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NÖRLUND SUMMABILITY OF FOURIER SERIES

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The Nörlund summability was first applied to the theory of Fourier series by E. Hille and J. D. Tamarkin. Many other mathematicians have since worked in this field. Recently T. Singh has proved a nice theorem concerning the Nörlund summability of Fourier series. In Part I, we shall give a generalization.

Absolute Nörlund summability was defined by L. McFadden and he proved a theorem concerning the absolute Nörlund summability of the Fourier series of functions of the Lipschitz class which was generalized by S. N. Lal. We shall give another generalization of McFadden's theorem in Part II.

PART I.

1. Let $\sum_{n=0}^{\infty} a_n$ be a given series and (s_n) be the sequence of its partial sums. Let (p_n) be a sequence of real numbers and $P_n = p_0 + p_1 + \cdots + p_n$. We suppose that $P_n \neq 0$ for all n. The series $\sum a_n$ is called to be summable (N, p_n) to s when $\lim_{n\to\infty} t_n$ exists and is equal to s, where

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_{n-k} \ .$$

t_n is called the nth (N, p_n) mean or nth Nörlund mean.

In the special case in which $p_n = \binom{n+\alpha-1}{n} = A_n^{\alpha-1} (\alpha > 0)$, the Nörlund mean reduces to the (C, α) mean. Another special case that $p_n = 1/(n+1)$, is called the Harmonic mean.

The condition for the regularity of summability (N, p_n), is

$$p_n/P_n \to 0 \quad \text{and} \quad \sum_{k=0}^{n} |p_k| = O(|P_n|) \qquad \text{as } n \to \infty \ .$$

If (p_n) is a positive sequence, then the second condition is satisfied. It is also easy to see that, if (p_n) is an increasing sequence, then a $(C, 1)$ summable sequence is summable (N, p_n).

We shall define $p(t)$ on the interval $(0, \infty)$ such that $p(n) = p_n$ for $n = 0, 1, 2, \cdots$ and that $p(t)$ is continuous on $(0, \infty)$ and is linear in each interval $(k, k+1) (k = 0, 1, 2, \cdots)$. We put $P(t) = \int_0^t p(u)du$, then, $P(n) = (1/2)p_0 + p_1 + \cdots + p_{n-1} + (1/2)p_n \cong P_n$ as $n \to \infty$ when $P_n \to \infty$ and $p_n/P_n \to 0$.

T. Singh [4] has proved the following theorems:

THEOREM S1. *If* (p_n) *is a positive sequence such that* $p_n \downarrow$ *and* $P_n \to \infty$ *and further if*

$$\Phi(t) = \int_0^t |\varphi_x(u)| \, du = o(p(1/t)/P(1/t)) \qquad \text{as } t \to 0 , \tag{1}$$

then the Fourier series of f *is summable* (N, p_n) *to* $f(x)$ *at the point* x, *where* $\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)$.

THEOREM S2. *If* (p_n) *satisfies the conditions of Theorem* S1 *and*

$$\Psi(t) = \int_0^t |\psi_x(u)| \, du = o(p(1/t)/P(1/t)) \qquad \text{as } t \to 0 , \tag{2}$$

then the conjugate Fourier series of f *is summable* (N, p_n) *to*

$$-\frac{1}{\pi}\int_0^\pi \frac{\psi_x(u)}{2\tan u/2} du$$

at the point x *when the last integral exists, where*

$$\psi_x(u) = f(x+u) - f(x-u) .$$

We shall prove that we can replace the condition $p_n \downarrow$ by the more general one

$$\int_1^n u \, |p'(u)| \, du = O(P_n) \qquad \text{as } n \to \infty . \tag{3}$$

If (p_n) is monotone, then the condition (3) is equivalent to

$$np_n = O(P_n) , \tag{4}$$

since

$$\int_1^n up'(u)du = [up(u)]_1^n - \int_1^n p(u)du = np(n) - P_n + O(1) .$$

If (p_n) is decreasing, then (4) is satisfied automatically. In general, condition (3) implies (4).

If the condition (4) is satisfied, then (1) implies

$$\Phi(t) = o(t) \qquad \text{as } t \to 0 . \tag{5}$$

2. Our first theorem is as follows.

THEOREM 1. *If* (p_n) *is a positive sequence such that* $P_n \to \infty$, (3) *holds, and condition* (1) *is satisfied, then the Fourier series of* f *is summable* (N, p_n) *to* $f(x)$ *at the point* x.

