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# BOUNDARY VALUE PROBLEMS FOR ELLIPTIC CONVOLUTION EQUATIONS OF WIENER-HOPF TYPE IN A BOUNDED REGION

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**Let $A$ be an elliptic convolution operator of order $\alpha$ on a bounded open set $G$ of $R^n$, $\alpha > 0$. Let $A_j$ be the principal part of $A$ in a local coordinates system and $\tilde{A}_j(x^j, \xi)$ be its symbol with a Wiener-Hopf type of factorization with respect to $\xi_n$: $\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$ for $x_n^j = 0$. $\tilde{A}_j^+$ is analytic in $\operatorname{Im} \xi_n > 0$, is homogeneous of order $k$ in $\xi$, $k$ is a positive integer, $k<\alpha$. $\tilde{A}_j^-$ is analytic in $\operatorname{Im} \xi_n \leqq 0$. Let $B_r$; $r=1,\cdots,k$ be a system of convolution operators on $\partial G$, of orders $\alpha_r$; $0 \leqq \alpha_r < \alpha$ and let $B_{rj}$ be the principal part of $B_r$ in a local coordinates system. The $\tilde{A}_j^+, \tilde{B}_{rj}$ are assumed to satisfy a Shapiro-Lopatinskii type of condition for each $j$.**

**Visik and Eskin have shown that the operator $U$ from $H_+^s(G)$ into**

$$H^{s-\alpha}(G, \partial G) = H^{s-\alpha}(G)x \times \prod_{r=1}^{k} H^{s-\alpha_r-1/2}(\partial G)\,; \qquad \alpha \leqq s\ ,$$

**defined by: $Uu = \{Au, B_1u, \cdots, B_ku\}$ is of Fredholm type. In this paper, we show the smoothness in the interior of the solutions of $Uu = (f, g_1, \cdot, g_k)$. We prove that if $\tilde{A}_j^+, \tilde{B}_{rj}$ satisfy a strengthened form of the Shapiro-Lopatinskii condition, then the operator $U_\lambda u = \{(A + \lambda^\alpha)u, B_1u, \cdots, B_ku\}$ is one-to-one and onto. The nonlinear problem:**

$$U_\lambda u = \{f(x, S_0u, \cdots, S_{\alpha-1}u), g_1, \cdots, g_k\}$$

**has a solution in $H_+^\alpha(G)$. $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ is continuous in all the variables and has at most a linear growth in $(\zeta_0, \cdots,\zeta_{\alpha-1})$. If the set $\Omega = \{u: u \in H_+^\alpha(G), B_ru = 0$ on $\partial G, r = 1, \cdots, k\}$ is dense in $L^2(G)$, then the completeness in $L^2(G)$ of the generaliled eigenfunctions of the operator $A_2$ associated with $Uu = \{f, 0, \cdots, 0\}$ is established.**

Boundary-value problems for elliptic convolution operators have been considered recently by Visik-Eskin [4].

In § I, we give the notation and terminology which are those of Visik-Eskin and state the assumptions. The main results are given without proofs in § 2. The proofs are carried out in § 3.

1. Let $s$ be an arbitrary real number and $H^s(R^n)$ be the Sobolev Slobodetskii space of (generalized) functions $f$ such that:

$$||f||_s^2 = \int_{E^n}(1 + |\xi|^2)^s\,|\tilde{f}(\xi)|^2\,d\xi$$

$\tilde{f}(\xi)$ is the Fourier transform of $f$.

By $H^s(R^n_+)$, we denote the space consisting of functions defined on $R^n_+ = \{x: x_n > 0\}$ and which are the restrictions to $R^n_+$ of functions in $H^s(R^n)$. Let $lf$ be an extension of $f$ to $R^n$. Then:

$$||f||^+_s = ||f||_{H^s(R^n_+)} = \inf ||lf||_s .$$

The infimum is taken over all the extensions $lf$ of $f$.

Let $\theta(x_n)$ be the function equal to 1 if $x_n > 0$ and to 0 if $x_n \leqq 0$. Every function $f$ in $L^2(R^n)$ may be written as $f = \theta f + (1 - \theta)f$. Hence $L^2(R^n)$ has the following orthogonal decomposition:

$$L^2(R^n) = \mathring{H}^+_0 + \mathring{H}^-_0 .$$

We denote by $H^+_s$, the space of functions $f_+$ with $f_+$ in $\mathring{H}^+_0$ and such that $f_+$ belongs to $H^s(R^n_+)$. $\mathring{H}^+_s$ is the subspace of $H^s(R^n)$ consisting of functions with supports in $\mathrm{cl}\,(R^n_+)$. $\widetilde{H}^+_s, \widetilde{H}_s, \widetilde{\mathring{H}}{}^+_s$ denote respectively the spaces which are the Fourier images of $H^+_s, H_s, \mathring{H}^+_s$.

Let $\tilde{f}(\xi)$ be a smooth decreasing function (i.e. $\tilde{f}(\xi) \leqq M|\xi_n|^{-1-\varepsilon}$ for large $|\xi_n|$ and $\varepsilon > 0$). The operator $\Pi^+$ is defined as:

$$\Pi^+\tilde{f}(\xi) = \frac{1}{2}\tilde{f}(\xi', \xi_n) + i(2\pi)^{-1}\,\mathrm{v.p.}\int\tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1}d\eta_n ,$$

$$\xi' = (\xi_1, \cdots, \xi_{n-1})$$

For any $\tilde{f}$, then the above relation is understood as the result of the closure of the operator $\Pi^+$ defined on the set of smooth and decreasing functions.

$\Pi^+$ is a bounded mapping from $\widetilde{H}_s$ into $\widetilde{\mathring{H}}{}^+_s$ if $0 \leqq s < \frac{1}{2}$ and a mapping from $\widetilde{H}_s$ into $\widetilde{H}^+_s$ if $\frac{1}{2} \leqq s$. $\Pi^-$ is defined similarly.

Set: $\xi_- = \xi_n - i|\xi'|$; $(\xi_- - i)^s$ is analytic in $\mathrm{Im}\,\xi_n < 0$. Then:

$$||f||^+_s = ||\Pi^+(\xi_- - i)^s\widetilde{lf}(\xi)||_0$$

where $lf$ is any extension of $f$ to $R^n$ (Cf. [4], p. 93, relation (8.1))

Let $G$ be a bounded open set of $R^n$ with a smooth boundary $\partial G$. We denote by $H^s(G)$ the restriction to $G$ of functions in $H^s(R^n)$ with the norm: $||f||_s = \inf ||g||_{H^s(R_n)}$; $g = f$ on $G$; $s \geqq 0$.

By $H^s_+(G)$, we denote the space of functions $f$ defined on all of $R^n$, equal to 0 on $R^n/\mathrm{cl}\,(G)$ and coinciding in $\mathrm{cl}\,(G)$ with functions in $H^s(G)$.

$H^s(\partial G)$ is defined as the completion of $C^\infty(\partial G)$ with respect to:

$$||f||'_s = \left\{\sum_j ||\varphi_j f||^2_{H^s(R^{n-1})}\right\}^{1/2}; s \geqq 0$$

where $\| \varphi_j f \|_{H^s(R^{n-1})}$ is taken in local coordinates and the $\varphi_j$ are those functions of a finite partition of unity corresponding a covering of $\mathrm{cl}\, G$, whose supports intersect the boundary $G$. We may show that different partitions of unity give rise to equivalent norms (cf. [3]).

DEFINITION 1. $\tilde{A}(\xi)$ is in $0_\alpha$ if and only if:
(i) $\tilde{A}$ is homogeneous in $\xi$ of order $\alpha$.
(ii) $\tilde{A}$ is continuous for $\xi \neq 0$.

DEFINITION 2. $\tilde{A}_+(\xi', \xi_n)$ is in $0_\alpha^+$ if and only if:
(i) $\tilde{A}_+$ is in $0_\alpha$.
(ii) $\tilde{A}_+(\xi', \xi_n)$ has an analytic continuation with respect to $\xi_n$ in $\mathrm{Im}\, \xi_n > 0$ for each $\xi'$.
Similar definition for $0_\alpha^-$.

