}(t_{j(n_k)})y_{n_k} \in G_{n_k}$ since G_{n_k} is invariant for $t_{j(n_k)}$, and so

$$d(y, G_{n_k}) \leqq \| y - p_{n_0}(t_{j(n_k)})y_{n_k} \| < 2\varepsilon \quad \text{if} \quad k \geqq k_1 .$$

Therefore $d(y, G_{n_k}) \to 0$ as $k \to \infty$, and $y \in \liminf G_{n_k}$.

Now by (11) $y \neq 0$, and so $\liminf G_{n_k}$ will be a proper closed invariant subspace for t unless $\liminf G_{n_k} = X$. Thus we may suppose that $\liminf G_{n_k} = X$, and hence that $e, se \in \liminf G_{n_k}$, i.e.

$$d(e, G_{n_k}) = \| e - y_{n_k} \| \to 0 \quad \text{as} \quad k \to \infty$$

and

$$d(se, G_{n_k}) = \| se - z_{n_k} \| \to 0 \quad \text{as} \quad k \to \infty .$$

Therefore

$$x_{n_k} + \beta_{n_k} v_{n_k} \to e \quad \text{and} \quad x'_{n_k} + \beta'_{n_k} v_{n_k} \to se \quad \text{as} \quad k \to \infty .$$

Hence

$$\beta'_{n_k} x_{n_k} - \beta_{n_k} x'_{n_k} \to \beta' e - \beta se \quad \text{as} \quad k \to \infty ,$$

and so

$$\beta' e - \beta s e \in \liminf F_{n_k} .$$

If $\beta = 0$ then $x_{n_k} \to e$ and $e \in \liminf F_{n_k}$, contradicting (10). So $\beta \neq 0$. If $\beta' e - \beta s e = 0$ then $(\beta'/\beta)e = se \neq 0$ and so $\beta' \neq 0$. Then $s \neq (\beta'/\beta)\mathscr{I}$ since s is compact and X has infinite dimension ($\mathscr{I}$ being the identity operator on X). Therefore

$$0 \neq e \in \left(s - \frac{\beta'}{\beta}\mathscr{I}\right)^{-1}(0)$$

and $\{s - (\beta'/\beta)\mathscr{I}\}^{-1}(0)$ is a proper closed invariant subspace for t. Finally, if $\beta' e - \beta s e \neq 0$ then $\liminf F_{n_k} \neq (0)$, and so, by (10), $\liminf F_{n_k}$ is a proper closed invariant subspace for t.

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