Proof. We write $\varphi_x(u) = \varphi(u)$ and by $t_n(x)$ we denote the nth Nörlund mean of the Fourier series of f at the point x, then

$$\begin{aligned} t_n(x) - f(x) &= \frac{1}{2\pi P_n}\int_0^\pi \frac{\varphi(t)}{\sin t/2}\Big(\sum_{k=0}^n p_{n-k}\sin(k+1/2)t\Big)dt \\ &= \frac{1}{2\pi P_n}\int_0^\pi \frac{\varphi(t)}{2\sin t/2}L_n(t)dt \\ &= \frac{1}{2\pi P_n}\Big(\int_0^{1/n} + \int_{1/n}^\pi\Big) = \frac{1}{2\pi}(I+J)\ . \end{aligned}$$

By (5),

$$\begin{aligned} |I| &\leqq \frac{A}{P_n}\int_0^{1/n}\frac{|\varphi(t)|}{t}\Big(\sum_{k=0}^n p_{n-k}\cdot kt\Big)dt \\ &\leqq An\int_0^{1/n}|\varphi(t)|\,dt = o(1)\ . \end{aligned}$$

We write

$$\begin{aligned} |L_n(t)| &= \Big|\sum_{k=0}^n p_k \sin(n-k+1/2)t\Big| \\ &\leqq \sum_{k=0}^{[1/t]} p_k + \Big|\sum_{k=[1/t]}^n p_k\sin(n-k+1/2)t\Big| \\ &= L_n'(t) + L_n''(t)\ , \end{aligned}$$

then we have, by Abel's lemma,

$$|L_n''(t)| \leqq A\Big\{\frac{p([1/t])}{t} + \frac{p(n)}{t} + \frac{1}{t}\int_{1/t}^n |p'(u)|\,du\ .$$

Therefore

$$\begin{aligned} |J| &= \Big|\frac{1}{P_n}\int_{1/n}^\pi \frac{\varphi(t)}{\sin t/2}L_n(t)dt\Big| \\ &\leqq \frac{A}{P_n}\Big\{\int_{1/n}^\pi \frac{|\varphi(t)|}{t}L_n'(t)dt + \int_{1/n}^\pi\frac{|\varphi(t)|}{t}L_n''(t)dt\Big\} \\ &= \frac{A}{P_n}(J_1 + J_2)\ , \end{aligned}$$

where

$$\begin{aligned} |J_1| &\leqq \int_{1/n}^\pi \frac{|\varphi(t)|}{t}P(1/t)dt \leqq \Big[\frac{\Phi(t)}{t}P(1/t)\Big]_{1/n}^\pi \\ &\quad + \int_{1/n}^\pi \frac{\Phi(t)}{t^2}P(1/t)dt + \int_{1/n}^\pi \frac{\Phi(t)}{t}\,\frac{p(1/t)}{t^2}dt \\ &= o(P_n) + o(1)\int_{1/n}^\pi \frac{p(1/t)}{t^2}dt + o(1)\int_{1/n}^\pi \frac{p(1/t)}{t^2}dt \\ &= o(P_n) \end{aligned}$$

and

$$
\begin{aligned}
J_2 &\leqq A\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}\frac{p([1/t])}{t}dt + \int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}\frac{p(n)}{t}dt\right. \\
&\quad \left. + \int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^2}dt\int_{1/t}^{n}|p'(u)|\,du\right\} \\
&\leqq A\left\{\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}P(1/t)dt + p_n\left[\frac{\Phi(t)}{t^2}\right]_{1/n}^{\pi}\right\} \\
&\quad + p_n\int_{1/n}^{\pi}\frac{\Phi(t)}{t^3}dt + \left[\frac{\Phi(t)}{t^2}\int_{1/t}^{n}|p'(u)|\,du\right]_{1/n}^{\pi} \\
&\quad + \int_{1/n}^{\pi}\frac{\Phi(t)}{t^3}dt\int_{1/t}^{n}|p'(u)|\,du + \int_{1/n}^{\pi}\frac{\Phi(t)}{t^2}\frac{|p'(1/t)|}{t^2}dt \\
&\leqq AJ_1 + o(np_n) + o(np_n) + O\left(\int_1^n |p'(u)|\,du\right) \\
&\quad + o\left(\int_1^n u\,|p'(u)|\,du\right) + o\left(\int_1^n u\,|p'(u)|\,du\right) \\
&= o(P_n)\,.
\end{aligned}
$$

Thus we get $J = o(1)$ and then we have proved the theorem.