DEFINITION 3. $\tilde{A}(\xi)$ is in $E_\alpha$ if and only if:
(i) $\tilde{A}(\xi)$ is in $0_\alpha$.
(ii) $\tilde{A}(\xi)$ satisfies the ellipticity condition, i.e. $\tilde{A}(\xi) \neq 0$ for $\xi \neq 0$.
(iii) $\tilde{A}(\xi)$ has for $\xi' \neq 0$, continuous first order derivatives, bounded if $|\xi| = 1, \xi' \neq 0$.

DEFINITION 4. $\tilde{A}_+(\xi)$ is in $C_k^+$ if and only if:
(i) $\tilde{A}_+(\xi)$ is in $0_k^+$ and $\tilde{A}_+(\xi) \neq 0$ for $\xi \neq 0$; $k$ is a positive integer.
(ii) For any integer $p > 0$, there is an expansion:

$$\tilde{A}_+(\xi) = \sum_{s=0}^{p} c_s(\xi') \xi_+^{k-s} + R_{k,p+1-k}(\xi', \xi_n)$$

where $\xi_+ = \xi_n + i |\xi'|$; all the terms are in $0_k^+$ and:

$$|R_{k,p+1-k}(\xi', \xi_n)| \leqq C |\xi'|^{p+1} (|\xi'| + |\xi_n|)^{k-p-1} .$$

DEFINITION 5. $\tilde{A}(\xi)$ is in $D_\alpha$ if and only if:
(i) $\tilde{A}(\xi)$ is in $0_\alpha$.
(ii) For each $s \geqq \alpha$; there is a decomposition:

$$\xi_-^s \tilde{A}(\xi) = \tilde{A}_-(\xi) + R_{s+\alpha,-1}(\xi)$$

where $\tilde{A}_-(\xi)$ is in $0_{\alpha+s}^-$, $|R_{s+\alpha,-1}(\xi)| \leqq C |\xi'|^{s+1+\alpha} (|\xi'| + |\xi_n|)^{-1}$.

DEFINITION 6. $\tilde{A}(\xi)$ is in $D_{\alpha,1}$ if and only if:
(i) $\tilde{A}(\xi)$ is in $D_\alpha$.
(ii) $\tilde{A}_-(\xi)$ and $R_{s+\alpha,-1}(\xi)$ are continuously differentiable for $\xi' \neq 0$.
(iii) $|\tilde{A}_-(\xi)| \leqq C |\xi|^{\alpha-1}$; $|R_{s+\alpha,-1}(\xi)| \leqq C |\xi'|^{s+\alpha} (|\xi'| + |\xi_n|)^{-1}$.

DEFINITION 7. Let $A$ be a linear, bounded operator from $H_s^+$ into

$H^{s-\alpha}(R^n_+)$. Then any bounded, linear operator $T$ from $H^+_{s-1}$ into $H^{s-\alpha}(R^n_+)$ (or from $H^+_s$ into $H^{s-\alpha+1}(R^n_+)$) is called a right (left) smoothing operator with respect to $A$.

$T$ is a smoothing operator with respect to $A$ if $T$ is both a left and right smoothing operator.

Let $\tilde{A}(\xi)$ be in $E_\alpha$ for $\alpha > 0$ and $u_+$ be in $H^+_s$, $s \geqq 0$. Then we define: $Au_+ = F^{-1}(\tilde{A}(\xi)\tilde{u}_+(\xi))$ where the inverse Fourier transform is understood in the sense of the theory of distributions. $Au_+$ is well-defined.

Let $\tilde{A}(x, \xi)$ be in $E_\alpha$ for $x$ in $\mathrm{cl}\, G$ and $\tilde{A}(x, \xi)$ be infinitely differentiable with respect to $x$ and to $\xi$. We extend $\tilde{A}(x, \xi)$ with respect to $x$, to all of $R^n$ by setting $\tilde{A}(x, \xi) = 0$ for $|x| \geqq p - \varepsilon, \varepsilon > 0$. The homogeneity with respect to $\xi$ is preserved. We expand $\tilde{A}(x, \xi)$ into a Fourier series:

$$\tilde{A}(x, \xi) = \sum_{k=-\infty} \psi_0(x) \exp(ikx\pi/p)\tilde{L}_k(\xi)\,; \qquad k = (k_1, \cdots, k_n)$$

and:

$$\tilde{L}_k(\xi) = (2p)^{-n}\int_{-p}^{p} \exp(-ikx\pi/p)\tilde{A}(x, \xi)dx$$

$\psi_0(x) \in C^\infty_c(R^n)$ with $\psi_0(x) = 1$ for $|x| \leqq p - \varepsilon$; $\psi_0(x) = 0$ for $|x| \geqq p$. For $u_+$ in $H^s_+(G)$, we define:

$$P^+Au_+ = P^+\Big(\sum_{k=-\infty} \psi_0(x) \exp(ikx\pi/p)L_k * u_+\Big)\,.$$

$P^+$ is the restriction operator of functions defined on $R^n$ to $G$, $L_k u_+$ is defined as before since its symbol $\tilde{L}_k(\xi)$ is independent of $x$ and $|\tilde{L}_k(\xi)| \leqq (1 + |k|)^{-M}|\xi|^\alpha$ for large positive $M$.

DEFINITION 8. $\tilde{A}(x, \xi)$ is in $D^0_\alpha$ if and only if:

(i) $\tilde{A}(x, \xi)$ is infinitely differentiable with respect to $x$ and $\xi \neq 0$.

(ii) $\tilde{A}(x, \xi)$ is in $0_\alpha$ for $x$ in $R^n$.

(iii) $a_{k2}(x) = (\partial^k/\partial\xi'^k)\tilde{A}(x, 0, -1) = (-1)^k \exp(-i\pi\alpha)(\partial^k/\partial\xi'^k)\tilde{A}(x, 0, 1)$ $x$ in $R^n$, $0 \leqq |k| < +\infty$.

DEFINITION 9. $\tilde{A}(x, \xi)$ is in $\hat{D}^1_{\alpha,1}$ if and only if:

(i) $|D^p_x\tilde{A}(x, \xi)| \leqq C_p(1 + |\xi|)^\alpha$; $0 \leqq |p| < +\infty$.

(ii) For each $x$ in $R^n$ and for any $s \geqq -\alpha$, there is a decomposition: $(\xi_- - i)^s\tilde{A}(x, \xi) = \tilde{A}_-(x, \xi) + R(x, \xi)$

$\tilde{A}_-(x, \xi)$ and $R(x, \xi)$ are infinitely differentiable with respect to $x$ $\tilde{A}_-(x, \xi)$ is analytic in $\mathrm{Im}\, \xi_n \leqq 0$ and:

$$| D_x^p \tilde{A}_-(x, \xi) | \leqq C_p(1 + |\xi|)^{s+\alpha}; | D_x^p D_\xi \tilde{A}_-(x, \xi) | \leqq c_p(1 + |\xi|)^{s-1+\alpha}$$
$$| D_x^p R(x, \xi) | \leqq C_p(1 + |\xi'|)^{s+1+\alpha}(1 + |\xi|)^{-1}$$
$$| D_x^p D_\xi R(x, \xi) | \leqq c_p(1 + |\xi'|)^{s+\alpha}(1 + |\xi|)^{-1} ; \qquad 0 \leqq |p| < +\infty .$$

Let $B_r; r = 1, \cdots, k$ be a system of convolution operators on $\partial G$. We introduce the definition of a regular elliptic convolution boundary value problem on $G$:

DEFINITION 10. Let $G$ be a bounded open set of $R^n$ and $\varphi_j$ be a finite partition of unity corresponding to a covering $N_j$ of $\mathrm{cl}\, G$. Let $\psi_j$ be the infinitely differentiable functions with compact supports in $N_j$ and such that: $\varphi_j \psi_j = \varphi_j$

(1) Let: $P^+A = \sum_j P^+\varphi_j A \psi_j + \sum_j P^+\varphi_j A(1 - \psi_j)$ be an elliptic convolution operator of order $\alpha$ on $G$ with the following properties:

(a) The operator $\varphi_j A \psi_j$ transformed in local coordinates, is the sum of a convolution operator $A_j$ and a smoothing operator. The symbol $\tilde{A}_j(x^j, \xi)$ is homogeneous of order $\alpha$ in $\xi; \alpha > 0$

(b) $\tilde{A}_j(x^j, \xi) \in E_\alpha$ and for $x_n^j = 0$ admits the factorization:

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+, \tilde{A}_j^-$ belong to $0_k^+, 0_{\alpha-k}^-$ respectively and $k$ is a positive integer.