3. THEOREM 2. *If* (p_n) *is a positive sequence satisfying the conditions in Theorem* 1 *and*

$$
\Psi(t) = \int_0^t |\psi_x(u)|\,du = o(p(1/t)/P(1/t)) \qquad \textit{as } t \to 0\,,
$$

then the conjugate Fourier series of f *is summable* (N, p_n) *to*

$$
-\frac{1}{\pi}\lim_{n\to\infty}\int_{1/n}^{\pi}\frac{\psi_x(t)}{2\tan t/2}dt
$$

when the last limit exists.

Proof. Let $\psi_x(t) = \psi(t)$ and $\tilde{t}_n(x)$ be the nth (N, p_n) mean of the conjugate Fourier series of f, then

$$
\begin{aligned}
&\tilde{t}_n(x) - \left(-\frac{1}{\pi}\int_{1/n}^{\pi}\frac{\psi_x(t)}{2\tan t/2}dt\right) \\
&\quad = \frac{-1}{2\pi P_n}\int_0^{1/n}\frac{\psi_x(t)}{\sin t/2}\left(\sum_{k=0}^{n}\left(\cos\frac{1}{2}t - \cos\left(k+\frac{1}{2}\right)t\right)\right)dt \\
&\qquad + \frac{1}{2\pi P_n}\int_{1/n}^{\pi}\frac{\psi(t)}{\sin t/2}\left(\sum_{k=0}^{n}p_{n-k}\cos\left(k+\frac{1}{2}\right)t\right)dt\,.
\end{aligned}
$$

Applying the method of proof of Theorem 1 to above integrals, we obtain the theorem.

PART II.

4. Let (t_n) be the sequence of Nörlund means of the series $\sum a_n$. If $\sum |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is called to be summable $|N, p_n|$ or absolutely summable (N, p_n).

L. McFadden [6] proved the following theorem.

THEOREM M. *Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If $f \in \mathrm{Lip}\,\alpha (0 < \alpha < 1)$ and*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1/2} P_n} < \infty , \tag{6}$$

then the Fourier series of f is $|N, p_n|$ summable.

This was generalized in the following form by S. L. Lal [3]:

THEOREM L. *Let (p_n) be a nonnegative, decreasing and convex sequence tending to zero such that $\sum P_n^2/n^2 < \infty$. If the continuity modulus $\omega(t) = \omega(t; f)$ of f satisfies the conditions*

$$\sum_{n=1}^{\infty} \frac{\omega(1/n^{\delta})}{n} < \infty \ (0 < \delta < 1) \tag{7}$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/2} P_n} < \infty , \tag{8}$$

then the Fourier series of f is $|N, p_n|$ summable.

We shall prove the following theorem:

THEOREM 3. *Let $2 \geqq p > 1$, $1/p + 1/q = 1$ and let (p_n) be a positive, decreasing and convex sequence tending to zero. If*

$$\sum_{n=1}^{\infty} p_n^p n^{p-2} < \infty \tag{9}$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty \tag{10}$$

where $\omega(\delta)$ is the continuity modulus of f, and further if

$$\sum_{m=n}^{\infty} \frac{1}{m^p(\omega(1/m))^{p-1}} \leqq \frac{A}{(n\omega(1/n))^{p-1}}$$

(or more specially, if $u^{-\delta}\omega(u)\downarrow$ as $u\uparrow$ for a positive $\delta < 1$), then the Fourier series of f is $|N, p_n|$ summable.

If (p_n) decreases monotonically then (9) is equivalent to

$$\xi_1(x) = \sum_{n=1}^{\infty} p_n \cos nx \quad \text{and} \quad \xi_2(x) = \sum_{n=1}^{\infty} p_n \sin nx$$

belong to L^p.

From (9) we have

$$\begin{aligned} P_n &= \sum_{k=1}^{n} p_k = \sum_{k=1}^{n} p_k k^{1-2/p} \cdot k^{2/p-1} \\ &\leqq \Big(\sum_{k=1}^{n} p_k^p k^{p-2}\Big)^{1/p} \Big(\sum_{k=1}^{n} k^{(2-p)q/p}\Big)^{1/q} \leqq A n^{1/p} \end{aligned}$$

and then

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} \cdot n^{1/p}} \leqq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/q} P_n} < \infty . \tag{11}$$

Therefore, under the condition (9), the condition (10) is stronger than (11). If p decreases from 2, then (9) becomes weaker but (10) becomes stronger.

In the case $p = 1$, we have the following.