(c) $\tilde{A}_j(x^j, \xi)$ is in $D_\alpha^0 \cap \hat{D}_{\alpha,1}^1$ for $x \in N_j \cap \partial G \neq 0$.

(2) Let $\gamma$ denote a passage to the boundary $\partial G$ and:

$$P^+B_r = \sum_j P^+\varphi_j B_r \psi_j + \sum_j P^+\varphi_j B_r(1 - \psi_j) ; \qquad r = 1, \cdots, k$$

be a system of convolution operators on $\partial G$ with the following properties:

(a) The operator $\varphi_j B_r \psi_j$ taken in local coordinates, is the sum of a convolution operator $B_{rj}$ with symbol $\tilde{B}_{rj}$, homogeneous of order $\alpha_r$ in $\xi$ and a smoothing operator. $0 \leqq \alpha_r < \alpha - \frac{1}{2}$.

(b) $\tilde{B}_{rj}(x^j, \xi) \in D_{\alpha_r}^0 \cap \hat{D}_{\alpha_r,1}^1$ for $x \in N_j \cap \partial G \neq 0$.

The boundary-value problem: $\{P^+Au_+, \gamma P^+\beta_1 u_+, \cdots, \gamma P^+B_k u_+\}$ is said to be uniformly regular on $G$ if:

$$\mathrm{Det}\,((b_{rs}(x^j, \xi'))) \neq 0 \quad \text{for all} \quad x^j \in N_j \cap \partial G \neq 0$$

and:

$$\Pi^+\tilde{B}_{rs}(x^j, \xi)\xi_n^{s-1}(\tilde{A}_j^+(x^j, \xi))^{-1} = ib_{rs}(x^j, \xi')\xi_+^{-1} + R_{rs}(x^j, \xi)$$
$$\mathrm{ord}\,(b_{rs}(\xi')) = \alpha_r + k - s , \qquad r, s = 1, \cdots, k .$$

*Assumption* (1); Let $\{P^+A, \gamma P^+B_1, \cdots, \gamma P^+B_k\}$ be a uniformly regular elliptic convolution boundary-value problem on $G$ in the sense of Definition 10.

We assume there exists a ray $\arg \lambda = \theta$ such that:

(i) If: $\tilde{A}_j(x^j, \xi, \lambda) = \tilde{A}_j(x^j, \xi) + \lambda^\alpha = \tilde{A}_j^+(x^j, \xi, \lambda)\tilde{A}_j^-(x^j, \xi, \lambda)$; then: $\tilde{A}_j^+(x^j, \xi, \lambda)$ is in $C_k^+$.

(ii) $\operatorname{Det}((b_{rs}(x^j, \xi', \lambda))) \neq 0$ for all $x^j$ with $N_j \cap \partial G \neq 0$ and $\arg \lambda = \theta, |\lambda| > \lambda_0 > 0$

$$\Pi^+ \tilde{B}_{rj}(x^j, \xi)\xi_n^{s-1}(\tilde{A}_j^+(x^j, \xi, \lambda))^{-1} = ib_{rs}(x^j, \xi', \lambda)(\xi_+^\lambda)^{-1} + R_{rs}(x^j, \xi, \lambda)$$
$$\xi_+^\lambda = \xi_n + i(|\lambda| + |\xi'|) ; \qquad r, s = 1, \cdots, k .$$

2. In this section, we shall state the results of the paper. First, we have an interior regularity theorem:

THEOREM 2.1. *Let* $\{P^+A, \gamma P^+B_1, \cdots, \gamma P^+B_k\}$ *be a uniformly regular elliptic convolution boundary-value problem on G in the sense of Definition* 10. *Let* $u_+$ *be an element of* $H_+^\alpha(G)$ *and* $Uu_+ = \{f, g_1, \cdots, g_k\}$ *with* $\{f, g_1, \cdots, g_k\}$ *in* $H^0(G, \partial G)$ *and* $\alpha \geqq 0$. *Suppose that f is in* $H^{s-\alpha}(G)$, $s \geqq \alpha$ *then* $u_+$ *is in* $H_+^\alpha(G) \cap H_{\mathrm{loc}}^s(G)$.

*If f is in* $C^\infty(cl\, G)$, *then*: $u_+$ *is in* $C_{\mathrm{loc}}^\infty(G)$.

With an additional hypothesis, we show that the operator associated with the problem is one-to-one and onto:

THEOREM 2.2. *Let* $\{P^+A, \gamma P^+B_r; r = 1, \cdots, k\}$ *be a uniform uniformly regular elliptic convolution boundary value problem on G in the sense of Definition* 10. *Suppose that Assumption* (1) *is satisfied. Then for every* $(f, g_1, \cdots, g_k)$ *in* $H^{s-\alpha}(G, \partial G)$, *there exists a unique solution* $u_+$ *in* $H_+^s(G)$ *of*:

$$P^+(A + \lambda^\alpha)u_+ = f \text{ on } G, \qquad \gamma P^+B_r u_+ = g_r \text{ on } \partial G; r = 1, \cdots, k$$

$s \geqq \alpha$ *and* $s, \alpha, \alpha_r$ *are all assumed to be nonnegative integers.*

*Moreover, there exists a positive constant M independent of* $\lambda, u_+$ $f, g_r$ *such that*:

$$\|u_+\|_s + |\lambda|^s \|u_+\|_0 \leqq M\Big\{\|f\|_{s-\alpha} + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^k \|g_r\|'_{s-\alpha_r-(1/2)} + |\lambda|^{s-\alpha_r-(1/2)}\|g_r\|'_0\Big\}$$

*for all* $u_+$ *in* $H_+^s(G), \arg \lambda = \theta; |\lambda| \geqq \lambda_0 > 0$.

Now, we have a global regularity theorem for the solutions of $Uu_+ = (f, g_1, \cdots, g_k)$.

THEOREM 2.3. *Suppose the hypotheses of Theorem* 2.2 *are satisfied. Let* $u_+$ *be a solution in* $H_+^\alpha(G)$ *of* $Uu_+ = (f, g_1, \cdots, g_k)$. *If* $(f, g_1, \cdots, g_k)$ *is in* $H^{s-\alpha}(G, \partial G)$, $s \geqq \alpha$, *then*: $u_+ \in H_+^s(G)$. *More generally if* $f$ *is in* $C^\infty(\mathrm{cl}\, G)$, $g_r$ *are in* $C^\infty(\partial G)$; *then* $u_+$ *is in* $C^\infty(G)$.

We shall now consider problems related to the spectral theory of the operator associated with $Uu_+ = (f, 0, \cdots, 0)$.

COROLLARY 2.1. (i) *Suppose the hypotheses of Theorem* 2.2 *are satisfied. Let*

$$\Omega = \{u_+ : u_+ \in H_+^\alpha(G), \gamma P^+ B_r u_+ = 0 \text{ on } \partial G; r = 1, \cdots, k\}$$

*Suppose that* $\Omega$ *is dense in* $L^2(G)$. *Let* $A_2$ *be the operator on* $L^2(G)$ *with* $D(A_2) = \Omega$; $A_2 u_+ = P^+ A u_+$ *on* $G$.

*Then*: $(A_2 + \lambda^\alpha I)^{-1}$ *exists, is defined on all of* $L^2(G)$ *and is a compact operator. The spectrum of* $A_2$ *is discrete.*

(ii) *Suppose further that Assumption* (1) *is satisfied by rays* $\arg \lambda = \theta_r$; $r = 1, \cdots, N$ *and that the plane is divided by those rays into angles less than* $2\alpha\pi/n$. *Then the generalized eigenfunctions of* $A_2$ *are complete in* $L^2(G)$.