THEOREM 4. *Let (p_n) be a decreasing and convex sequence tending to zero, such that*

$$\sum_{n=1}^{\infty} \frac{p_n}{n} < \infty \tag{12}$$

and

$$\sum_{n=k}^{\infty} \frac{1}{n P_n} \leqq \frac{A}{P_k} . \tag{13}$$

If f has the continuity modulus $\omega(\delta)$ such that

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_n} < \infty , \tag{14}$$

then the Fourier series of f is summable $|N, p_n|$.

It is known ([7], Chap. V, § 1) that if (p_n) is a decreasing and convex sequence tending to zero, then $\xi_1(x)$ is integrable and that if (p_n) is decreasing and satisfies the condition (12), then $\xi_2(x)$ is integrable. We have also

$$P_n = \sum_{k=1}^{n} p_k = \sum_{k=1}^{n} (p_k/k)k \leqq An ,$$

and then we have $\sum \omega(1/n)/n < \infty$, i.e., (11) holds also in this case.

5. For the proof of Theorem 3, we use the following lemmas.

LEMMA 1. ([5]) *If (p_n) is a positive and decreasing sequence, then*

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqq AP(1/t)$$

for any a and $b > a$ and for any integer n.

LEMMA 2. ([6]) *If (p_n) is a positive and decreasing sequence and $\xi(t) = \sum_{k=0}^{\infty} p_k e^{ikt}$, then*

$$|\xi(x + 2t) - \xi(x)| \leqq A\frac{t}{x}P\left(\frac{1}{t}\right) \qquad \textit{for all } x \textit{ in } (t, \pi) .$$

LEMMA 3. ([2] *and* [1]) *If $\lambda(t)$ is a positive increasing function on $(1, \infty)$ and $f \in L^p$ $(1 < p \leqq 2)$, then*

$$\sum_{n=2}^{\infty} \frac{1}{\lambda(2n)} \left(\sum_{m=n}^{\infty} \rho_m^q \right)^{1/q} \leqq A\int_0^1 \frac{dt}{t^2\lambda(1/t)} \left(\int_0^{2\pi} |f(x + t) - f(x - t)|^p \, dx \right)^{1/p}$$

where $\rho_m^2 = a_m^2 + b_m^2$, a_m and b_m being the mth *Fourier coefficients of f and $1/p + 1/q = 1$.*

6. We shall now prove Theorem 3. By the definition, we have

$$\begin{aligned} t_n - t_{n-1} &= \frac{1}{\pi}\int_0^{\pi} \varphi(t) \left\{ \sum_{k=0}^{n} \left(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} \right) D_k(t) \right\} dt \\ &= \frac{1}{\pi}\int_0^{\pi} \varphi(t) \left\{ \sum_{k=1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \cos kt \right\} dt \\ &= \frac{1}{\pi}\int_0^{\pi} \varphi(t) \left\{ \sum_{k=0}^{n-1} \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) \cos (n - k)t \right\} dt \\ &= \frac{1}{\pi} \frac{1}{P_n P_{n-1}} \int_0^{\pi} \varphi(t) \left\{ \sum_{k=0}^{n-1} (p_k P_n - p_n P_k) \cos (n - k)t \right\} dt \end{aligned}$$

where $p_{-1} = P_{-1} = 0$, and then

$$
\begin{aligned}
\pi\,|\,t_n - t_{n-1}\,| \leqq\ & \frac{1}{P_{n-1}}\left|\int_0^{\pi}\varphi(t)\Big(\sum_{k=0}^{\infty} p_k \cos (n-k)t\Big)dt\right| \\
& + \frac{1}{P_{n-1}}\left|\int_0^{1/n}\varphi(t)\Big(\sum_{k=n}^{\infty} p_k \cos (n-k)t\Big)dt\right| \\
& + \frac{p_n}{P_n P_{n-1}}\left|\int_0^{1/n}\varphi(t)\Big(\sum_{k=0}^{n-1} P_k \cos (n-k)t\Big)dt\right| \\
& + \frac{1}{P_{n-1}}\left|\int_{1/n}^{\pi}\varphi(t)\Big(\sum_{k=n}^{\infty} p_k \cos (n-k)t\Big)dt\right. \\
& \qquad \left. + \frac{p_n}{P_n}\int_{1/n}^{\pi}\varphi(t)\Big(\sum_{k=0}^{n-1} P_k \cos (n-k)t\Big)dt\right| \\
=\ & I_n + J_n + K_n + L_n\,,
\end{aligned}
\tag{15}
$$

following McFadden [6]. We shall begin to estimate I_n.