COROLLARY 2.2. *Suppose that the hypotheses of Theorem* 2.2 *are satisfied. Let* $S_r$; $r = 0, \cdots, \alpha - 1$ *be bounded linear operators from* $H_+^r(G)$ *into* $L^2(G)$. *Let* $f(x, \zeta_0, \cdots, \zeta_{\alpha-1})$ *be a function measurable in* $x$ *on* $G$, *continuous in all the other variables and such that*:

$$|f(x, \zeta_0, \cdots, \zeta_{\alpha-1})| \leqq M\left\{1 + \sum_{j=0}^{\alpha-1} |\zeta_j|\right\}.$$

*Then for* $(g_1, \cdots, g_k)$ *in* $\prod_{r=1}^k H^{\alpha-\alpha_r-(1/2)}(\partial G)$ *and* $|\lambda| \geqq \lambda_0 > 0$, $\arg \lambda = \theta$ *there exists a solution* $u_+$ *in* $H_+^\alpha(G)$ *of*:

$$P^+(A + \lambda)u_+ = f(x, S_0 u_+, \cdots, S_{\alpha-1} u_+) \text{ on } G;$$
$$\gamma P^+ B_r u_+ = g_r \text{ on } \partial G; r = 1, \cdots, k$$

3. *Proof of Theorem* 2.1. (1) First, we show the existence of a left regularizer of $U$.

From Theorem 2.9 of [4], the operator $U$ has a right regularizer $S$, i.e. $US = I + R$, where $S$ is a bounded linear mapping from $H^{s-\alpha}(G, \partial G)$ into $H_+^s(G)$ and $R$ is a bounded linear mapping from $H^{s-\alpha}(G, \partial G)$ in $H^{s+1-\alpha}(G, \partial G)$.

Let $R_1$ be the operator from $H_+^s(G)$ into itself defined by the ralation: $R_1 u_+ = SUu_+ - u_+$.

We show that: $\| R_1 u_+ \|_{s+1} \leqq C \| u_+ \|_s$ for all $u_+$ in $H_+^{s+1}(G)$.

Consider:

$$UR_1u_+ = USUu_+ - Uu_+ = Uu_+ + RUu_+ - Uu_+ = RUu_+ .$$

From Theorem 2.9 of [4], we have:

$$\| R_1u_+ \|_{s+1} \leqq M\Big\{\| R_1u_+ \|_0 + \| P^+AR_1u_+ \|_{s+1-\alpha} + \sum_{r=1}^{k} \| \gamma P^+B_rR_1u_+ \|'_{s-\alpha_r+(1/2)}\Big\} .$$

But $RUu_+ = UR_1u_+$ and $R$ is a bounded mapping from $H^{s-\alpha}(G, \partial G)$ into $H^{s+1-\alpha}(G, \partial G)$. Therefore:

$$\| R_1u_+ \|_{s+1} \leqq M\Big\{\| R_1u_+ \|_0 + \| P^+Au_+ \|_{s-\alpha} + \sum_{r=1}^{k} \| \gamma P^+B_ru_+ \|'_{s-\alpha_r-(1/2)}\Big\} .$$

Since we assume that in all the local coordinates system, the principal parts of $A, B_r$ have symbols belonging to $\hat{D}^1_{\alpha,1}; \hat{D}^1_{\alpha_r,1}$ respectively with $0 \leqq \alpha_r < \alpha$; we have:

$$\| P^+Au_+ \|_{s-\alpha} \leqq C \| u_+ \|_s \quad \text{and} \quad \| \gamma P^+B_ru_+ \|'_{s-\alpha_r-1/2} \leqq C \| u_+ \|_s$$

(Cf. [4], Th. 1.4; p. 104).

Hence: $\| R_1u_+ \|_{s+1} \leqq M \| u_+ \|_s$ for all $u_+$ in $H^{s+1}_+(G)$.

(2) (a) We show that: $\| R_1(\varphi u_+) \|_{s+1} \leqq M \| \varphi u_+ \|_s$ for all $u_+$ in $H^s_+(G)$ and $\varphi$ in $C^\infty_c(G)$.

Let $\zeta(x)$ be an infinitely differentiable function with compact support in $G$ and such that: $0 \leqq \zeta(x) \leqq 1$; $\zeta(x) = 1$ on $G_1$, $\zeta(x) = 0$ outside of $G_0$ with $\operatorname{cl} G_1 \subset G_0 \subset \operatorname{cl} G_0 \subset G$.

Let $u_+$ be an element of $H^s_+(G)$. Then $u_+$ is in $H^s(G)$ and there exists a sequence $\varphi_n$ of elements in $C^\infty(\operatorname{cl} G)$ such that:

$$\varphi_n \longrightarrow u_+ \quad \text{in} \quad H^s(G) .$$

One can check easily that: $\zeta\varphi_n \to \zeta u_+$ in $H^s_+(G)$; $s \geqq 0$. Consider $\zeta\varphi_n$ It is an element of $H^{s+1}_+(G)$, so from the first part we get:

$$\| R_1(\zeta\varphi_n) \|_{s+1} \leqq M \| \zeta\varphi_n \|_s .$$

$M$ is independent of $n$. Hence $R_1(\zeta\varphi_n) \to v$ in $H^{s+1}(G)$. Since $\zeta\varphi_n \to \zeta u_+$ in $H^s_+(G)$ and $R_1$ is a bounded linear mapping from $H^s_+(G)$ into itself, we obtain: $v = R_1(\zeta u_+)$.

Therefore: $\| R_1(\zeta u_+) \|_{s+1} \leqq C \| \zeta u_+ \|_s$ for all $u_+$ in $H^s_+(G)$.

(b) We shall deduce the smoothness in the interior of the solutions of $Uu_+ = (f, g_1, \cdots, g_k)$ from the above argument.

Let $u_+$ be a solution in $H^\alpha_+(G)$ of $Uu_+ = (f, g_1, \cdots, g_k)$ where $(f, g_1, \cdots, g_k)$ is in $H^0(G, \partial G)$ and $f$ is in $H^1(G)$.

Consider:

$$P^+A(\zeta u_+) = \sum_j P^+\varphi_j A(\zeta\varphi_j u_+^+) + \sum_j P^+\varphi_j A(1-\psi_j)(\zeta u_+) .$$

Transforming $\varphi_j A(\zeta\psi_j u_+)$ in local coordinates and applying Lemma 4.D.1 of [4], (p. 145), we get:

$$\varphi_j A(\zeta\psi_j u) = \zeta\varphi_j A_j(\psi_j u_+) + T_j^1(\psi_j u_+) + \zeta T_j^2(\psi_j u_+)$$

where $T_j$ are smoothing operators with respect to $A_j$, i.e. with respect to a bounded linear mapping from $H_+^s(B_+)$ into $H^{s-\alpha}(B_+)$.

On the other hand, since the kernel of $A$ has a point singularity and $\varphi_j(1-\psi_j) = 0$, the operator $\varphi_j A(1-\psi_j)u_+$ has an infinitely differentiable kernel and hence may be estimated in any norm (Cf. [4], p. 125).

So:

$$P^+A(\zeta u_+) = \zeta A u_+ + T_0 u_+$$

where $T_0$ is a smoothing operator with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$.

Doing in a similar fashion for $B_r(\zeta u_+)$, we obtain:

$$\gamma P^+B_r(\zeta u_+) = \zeta B_r u_+ + T_r u_+ \; ; \qquad r = 1, \cdots, k$$

where $T_r$ are smoothing operators with respect to a bounded, linear mapping from $H_+^s(G)$ into $H^{s-\alpha_r-(1/2)}(\partial G)$.

Combining the results and taking into account the fact that $\zeta$ has compact support in $G_0$ whose closure is in $G$, we get:

$$U(\zeta u_+) = (\zeta f + T_0 u_+, \gamma T_r u_+; r = 1, \cdots, k) .$$

We have: $SU(\zeta u_+) = \zeta u_+ + R_1(\zeta u_+)$.

Consider $U(\zeta u_+)$. Since $u_+$ is in $H_+^\alpha(G)$ and the $T_s$ are all smoothing operators, $U(\zeta u_+)$ is in $H^1(G, \partial G)$. Therefore $SU(\zeta u_+)$ is in $H_+^{\alpha+1}(G)$.

From the first part of the proof, we get: $R_1(\zeta u_+) \in H_+^{\alpha+1}(G)$. Hence $\zeta u_+$ is in $H_+^{\alpha+1}(G)$.

(c) We prove by induction for the general case.