$$
\begin{aligned}
I_n &\leqq \frac{1}{P_{n-1}}\left\{\left|\int_0^{\pi}\varphi(t)\Big(\sum_{k=0}^{\infty} p_k \cos kt\Big)\cos nt\,dt\right|\right. \\
&\qquad \left. + \left|\int_0^{\pi}\varphi(t)\Big(\sum_{k=0}^{\infty} p_k \sin kt\Big)\sin nt\,dt\right|\right\} \\
&= \frac{1}{P_{n-1}}\left\{\left|\int_0^{\pi}\varphi(t)\xi_1(t)\cos nt\,dt\right|\right. \\
&\qquad \left. + \left|\int_0^{\pi}\varphi(t)\xi_2(t)\sin nt\,dt\right|\right\} \\
&= I_{n,1} + I_{n,2}\,.
\end{aligned}
$$

Let

$$
A_n = \int_0^{\pi}\varphi(t)\xi_1(t)\cos nt\,dt \quad \text{and} \quad E_n = \Big(\sum_{m=n}^{\infty} |\,A_m\,|^q\Big)^{1/q},
$$

then

$$
\begin{aligned}
\mathscr{I} &= \sum_{n=2}^{\infty} |\,I_{n,1}\,| = \sum_{n=2}^{\infty}\frac{|\,A_n\,|}{P_{n-1}} = \sum_{j=1}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}\frac{|\,A_n\,|}{P_{n-1}} \\
&\leqq \sum_{j=1}^{\infty}\Big(\sum_{n=2^j}^{2^{j+1}-1}|\,A_n\,|^q\Big)^{1/q}\Big(\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{P_{n-1}^p}\Big)^{1/p} \\
&\leqq \sum_{j=1}^{\infty} E_{2^j}\Big(\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{P_{n-1}^p}\Big)^{1/p} \\
&\leqq 4\sum_{j=1}^{\infty}\sum_{n=2^{j-1}}^{2^j-1}\frac{E_n}{n}\Big(\sum_{m=2n}^{4n-1}\frac{1}{P_{m-1}^p}\Big)^{1/p} \\
&= 4\sum_{n=1}^{\infty}\frac{E_n}{n}\Big(\sum_{m=2n}^{4n-1}\frac{1}{P_{m-1}^p}\Big)^{1/p} \leqq A\sum_{n=1}^{\infty}\frac{E_n}{n^{1/q}P_n}\,.
\end{aligned}
$$

If we put $\lambda(n) = n^{1/q}P_n$, then $\lambda(n) \uparrow \infty$ as n increases. By Lemma 3, we get

$$\begin{aligned}\mathscr{I} &\leqq A\int_0^1 \frac{dt}{t^2\lambda(1/t)}\Big(\int_{-\pi}^{\pi} |\Delta_t(\varphi\cdot\xi_1)|^p\,dx\Big)^{1/p} \\ &= A\int_0^1 \frac{dt}{t^{1+1/p}P(1/t)}\Big(\int_{-\pi}^{\pi} |\Delta_t(\varphi\cdot\xi_1)|^p\,dx\Big)^{1/p}\end{aligned}$$

where

$$\begin{aligned}\Delta_t(\varphi\cdot\xi_1) &= \varphi(x+t)\xi_1(x+t) - \varphi(x-t)\xi_1(x-t) \\ &= \xi_1(x+t)[\varphi(x+t)-\varphi(x-t)] + \varphi(x-t)[\xi_1(x+t)-\xi_1(x-t)] \\ &= \xi_1(x+t)\Delta_t\varphi + \varphi(x-t)\Delta_t\xi_1\,,\end{aligned}$$

and then

$$\begin{aligned}\mathscr{I} &\leqq A\int_0^1 \frac{dt}{t^{1+1/p}P(1/t)}\Big(\int_{-\pi}^{\pi} |\xi_1(x+t)\Delta_t\varphi|^p\,dx\Big)^{1/p} \\ &\quad + A\int_0^1 \frac{dt}{t^{1+1/p}P(1/t)}\Big(\int_{-\pi}^{\pi} |\varphi(x-t)\Delta_t\xi_1|^p\,dx\Big)^{1/p} \\ &= \mathscr{I}' + \mathscr{I}''\,.\end{aligned}$$

By (9) and (10), we have

$$\begin{aligned}\mathscr{I}' &\leqq A\int_0^1 \frac{\omega(t)dt}{t^{1+1/p}P(1/t)} \leqq A\int_1^\infty \frac{\omega(1/u)}{u^{1/q}P(u)}du \\ &\leqq A\sum_{n=1}^\infty \frac{\omega(1/n)}{n^{1/q}P_n} \leqq A\end{aligned}$$

and, by Lemma 2 and (11),

$$\begin{aligned}\mathscr{I}'' &\leqq A + \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)}\Big(\int_t^{\pi}\omega(x)^p\,|\xi_1(x+2t)-\xi_1(x)|^p\,dx\Big)^{1/p} \\ &\leqq A + \int_0^1 \frac{dt}{t^{1+1/p}P(1/t)}\Big(\int_t^{\pi}\omega(x)^p\Big(\frac{t}{x}P\Big(\frac{1}{t}\Big)\Big)^p dx\Big)^{1/p} \\ &\leqq A + \int_0^1 \frac{dt}{t^{1/p}}\Big(\int_t^{\pi}\frac{\omega(x)^p}{x^p}dx\Big)^{1/p} \\ &\leqq A + \int_1^\infty \frac{\omega(1/u)^{1/q}}{u^{1/q}}\cdot\frac{1}{u\omega(1/u)^{1/q}}\Big(\int_1^u \frac{\omega(1/v)^p}{v^{2-p}}dv\Big)^{1/p}du \\ &\leqq A + \Big(\int_1^\infty \frac{\omega(1/u)}{u}du\Big)^{1/q}\Big(\int_1^\infty \frac{du}{u^p\omega(1/u)^{p/q}}\int_1^u \frac{\omega(1/v)^p}{v^{2-p}}dv\Big)^{1/p} \\ &\leqq A + A\Big(\int_1^\infty \frac{\omega(1/v)^p}{v^{2-p}}dv\int_v^\infty \frac{du}{u^p\omega(1/u)^{p-1}}\Big)^{1/p} \\ &\leqq A + A\Big(\int_1^\infty \frac{\omega(1/v)}{v}dv\Big)^{1/p} \leqq A\,.\end{aligned}$$

Thus we have proved that $\mathscr{I} = \sum_{n=2}^\infty |I_{n,1}| < \infty$. Similarly we can prove that $\sum_{n=1}^{\infty}|I_{n,2}|<\infty$, and therefore $\sum_{n=1}^{\infty}I_n<\infty$.

Secondly, we shall estimate the sum $\sum_{n=1}^{\infty}J_n$. Since $\sum_{k=n}^{\infty}p_k\cos(k-n)x$ is a nonnegative function, we have

$$\begin{aligned}\sum_{n=1}^{\infty}J_n &\leqq \sum_{n=1}^{\infty}\frac{\omega(1/n)}{P_{n-1}}\int_0^{1/n}\Big(\sum_{k=n}^{\infty}p_k\cos(k-n)t\Big)dt\\ &\leqq \sum_{n=1}^{\infty}\frac{\omega(1/n)}{P_{n-1}}\Big(\frac{p_n}{n}+\sum_{k=n+1}^{\infty}\frac{p_n}{k-n}\sin\frac{k-n}{n}\Big)\\ &\leqq A+\sum_{n=1}^{\infty}\frac{\omega(1/n)}{P_{n-1}}p_n\leqq\sum_{n=1}^{\infty}\frac{\omega(1/n)}{n}+A\leqq A\,,\end{aligned}$$

since $\sum_{k=1}^{n}(\sin kx/k)$ is uniformly bounded.

Thirdly,

$$\begin{aligned}\sum_{n=1}^{\infty}K_n &\leqq \sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\frac{\omega(1/n)}{n}\sum_{k=0}^{n-1}P_k\\ &\leqq A\sum_{k=1}^{\infty}P_k\sum_{n=k+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\frac{\omega(1/n)}{n}\\ &\leqq A\sum_{k=1}^{\infty}P_k\frac{\omega(1/k)}{k}\sum_{n=k+1}^{\infty}\Big(\frac{1}{P_{n-1}}-\frac{1}{P_n}\Big)\\ &\leqq A\sum_{k=1}^{\infty}\frac{\omega(1/k)}{k}\leqq A\,.\end{aligned}$$

Finally,

$$\begin{aligned}&\sum_{k=n}^{\infty}p_k\cos(k-n)t+\frac{p_n}{P_n}\sum_{k=0}^{n-1}P_k\cos(n-k)t\\ &\quad=\Big\{\frac{p_n}{2}+\sum_{k=n}^{\infty}(p_k-p_{k+1})\frac{\sin(n-k+1/2)t}{2\sin t/2}\Big\}\\ &\qquad+\Big\{\sum_{k=0}^{n-1}p_k\frac{\sin(n-k+1/2)t}{2\sin t/2}-\frac{1}{2}P_{n-1}\Big\}\end{aligned}$$