Suppose that $\zeta u_+$ is in $H_+^{s-1}(G)$, $s-1 \geqq \alpha$. We show that it is true for $s$.

Let $\eta$ be an infinitely differentiable function with compact support in $G$ and such that: $0 \leqq \eta(x) \leqq 1$; $\eta(x) = 1$ on $G_3$, $\eta(x) = 0$ outside of $G_2$ with

$$\mathrm{cl}\, G_3 \subsetneqq G_2 \subsetneqq \mathrm{cl}\, G_2 \subsetneqq G_1 \subsetneqq \mathrm{cl}\, G_1 \subsetneqq G_0$$

and $\mathrm{cl}\, G_0 \subsetneqq G$. Consider $U(\zeta\eta u_+)$. We have:

$$P^+A(\zeta\eta u_+) = \sum_j P^+\varphi_j A(\zeta\eta\psi_j u_+) + \sum_j P^+\varphi_j A(1-\psi_j)(\zeta\eta u_+) .$$

We express $\varphi_j A(\zeta\eta\psi_j)$ in local coordinates and applying Lemma 4.D.1 of [4], we obtain:

$$\begin{aligned}\varphi_j A(\zeta\eta\psi_j u_+) &= \eta\varphi_j A_j(\zeta\psi_j u_+) + T_0^1(\zeta u_+)\\ &= \zeta\eta\varphi_j A_j(\psi_j u_+) + \eta T_0^2(u_+) + T_1^0(\zeta u_+)\ .\end{aligned}$$

So:

$$P^+\varphi_j A(\zeta\eta\psi_j u_+) = \zeta\eta A u_+ + \eta T_0^3 u_+ + T_0^4(\zeta u_+)$$

where $T_0^3$, $T_0^4$ are smoothing operators with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$.

Since $\zeta u_+ \in H_+^{s-1}(G)$, $T_0^4(\zeta u_+)$ lies in $H^{s-\alpha}(G)$ and:

$$\|\eta T_0^3 u_+\|_{s-\alpha} \leqq M\|T_0^3 u_+\|_{H^{s-\alpha}(G_2)} \leqq M\|u_+\|_{H^{s-1}(G_2)}\ .$$

So, $P^+A(\zeta\eta u_+)$ is in $H^{s-\alpha}(G)$.

We do in a similar fashion for $\gamma P^+B_r(\zeta\eta u_+)$.

An argument as above shows that $U(\zeta\eta u_+)$ is in $H^{s-\alpha}(G, \partial G)$. Therefore $SU(\zeta\eta u_+)$ belongs to $H_+^s(G)$. Moreover, since $\zeta u_+$ is in $H_+^{s-1}(G)$, $R_1(\zeta\eta u_+)$ lies in $H_+^s(G)$. Hence $\zeta\eta u_+$ belongs to $H_+^s(G)$.

(d) If $f$ is in $C^\infty(G)$, then by repeated use of the Scbolev imbedding theorem, we get: $u_+ \in C_{\text{loc}}^\infty(G)$.

*Proof of Theorem* 2.3 *using Theorem* 2.2. Let $u$ be a solution in $H_+^\alpha(G)$ of: $Uu = (f, g_1, \cdots, g_k)$ where $(f, g_1, \cdots, g_k)$ is an element of $H^{s-\alpha}(G, \partial G)$ for $s \geqq \alpha$.

From Theorem 2.2, there exists a unique element $v$ in $H_+^s(G)$, solution of:

$$U(\lambda)v = (f, g_1, \cdots, g_k)$$

where

$$U(\lambda)v = (P^+(A+\lambda^\alpha)v, \gamma P^+B_1 v, \cdots, \gamma P^+B_k v)\ .$$

Consider:

$$U(\lambda)(v-u) = (-\lambda^\alpha u, 0, \cdots, 0)\ .$$

Since $\lambda^\alpha u$ is in $H^\alpha(G)$, it follows from Theorem 2,2 that the unique solution $w = v - u$ of $U(\lambda)w = (-\lambda^\alpha u, 0, \cdots, 0)$ is in $H_+^{2\alpha}(G)$. Therefore $u = v - w$ belongs to $H_+^{\min(s,2\alpha)}(G)$.

If $\min(s, 2\alpha) = s$, then we are through. If $2\alpha < s$, then since $u$ is in $H_+^{2\alpha}(G)$, $w$ is in $H_+^{3\alpha}(G)$, so $u$ $H_+^{\min(s,3\alpha)}(G)$.

Repeating this boot-strap argument, we get finally $u$ in $H_+^s(G)$.

*Proof of Corollary* 2.1. (1) Let $A_2$ be the linear operator from $D(A_2) = \Omega$ into $L^2(G)$ with $A_2 u = P^+Au$ if $u \in D(A_2)$.

With the hypotheses of the corollary, it follows from the theorem that $(A_2 + \lambda^\alpha I)^{-1}$ exists, is defined on all of $L^2(G)$ and maps $L^2(G)$ into $H_+^\alpha(G)$. Since $G$ is bounded, the injection mapping from $H_+^\alpha(G)$ into $L^2(G)$ is compact. So $(A_2 + \lambda^\alpha I)^{-1}$ is a compact mapping of $L^2(G)$ into itself and therefore the spectrum of $A_2$ is discrete, and the eigenspaces are of finite dimension.

(2) We have the following estimate on the growth of $(A_2+\lambda^\alpha I)^{-1}$:

$$\|(A_2 + \lambda^\alpha I)^{-1}\| \leqq M/|\lambda|^\alpha .$$

If Assumption (1) is valid for rays $\arg\lambda = \theta_j; j = 1, \cdots, N$ and the plane is divided by these rays into angles less than $2\alpha\pi/n$, then it follows from Theorem 3.2 of Agmon [1] (p. 128–129) that the generalized eigenfunctions of $A_2$ are complete in $L^2(G)$. Indeed in the proof of the theorem, only the compactness of $(A_2 + \lambda^\alpha I)^{-1}$ and an estimate on the growth of the resolvent operator as in this paper are needed.

*Proof of Corollary* 2.2. Taking into account Theorem 2.2, we may prove without much modification Corollary 2.2 as in [2].

*Proof of Theorem* 2.2. The proof is long. It is technically simpler than in the case when $\lambda = 0$. First, we have the lemma:

LEMMA 3.1. *Let $\{P^+A; \gamma P^+B_r; r = 1, \cdots, k\}$ be a regular elliptic convolution boundary-value problem on $R_+^n$ in the sense of Definition* 10, *with constant symbols $\tilde{A}(\xi), \tilde{B}_r(\xi)$, homogeneous of orders $\alpha, \alpha_r$ repectively. $\alpha, \alpha_r$ are positive integers. Suppose that Assumption* (1) *is satisfied. Then for every $(f, g_1, \cdots, g_k)$ in $H^{s-\alpha}(R_+^n, R^{n-1})$, $s \geqq \alpha$, there exists a unique solution $u_+$ in $H_s^+$ of: $P^+(A + \lambda^\alpha)u_+ = f$ on $R_+^n$; $\gamma P^+B_r u_+ = g_r$ on $R^{n-1}$; $r = 1, \cdots, k$ Moreover*:

$$\|u_+\|_s^+ + |\lambda|^s \|u_+\|_0^+ \leqq M\Big\{\|f\|_{s-\alpha}^+ + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^k \|g_r\|'_{s-\alpha_r-(1/2)} + |\lambda|^{s-\alpha_r-(1/2)}\|g_r\|'_0\Big\} .$$

*$M$ is independent of $\lambda, u_+, f, g_r$, $u_+$ is the inverse Fourier transform of $\tilde{u}_+(\xi)$ with*:

$$\tilde{u}_+(\xi) = (\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\lambda)(\tilde{A}_-(\xi, \lambda))^{-1} + \sum_{r=1}^k \tilde{D}_r(\xi, \lambda)(\tilde{g}_r(\xi') - \tilde{f}_r(\xi', \lambda))$$

*where*:

$$\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda)$$

$$\tilde{D}_r(\xi, \lambda) = \sum_{m-1}^{k} b^1_{rm}(\xi', \lambda)\xi_n^{m-1}(\tilde{A}_+(\xi, \lambda))^{-1}$$

$b^1_{rm}$ *are the elements of the transpose of the inverse of the matrix* $((b_{rm}(\xi', \lambda)))$. $lf$ *is any extension of* $f$ *to* $R^n$ *and*

$$\tilde{f}_r(\xi', \lambda) = \Pi'\Pi^+\tilde{B}_r(\xi)(\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

*Proof.* Set $\tilde{A}(\xi, \lambda) = \tilde{A}(\xi) + \lambda^\alpha$. It is homogeneous of order $\alpha$ in $(\xi, \lambda)$ and belongs to $E_\alpha$. Since $\tilde{A}(\xi)$ is in $E_\alpha$; it has a factorization of the form: $\tilde{A}(\xi) = \tilde{A}_+(\xi)\tilde{A}_-(\xi)$ with $\tilde{A}_+ \in C^+_k$, $\tilde{A}_- \in 0^-_{\alpha-k}$. The factorization is unique up to a constant multiplier. The same proof as in Theorem 1.2 of [4], p. 95 with $\xi_+$ replaced by $\xi^\lambda_+ = \xi_n + i(|\lambda| + |\xi'|)$ and $\xi^\lambda_- = \xi_n - i(|\lambda| + |\xi'|)$ gives:

$$\tilde{A}(\xi, \lambda) = \tilde{A}_+(\xi, \lambda)\tilde{A}_-(\xi, \lambda) .$$

Moreover if $\tilde{A}_+ \in C^+_k$; then: $\tilde{A}_+(\xi, \lambda) \in C^+_k$, (with respect to $\xi, \lambda$). Similarly $\tilde{A}_-(\xi, \lambda) \in 0^-_{\alpha-k}$.

(1) First, we show that $\tilde{u}_+(\xi) \in \widetilde{\mathring{H}}{}^+_0$ so that $\Pi^+\tilde{u}_+(\xi) = \tilde{u}_+(\xi)$ (Cf. [4], p. 93, relation 10.1). $\tilde{u}_+(\xi)$ is analytic in $\operatorname{Im}\xi_n > 0$ for $|\lambda| \neq 0$. It suffices to show that:

$$\int |\tilde{u}_+(\xi', \xi_n + i\tau)|^2 d\xi' d\xi_n \leqq C .$$

$C$ is independent of $\tau > 0$.

(i) We write:

$$\tilde{u}_+(\xi) = \tilde{v}_+(\xi) + \tilde{w}_+(\xi) .$$

We have:

$$\begin{aligned}\int |\tilde{v}_+(\xi', \xi_n + i\tau)|^2 d\xi' d\xi_n &\leqq C\int (|\xi| + |\lambda| + \tau)^{-2k} |\tilde{g}(\xi', \xi_n + i\tau, \lambda)|^2 d\xi' d\xi_n \\ &\leqq C\int |\tilde{g}(\xi', \xi_n + i\tau, \lambda)|^2 d\xi' d\xi_n\end{aligned}$$

where:

$$\tilde{g}(\xi, \lambda) = \Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

But $\widetilde{lf}(\tilde{A}_-)^{-1}$ is in $\tilde{H}_0$, so $\Pi^+\widetilde{lf}(\tilde{A}_-)^{-1} = \tilde{g}$ is in $\widetilde{\mathring{H}}{}^+_0$, hence $\tilde{v}_+ \in \widetilde{\mathring{H}}{}^+_0$.

(ii) Since $\tilde{A}_+(\xi, \lambda) \in C^+_k$, $(\tilde{A}_+(\xi, \lambda))^{-1} \in C^+_{-k} \subset D_{-k}$ (Lemma 2.4 of [4]). So:

$$\tilde{D}_r(\xi, \lambda) \in D_{-1-\alpha_r} .$$

We have:

$$(\xi_-^\lambda)^M \tilde{D}_r(\xi, \lambda) = \tilde{P}_r(\xi, \lambda) + \tilde{R}_r(\xi, \lambda)$$

with:

$$\tilde{P}_r \in 0^-_{-1-\alpha_r+M} \quad \text{and} \quad |\tilde{R}_r| \leqq C(|\xi'| + |\lambda|)^{M-\alpha_r}(|\xi| + |\lambda|)^{-1} .$$

Therefore:

$$\tilde{P}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r)$$

is in $\tilde{\mathring{H}}_0^-$, and:

$$\Pi^+\tilde{P}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r) = 0 .$$

It remains to show that:

$$\tilde{R}_r(\xi_-^\lambda)^{-M}(\tilde{g}_r - \tilde{f}_r) \in \tilde{H}_0 .$$

We take $M$ large enough and the proof is trivial.

So:

$$\Pi^+\tilde{R}_r(\xi, \lambda)(\xi_-^\lambda)^{-M}(\tilde{g}_r(\xi') - \tilde{f}_r(\xi')) \in \tilde{\mathring{H}}_0^+ .$$

Therefore:

$$\tilde{w}_+ \in \tilde{\mathring{H}}_0^+ .$$

(2) Consider:

$$\| u_+ \|_s^+ = \| \Pi^+(\xi_- - i)^s \tilde{u}_+(\xi) \|_0 = \| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{u}_+(\xi) \|_0 .$$

It is majorized by:

$$\| \Pi^+(\xi_- - i)^s \Pi^+\{(\tilde{A}_+)^{-1}\Pi^+\widetilde{lf}(\tilde{A}_-)^{-1}\} \|_0 + \sum_{r=1}^{k} \| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{D}_r(\tilde{g}_r - \tilde{f}_r) \|_0 .$$

(i) Consider the first expression. It follows from [4] (footnote of p. 113) that the expression is equal to:

$$\| \Pi^+(\xi_- - i)^s (\tilde{A}_+)^{-1}\Pi^+\widetilde{lf}(\tilde{A}_-)^{-1} \|_0$$

which is majorized by:

$$C \| (\tilde{A}_+(\xi, \lambda))^{-1}\Pi^+\widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_s .$$

Since $\tilde{A}_+(\xi, \lambda)$ is in $0_k^+$, we may write:

$$\tilde{A}_+(\xi, \lambda) = (|\xi| + |\lambda|)^k \tilde{A}_+(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)) .$$

Let $c = \text{Min} |\tilde{A}_+(\xi, \lambda)|$ for $|\xi| + |\lambda| = 1$. Then $c > 0$ and is independent of $\xi, \lambda$. We obtain:

$$\| (\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_s \leqq c^{-1} \| \Pi^+ \widetilde{lf}(\tilde{A}_-)^{-1} \|_{s-k}$$

which is majorized by: $c^{-1} \| \widetilde{lf}(\tilde{A}_-)^{-1} \|_{s-k}$ (Cf. Remark 2 of [4], p. 105). We also have:

$$\| (\tilde{A}_+)^{-1} \Pi^+ \widetilde{lf}(\tilde{A}_-)^{-1} \|_0 \leqq C |\lambda|^{-k} \| \widetilde{lf}(\tilde{A}_-)^{-1} \|_0 .$$

Since:

$$\tilde{A}_-(\xi, \lambda) \in 0^-_{\alpha-k} .$$

We have:

$$\tilde{A}_-(\xi, \lambda) = (|\xi| + |\lambda|)^{-k} \tilde{A}_-(\xi/(|\xi| + |\lambda|), \lambda/(|\xi| + |\lambda|)) .$$

So as before, we get:

$$\| \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \leqq C \| \widetilde{lf}(\xi) \|_{s-\alpha} \leqq C \| f \|^+_{s-\alpha}$$

and:

$$|\lambda|^{-k} \| \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_0 \leqq C |\lambda|^{-\alpha} \| f \|^+_0 .$$

Therefore:

$$\| \Pi^+(\xi_- - i)^s \Pi^+\{(\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\} \|_s + |\lambda|^s \| \Pi^+\{(\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1}\} \|_0$$

is majorized by:

$$C\{\| f \|^+_{s-\alpha} + |\lambda|^{s-\alpha} \| f \|^+_0\} .$$

(ii) Consider:

$$\| \Pi^+(\xi_- - i)^s \tilde{D}_r(\xi, \lambda) \tilde{f}_r(\xi', \lambda) \|_0 + |\lambda|^s \| \Pi^+ \tilde{D}_r(\xi, \lambda) \tilde{f}_r \|_0 .$$