and then

$$\begin{aligned}L_n &\leqq \frac{1}{P_{n-1}}\Big|\int_{1/n}^{\pi}\frac{\varphi(t)}{2\sin t/2}\Big(\sum_{k=n}^{\infty}(p_k-p_{k+1})\sin(n-k+1/2)t\Big)dt\Big|\\ &\quad+\frac{p_n}{P_nP_{n-1}}\Big|\int_{1/n}^{\pi}\frac{\varphi(t)}{2\sin t/2}\Big(\sum_{k=0}^{n-1}p_k\sin(n-k+1/2)t\Big)dt\Big|\\ &\quad+\frac{1}{2}\frac{p_n}{P_{n-1}}\Big(1-\frac{P_{n-1}}{P_n}\Big)\int_0^{\pi}|\varphi(t)|\,dt\\ &=L_n'+L_n''+L_n''',\end{aligned}$$

as in McFadden [6]. Now

$$\sum_{n=1}^{\infty} L'_n \leqq \sum_{n=1}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t^2} dt$$
$$\leqq \sum_{n=1}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} dt + A$$
$$\leqq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} dt \sum_{n=k}^{\infty} \frac{p_n - p_{n+1}}{P_{n+1}} + A$$
$$\leqq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t^2} \frac{p(1/t)}{P(1/t)} dt + A \leqq A \sum_{k=1}^{\infty} \frac{\omega(1/n) p_n}{P_n} + A$$
$$\leqq A \sum_{k=1}^{\infty} \frac{\omega(1/n)}{n} + A \leqq A ,$$

$$\sum_{n=1}^{\infty} L''_n \leqq \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt$$
$$= \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt + A$$
$$\leqq \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} \frac{\omega(t)}{t} P\left(\frac{1}{t}\right) dt \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}} + A$$
$$\leqq \int_0^1 \frac{\omega(t)}{t} dt + A \leqq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} + A \leqq A$$

and

$$\sum_{n=1}^{\infty} L''_n \leqq A \sum_{n=1}^{\infty} \frac{p_n^2}{P_n P_{n-1}} \leqq A \sum_{n=1}^{\infty} \frac{1}{n^2} \leqq A .$$

Collecting above estimations, we get $\sum_{n=1}^{\infty} | t_n - t_{n-1} | < \infty$.

7. We shall now prove Theorem 4. We start from (15). First,

$$I_n = \frac{1}{P_{n-1}} \left| \int_0^{\pi} \varphi(t) \left(\sum_{k=0}^{\infty} p_k \cos (n-k) t \right) dt \right|$$
$$\leqq \frac{1}{P_{n-1}} \left\{ \left| \int_0^{\pi} \varphi(t) \xi_1(t) \cos nt \, dt \right| + \left| \int_0^{\pi} \varphi(t) \xi_2(t) \cos nt \, dt \right| \right\}$$
$$= I_{n,1} + I_{n,2} .$$

Hence

$$\sum_{n=1}^{\infty} I_{n,1} \leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} | \varphi(t + \pi/2n) \xi_1(t + \pi/2n) - \varphi(t - \pi/2n) \xi_1(t - \pi/2n) | dt$$
$$\leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} | \xi_1(t + \pi/2n) | \cdot | \varphi(t + \pi/2n) - \varphi(t - \pi/2n) | \, dt$$
$$+ \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{-\pi}^{\pi} | \varphi(t - \pi/2n) | \cdot | \xi_1(t + \pi/2n) - \xi_1(t - \pi/2n) | \, dt$$
$$= \mathscr{I}_1 + \mathscr{I}_2 ,$$

where, since ξ_1 is integrable and $| \varphi(t + \pi/2n) - \varphi(t - \pi/2n) | \leqq A\omega(1/n)$,

$$\mathscr{I}_1 \leqq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} < \infty$$

by the condition (14) and, since[1)]

$$\xi_1(x) = \sum_{\nu=0}^{\infty} (\nu + 1)\Delta^2 p_\nu \cdot K_\nu(x) ,$$

we get

$$\begin{aligned}
\mathscr{I}_2 &\leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \int_{\pi/n}^{\pi} \omega(x) \mid \xi_1(x + \pi/n) - \xi_1(x) \mid dx + A \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^{\infty} (\nu + 1)\Delta^2 p_\nu \int_{\pi/n}^{\pi} \omega(x) \mid K_\nu(x + \pi/n) - K_\nu(x) \mid dx + A \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \Big(\sum_{\nu=0}^{n} (\nu + 1)\Delta^2 p_\nu + \sum_{\nu=n+1}^{\infty} (\nu + 1)\Delta^2 p_\nu \Big) \\
&\qquad \times \int_{\pi/n}^{\pi} \omega(x) \mid K_\nu(x + \pi/n) - K_\nu(x) \mid dx + A \\
&= \mathscr{I}_2' + \mathscr{I}_2'' + A .
\end{aligned}$$