From this first part, we know that $\tilde{D}_r(\xi, \lambda) \in D_{-1-\alpha_r}$. Let $M$ be a large positive integer. We have from the definition of $D_{-1-\alpha_r}$:

$$(\xi^\lambda_-)^M \tilde{D}_r(\xi, \lambda) = \tilde{P}_r(\xi, \lambda) + R_r(\xi, \lambda)$$

with:

$$\tilde{P}_r(\xi, \lambda) \in 0^-_{-1-\alpha_r+M} ; \qquad |R_r(\xi, \lambda)| \leqq C(|\xi'| + |\lambda|)^{M-\alpha_r}(|\xi| + |\lambda|)^{-1} .$$

We can show easily that: $(\xi^\lambda_-)^{-M} \tilde{P}_r(\xi, \lambda) \in \widetilde{\overset{\circ}{H}{}^-_0}$, so: $\Pi^+(\xi^\lambda_-)^{-M} \tilde{P}_r = 0$. From [4] (footnote of p. 113), we get:

$$\Pi^+(\xi_- - i)^s \Pi^+(\xi^\lambda_-)^{-M} \tilde{P}_r(\xi, \lambda) = 0 .$$

Hence:

$$\begin{aligned}\| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{D}_r(\xi, \lambda)\tilde{f}_r(\xi', \lambda) \|_0 &= \| \Pi^+(\xi_- - i)^s \Pi^+(\xi_-^\lambda)^{-M} R_r \tilde{f}_r \|_0 \\ &= \| \Pi^+(\xi_- - i)^s(\xi_-^\lambda)^{-M} R_r \tilde{f}_r \|_0 \\ &\leqq C \| (\xi_- - i)^s(\xi_-^\lambda)^{-M} R_r \tilde{f}_r \|_0 .\end{aligned}$$

Consider:

$$\begin{aligned}&\int | (\xi_- - i) |^{2s} | \xi_-^\lambda |^{-2M} | R_r(\xi, \lambda)\tilde{f}_r(\xi', \lambda) |^2 d\xi_n d\xi' \\ &\quad \leqq C\int (| \xi' | + | \lambda |)^{2M-2\alpha_r}(| \xi | + | \lambda |)^{2s-2M-2} | \tilde{f}_r(\xi', \lambda) |^2 d\xi_n d\xi' \\ &\quad \leqq C\int (| \xi' | + | \lambda |)^{2s-2\alpha_r-1} | \tilde{f}_r(\xi', \lambda) |^2 d\xi'\end{aligned}$$

for $M$ sufficiently large.
So:

$$\| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{D}_r \tilde{f}_r \|_0 \leqq C\{\| \tilde{f}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \tilde{f}_r \|'_0\}$$

and:

$$\| \Pi^+ \tilde{D}_r \tilde{f}_r \|_0 \leqq C | \lambda |^{-\alpha_r-(1/2)} \| \tilde{f}_r \|'_0 .$$

(iii) Similarly, we have:

$$\begin{aligned}&\| \Pi^+(\xi_- - i)^s \Pi^+ \tilde{D}_r \tilde{g}_r \|_s + | \lambda |^s \| \Pi^+ \tilde{D}_r \tilde{g}_r \|_0 \\ &\quad \leqq C\{\| \tilde{g}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \tilde{g}_r \|'_0\} .\end{aligned}$$

(iv) Since $s, \alpha, \alpha_r$ are positive integers, we have from [3] (relation 1.14, p. 63):

$$\| \tilde{f}_r \|'_{s-\alpha_r-(1/2)} + | \lambda |^{s-\alpha_r-(1/2)} \| \tilde{f}_r \|'_0 \leqq M\{\| \tilde{f}_r \|_{s-\alpha_r} + | \lambda |^{s-\alpha_r} \| \tilde{f}_r \|_0\}$$

with

$$\tilde{f}_r = \Pi^+ \tilde{B}_r(\xi)(\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} .$$

Since $\tilde{B}_r(\xi)$ is homogeneous of order $\alpha_r$ in $\xi$ with $\alpha_r \geqq 0$; we get:

$$\| \tilde{f}_r \|_{s-\alpha_r} \leqq C \| (\tilde{A}_+(\xi, \lambda))^{-1} \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_s .$$

Again as before, we write:

$$\tilde{A}_+(\xi, \lambda) = (| \xi | + | \lambda |)^k \tilde{A}_+(\xi/(| \xi | + | \lambda |, \lambda/(| \xi | + | \lambda |)) .$$

So:

$$\begin{aligned}\| f_r \|_{s-\alpha_r} &\leqq C \| \Pi^+ \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \\ &\leqq C \| \widetilde{lf}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \|_{s-k} \\ &\leqq C \| f \|^+_{s-\alpha} .\end{aligned}$$

Similarly, we obtain:

$$\|\tilde{f}_r\|_0 \leqq C|\lambda|^{-\alpha-\alpha_r}\|f\|_0^+ .$$

Combining all the results, we get the *a priori* estimate

(3) A direct verification shows that $u_+$ is a solution of the problem. It remains to show that the solution is unique.

Let $v_+$ be a solution of the problem with $v_+ \in H_s^+$. Then $\tilde{v}_+(\xi)$, its Fourier transform has the same form as $\tilde{u}_+(\xi)$ with $\widetilde{lf}(\xi)$ replaced by $\widetilde{l_1 f}(\xi)$. $l_1 f$ is an extension of $f$ to $R^n$.

Set $l_2 f = lf - l_1 f$. Then $l_2 f \in \overset{\circ}{H}{}_0^-$, so $l_2 f(\xi) \in \widetilde{\overset{\circ}{H}{}_0^-}$.

Now a verification as in the first part shows that:

$$\widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} \in \widetilde{\overset{\circ}{H}{}_0^-} ,$$

hence:

$$\Pi^+ \widetilde{l_2 f}(\xi)(\tilde{A}_-(\xi, \lambda))^{-1} = 0 .$$

Taking into account the ellipticity of $\tilde{A}(\xi, \lambda)$, we get: $\tilde{u}_+(\xi) = \tilde{v}_+(\xi)$.

Let:

$$A_0 u_+ = \sum_k \psi_0(x_0) \exp(ikx_0\pi)/pL_k u$$

$$A_1 u_+ = \sum_k \psi_0(x) \exp(ikx\pi)/pL_k u_+$$

where $\psi_0(x)$, $L_k$ are as in § 1.

We have the Lemma:

LEMMA 3.2. *Let $\psi(x)$ be in $C_c^\infty(R^n)$, $\psi(x) = 0$ outside of $|x - x_0| \leqq \delta$ $|\psi(x)| \leqq K$ where $K$ is independent of $\delta$. Suppose that $\tilde{A}_1(x, \xi)$ is in $D_\alpha^1$. Then:*

$$\|\psi(A_1 - A_0)u_+\|_{s-\alpha}^+ \leqq C\delta \|u_+\|_{s-\alpha}^+ + C(\delta)\|u_+\|_{s-1-\alpha}^+$$

*$C(\delta) = 0$ if $s = \alpha$.*

*Proof.* Cf. Lemma 4.7 of [4] (p. 119).

*Proof of Theorem* 2.2 (*continued*).

(1) First, we establish the *a priori* estimate.

Let $N_j$ be a finite open covering of cl $G$ with diam $(N_j)$ sufficiently small; $\varphi_j$ be a finite partition of unity corresponding to $N_j$ and $\psi_j$ be the infinitely differentiable functions with compact supports in $N_j$ and such that: $\varphi_j \psi_j = \varphi_j$.

Let: $F = (f, g_1, \cdots, g_k)$ be an element of $H^{s-\alpha}(G, \partial G)$; $s \geqq \alpha$.

By definition, we have:

$$U(\lambda)u_+ = \sum_j P^+\varphi_j U(\lambda)(\psi_j u_+) + Tu_+ = F\,.$$

We express $\varphi_j U(\lambda)\psi_j$ in local coordinates. From Appendix 2 of [4], we get:

$$\varphi_j U(\lambda)\psi_j u_+ = \sum \varphi_j U_j(\lambda)(\psi_j u_+) + T_j u_+$$

where $T_j$ is a smoothing operator with respect to $U_j(\lambda)$.