It is well known that

$$\mid K_\nu'(x) \mid \leqq A\nu^2 \quad \text{and} \quad \mid K_\nu'(x) \mid \leqq A/x^2$$

and then

$$\int_0^{\pi/\nu} \mid K_\nu(x + \pi/n) - K_\nu(x) \mid dx \leqq \frac{\pi}{n} A\nu^2 \cdot \frac{\pi}{\nu} \leqq A\nu/n ,$$

$$\mid K_\nu(x + \pi/n) - K_\nu(x) \mid \leqq A/nx^2 \quad \text{in } (\pi/\nu, \pi) .$$

Therefore,

$$\begin{aligned}
\mathscr{I}_2' &\leqq \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=0}^{n} (\nu + 1)\Delta^2 p_\nu \Big(\int_{\pi/n}^{\pi/\nu} + \int_{\pi/\nu}^{\pi} \Big) \omega(x) \mid K_\nu(x + \pi/n) - K_\nu(x) \mid dx \\
&\leqq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \Big\{ \sum_{\nu=0}^{n} (\nu + 1)^2 \Delta^2 p_\nu \sum_{k=\nu}^{n} \frac{1}{k^2} \omega\Big(\frac{1}{k}\Big) + \sum_{\nu=0}^{n} (\nu + 1)\Delta^2 p_\nu \sum_{k=1}^{\nu} \omega\Big(\frac{1}{k}\Big) \Big\} \\
&\leqq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \Big\{ \sum_{k=1}^{n} \frac{1}{k^2} \omega\Big(\frac{1}{k}\Big) \cdot kP_k + \sum_{k=1}^{n} \omega\Big(\frac{1}{k}\Big)(k(p_k - p_{k+1}) + p_{k+1}) \Big\} \\
&\leqq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big) P_k \sum_{n=k}^{\infty} \frac{1}{nP_{n-1}} \\
&\qquad + A \sum_{k=1}^{\infty} \omega\Big(\frac{1}{k}\Big) \Big\{ k(p_k - p_{k+1}) + p_{k+1} \Big\} \sum_{n=k}^{\infty} \frac{1}{nP_{n-1}} \\
&\leqq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big) + A \sum_{k=1}^{\infty} \omega\Big(\frac{1}{k}\Big) \Big\{ \frac{k(p_k - p_{k+1})}{p_k} + \frac{p_{k+1}}{P_k} \Big\} \\
&\leqq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)
\end{aligned}$$

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1) $\Delta^2 p_\nu = p_\nu - 2p_{\nu+1} + p_{\nu+2}$ and K_ν is the νth Fejér kernel.

by the condition (11) and the relation ([6], p. 183)

$$k(p_k - p_{k+1})/P_k \leqq A/k .$$

Further,

$$\begin{aligned}\mathscr{J}_2'' &\leqq A \sum_{n=1}^{\infty} \frac{1}{nP_{n-1}} \sum_{\nu=n+1}^{\infty} (\nu + 1)\Delta^2 p_\nu \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) \\ &\leqq A \sum_{k=1}^{\infty} \omega\left(\frac{1}{k}\right) \sum_{n=k}^{\infty} \frac{(n+2)(p_{n+1} - p_{n+2}) + p_{n+2}}{nP_{n-1}} \\ &\leqq A \sum_{k=1}^{\infty} \frac{1}{k} \omega\left(\frac{1}{k}\right) < \infty .\end{aligned}$$

Thus we have proved that $\sum I_{n,1} < \infty$. The estimation of $I_{n,2}$ is similar to that of $I_{n,1}$ and thus $\sum I_n < \infty$.

Now, $\sum_{k=n}^{\infty} p_k \cos (k-n)x$ is a positive function and then

$$\begin{aligned}\sum_{n=1}^{\infty} J_n &\leqq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} \int_0^{1/n} \left(\sum_{k=n}^{\infty} p_k \cos (k-n)t \right) dt \\ &\leqq \sum_{n=1}^{\infty} \frac{\omega(1/n)}{P_{n-1}} p_n \leqq A \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} < \infty .\end{aligned}$$

Convergence of $\sum_{n=1}^{\infty} K_n$ and $\sum_{n=1}^{\infty} L_n$ is proved as in the proof of Theorem 3. Thus we have established Theorem 4.

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