So:

$$\varphi_j U_j^0(\lambda)(\psi_j u_+) = \varphi_j F + T_j u_+ + \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j u_+)\,.$$

$U_j^0(\lambda)$ corresponds to the case when $A_j$, $B_{rj}$ have constant symbols. From Lemma 4.D.1 of [4] (p. 145), we have:

$$\varphi_j U_j^0(\lambda)(\psi_j u_+) = U_j^0(\lambda)(\varphi_j\psi_j u_+) + T_j^2 u_+$$

where $T_j^2$ is again a smoothing operator.

Hence:

$$U_j^0(\lambda)(\varphi_j u_+) = \varphi_j F + \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j u_+) + T_j^2 u_+\,.$$

Applying Lemma 3.1, we obtain:

$$\begin{aligned}
\|\varphi_j u_+\|_s^+ + |\lambda|^s \|\varphi_j u_+\|_0^+ \leqq M\Big\{&\|\varphi_j f\|_{s-\alpha}^+ + |\lambda|^{s-\alpha}\|\varphi_j f\|_0^+ \\
&+ \|\varphi_j(A_j - A_{j0})(\psi_j u_+)\| + |\lambda|^{s-\alpha}\|\varphi_j(A_j - A_{j0})(\psi_j u_+)\|_0^+ \\
&+ \|u_+\|_{s-1} + |\lambda|^{s-\alpha}\|u_+\|_{\alpha-1} + \sum_{r=1}^k \|\varphi_j g_r\|'_{s-\alpha_r-(1/2)} \\
&+ |\lambda|^{s-\alpha_r-(1/2)}\|\varphi_j g_r\|'_0 + \|\gamma P^+\varphi_j(B_{rj} - B_{rj0})(\psi_j u_+)\|'_{s-\alpha^r-(1/2)} \\
&+ |\lambda|^{s-\alpha_r-(1/2)}\|\gamma P^+\varphi_j(B_{rj} - B_{rj0})(\psi_j u_+)\|'_0\Big\}\,.
\end{aligned}$$

Using Lemma 3.2, we get:

$$\begin{aligned}
\|\varphi_j u_+\|_s^+ + |\lambda|^s\|\varphi_j u_+\|_0^+ \leqq M\Big\{&\|u_+\|_{s-1} + |\lambda|^{s-\alpha}\|u_+\|_{\alpha-1} + \|\varphi_j f\|_{s-\alpha}^+ \\
&+ |\lambda|^{s-\alpha}\|\varphi_j f\|_0^+ + \delta\|\varphi_j u_+\|_{s-\alpha}^+ + \delta|\lambda|^{s-\alpha}\|\varphi_j u_+\|_0^+ \\
&+ \sum_{r=1}^k \|\varphi_j g_r\|'_{s-\alpha_r-(1/2)} + |\lambda|^{s-(1/2)-\alpha_r}\|\varphi_j g_r\|'_0\Big\}
\end{aligned}$$

(by using an inequality in [3] p. 63).

Summing with respect to $j$, we have:

$$\begin{aligned}
\|u_+\|_s + |\lambda|^s\|u_+\|_0 \leqq M\Big\{&\|u_+\|_{s-1} + |\lambda|^{s-\alpha}\|u_+\|_{\alpha-1} + \|f\|_{s-\alpha} \\
&+ |\lambda|^{s-\alpha}\|f\|_0 + \delta\|u_+\|_{s-\alpha} + \delta|\lambda|^{s-\alpha}\|u_+\|_0 \\
&+ \sum_{r=1}^k \|g_r\|'_{s-(1/2)-\alpha_r} + |\lambda|^{s-(1/2)-\alpha_r}\|g_r\|'_0\Big\}\,.
\end{aligned}$$

Taking $\delta$ small and $|\lambda|$ large, we obtain by taking into account an interpolation inequality of Visik-Agranovich [3] (p. 64, relation 1.21):

$$\|u_+\|_s + |\lambda|^s \|u_+\|_0 \leqq M\Big\{\|f\|_{r-\alpha} + |\lambda|^{s-\alpha}\|f\|_0 + \sum_{r=1}^{k} \|g_r\|'_{s-(1/2)-\alpha_r} + |\lambda|^{s-\alpha_r-(1/2)}\|g_n\|'_0\Big\} .$$

(2) It follows from the *a priori* estimate that if there exists a solution, then it is unique.

It remains to show the existence of a solution.

We know from Lemma 3.1 that $U_j^0(\lambda)$ has a right inverse $R_j$ Let $\hat{R}_j$ be the operator $R_j$ expressed in the global coordinates system of $G$.

Set:

$$RF = \sum_j P^+\varphi_j\hat{R}_j(\psi_j F) .$$

We have:

$$U(\lambda)RF = \sum_j U(\lambda)\varphi_j\hat{R}_j(\psi_j F) = \sum_j U(\lambda)\varphi_j\psi_j\hat{R}_j(\psi_j F) .$$

Passing into local coordinates (using Appendix 2 of [4]) and applying Lemma 4.D.1 of [4], we obtain:

$$\begin{aligned} U(\lambda)\varphi_j\psi_j R_j(\psi_j F) &= \varphi_j U_j(\lambda)(\psi_j R_j(\psi_j F)) + T_j^2 F \\ &= \varphi_j U_j^0(\lambda)(\psi_j R_j(\psi_j F)) + T_j^2 F \\ &\quad + \varphi_j(U_j(\lambda) - U_j^0(\lambda))(\psi_j R_j(\psi_j F)) \end{aligned}$$

where $T_j^2$ is a smoothing operator.

Applying again Lemma 4.D.1 of [4], we have:

$$\begin{aligned} \varphi_j U_j^0(\lambda)\psi_j R_j(\psi_j F) &= \varphi_j\psi_j U_j^0(\lambda)R_j(\psi_j F) + T_j^2 RF \\ &= \varphi_j\psi_j F + T_j^2 RF . \end{aligned}$$

Therefore:

$$U(\lambda)RF = F + T'RF + \sum_j \varphi_j\hat{T}'_j F$$

where $T'$ is a smoothing operator with respect to $U(\lambda)$; i.e. with respect to a bounded linear mapping from $H_+^s(G)$ into $H^{s-\alpha}(G)$; and $\hat{T}'_j$ is the operator $T'_j$ defined by:

$$T'_j F = (U_j^0(\lambda) - U_j(\lambda))(\psi_j R_j(\psi_j F))$$

expressed in the global coordinates system of $G$.

So: $U(\lambda)RF = (I + \mathscr{C}R)F$.

Denote by:

$$|||\cdot|||_{s-\alpha} = ||\cdot||_s + |\lambda|^{s-\alpha} ||\cdot||_0$$
$$|||\cdot|||'_{s-\alpha} = ||\cdot||'_{s-\alpha-1/2} + |\lambda|^{s-\alpha-1/2} ||\cdot||'_0$$
$$|||\cdot|||_{H^{s-\alpha}(G,\partial G)} = |||\cdot|||_{s-\alpha} + |||\cdot|||'_{s-\alpha} .$$

Since $T'$ is a smoothing operator, we get by taking into account the first part of the proof:

$$|||\, T'RF \,|||_{H^{s-\alpha}(G,\partial G)} \leqq C \,|||\, F \,||_{H^{s-1-\alpha}(G,\partial G)} .$$

Using Lemma 3.2, we obtain:

$$|||\, \varphi_j(U_j^0(\lambda) - U_j(\lambda))(\psi_j \hat{R}_j(\psi_j F)) \,|||_{H^{s-\alpha}(G,\partial G)} \leqq |||\, F \,|||_{H^{s-1-\alpha}(G,\partial G)} + C(\delta)/\lambda \,|||\, F \,|||_{H^{s-\alpha}(G,\partial G)} .$$

So for small $\delta$, large $|\lambda|$, by using an interpolation inequality of [3], we have:

$$|||\, \mathscr{C}RF \,||_{H^{s-\alpha}(G,\partial G)} < 1/2 \,|||\, F \,|||_{H^{s-\alpha}(G,\partial G)} .$$

Hence: $(I + \mathscr{C}R)^{-1}$ exists and $U(\lambda)^{-1} = R(I + \mathscr{C}R)^{-1}$.

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