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BOUNDED GENERATORS OF LINEAR SPACES

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Let $S_\varphi = \{x \in X : \sup_\alpha \varphi_\alpha(x) < \infty\}$ where $\varphi = \{\varphi_\alpha\}$ is a family of semi-norms determining the topology of X. It is shown that φ may be chosen so S_φ is dense if and only if X has a bounded generating set if and only if there is a continuous norm on X^*. It is shown that these conditions hold for separable Fréchet spaces and for quotients of products of Banach spaces. An example is given of a Fréchet space containing no bounded generating set thus contradicting an assertion of L. Maté that S_φ is dense for Fréchet spaces.

All spaces, $X, Y, \cdots$, in this paper are linear spaces with locally convex Haussdorf linear topologies. Given X, let $\Phi = \Phi(X)$ be the set of all families $\varphi = \{\varphi_\alpha\}$ of continuous semi- norms on X determining the topology; for $\varphi \in \Phi$, let $S_\varphi = \{x \in X : N_\varphi(x) = \sup_\varphi \varphi_\alpha(x) < \infty\}$. For subsets $\Gamma \subset X$, let $[\Gamma]$ be the closed linear hull of Γ; if X has a bounded generator, i.e., if there is a bounded set $\Gamma \subset X$ with $[\Gamma] = X$, we call X a 'BG space'. Again, for a subset $\Gamma \subset X$, let $\Gamma^\perp$ be its annihilator, $\Gamma^\perp = \{\xi \in X^* : \xi(x) = 0 \text{ for } x \in \Gamma\}$, let Γ^0 be its polar set, $\Gamma^0 = \{\xi \in X^* : | \xi(x) | \leqq 1 \text{ for } x \in \Gamma\}$, and let ν_Γ be the Minkowski gauge of Γ^0 so, for $\xi \in X^*$, $\nu_\Gamma(\xi) = \inf\{t : \xi \in t\Gamma^0\} = \sup\{| \xi(x) | : x \in \Gamma\}$. By the definition of the strong topology on X^*, $\nu_\Gamma(\xi) = \inf\{t : \xi \in t\Gamma^0\} \sup\{| \xi(x) | : x \in \Gamma\}$. By the definition of the strong topology on X^*, ν_Γ is (strongly) continuous (and everywhere finite) if and only if Γ is bounded in X; it is a norm if and only if $[\Gamma] = X$.

It is asserted in Maté [1] that, given a Fréchet space X and $\varphi \in \Phi(X)$, $\varphi = \{\varphi_1, \varphi_2, \cdots\}$, the set S_φ is dense in X. That this need not hold in general may be seen from the following elementary example.

EXAMPLE 1. Let $X = s$ be the space of all sequences $x = [x_1, x_2, \cdots]$ with the topology of coordinate-wise convergence. We may take $\varphi \in \Phi$ to be given by $\varphi_n(x) = n \max\{| x_k | : k \leqq n\}$ $(n = 1, 2, \cdots)$ or $\psi \in \Phi$ with $\psi_n(x) = | x_n |$. Then S_ψ is the set of all bounded sequences (which *is* dense in X) but $S_\varphi = \{0\}$ since, for $k \geqq n$ and any $x \in X$, $| x_k | \leqq \varphi_n(x)/n \leqq N_\varphi(x)/n$.

This makes it clear that the density of S_φ in X depends on the choice of $\varphi \in \Phi$ and raises the question: Can one (in particular, if X is a Fréchet space) choose $\varphi \in \Phi(X)$ so that S_φ is dense in X? or, somewhat more generally: Can one choose $\varphi \in \Phi(X)$ so that the closure of S_φ is a specified closed subspace Y? We begin by showing the equivalence of certain conditions on X, Y.

THEOREM 1. *Given a space X and a closed subspace $Y \subseteqq X$, the following conditions are equivalent*:

(A) *There exists $\varphi \in \Phi(X)$ such that $\bar{S}_\varphi = Y$.*

(B) *There exists a bounded set $\Gamma \subset X$ with $[\Gamma] = Y$.*

(C) *Y is a BG space.*

(D) *There exists a (strongly) continuous semi-norm $\hat{\nu}$ on X^* such that $\hat{\nu}^{-1}(0) = Y^\perp$.*

(E) *There exists a (strongly) continuous norm ν on Y^*.*

Proof. The equivalence of (B) and (C) is immediate. The equivalence of (D) and (E) follows from the observation that we may take $\hat{\nu} = \nu \circ \tau$ (where τ: $X^* \rightarrow Y^*$ is the 'restriction of domain' map) and noting that τ is continuous, open, surjective, and linear with kernel $Y^\perp$. We now show the equivalence of (A) and (B) and of (C) and (E).

Given (A), set $\Gamma = \{x \in X: N_\varphi(x) \leqq 1\}$. Clearly Γ is bounded; as Γ is balanced and convex, $[\Gamma] = \overline{\cup_k k\Gamma}$ which is just $\bar{S}_\varphi = Y$. Conversely, suppose we are given (B). Let $\hat{\psi}$ be a set of continuous semi-norms on X such than $\cap \{|\hat{\psi}_\alpha^{-1}(0)|: \hat{\psi}_\alpha \in \hat{\psi}\} = Y$; such a set $\hat{\psi}$ may be obtained from any element of $\Phi(X/Y)$. Let A be the index set of $\hat{\psi} = \{\hat{\psi}_\alpha\}$, which we may assume infinite and let B be the set of all finite subsets of A; for $\beta \in B$, $x \in X$, set $\tilde{\psi}_\beta(x) = n_\beta \max\{\hat{\psi}_\alpha(x): \alpha \in \beta\}$ (where n_β is the cardinality of β) and let $\tilde{\psi} = \{\tilde{\psi}_\beta\}$. Next choose any $\hat{\varphi} \in \Phi(X)$ and define $\tilde{\varphi} \in \Phi$ as follows: for $\hat{\varphi}_\gamma \in \hat{\varphi}$ there is, as Γ is bounded, $c_\gamma > 0$ such that $\Gamma \subseteqq c_\gamma \{x \in X: \hat{\varphi}_\gamma(x) < 1\}$; set $\tilde{\varphi}_\gamma(x) = \hat{\varphi}_\gamma(x)/c_\gamma$ and $\tilde{\varphi} = \{\tilde{\varphi}_\gamma\}$. Finally, let $\varphi = \tilde{\psi} \cup \tilde{\varphi}$; since each $\tilde{\psi}_\beta \in \tilde{\psi}$ is continuous and $\tilde{\varphi} \in \Phi$, we have $\varphi \in \Phi$. For $x \in \Gamma$ and $\varphi_* \in \varphi$, we have $\varphi_*(x) = 0$ if $\varphi_* \in \tilde{\psi}$ (as $\Gamma \subset Y$) and $\varphi_*(x) \leqq 1$ if $\varphi_* \in \tilde{\varphi}$ so $N_\varphi(x) \leqq 1$. Thus $\Gamma \subset S_\varphi$ whence $Y = [\Gamma] \subseteqq \bar{S}_\varphi$. On the other hand, the construction of ψ guarantees that $\sup_\beta \tilde{\psi}_\beta(x) = \infty$ for $x \notin Y$ so $S_\varphi \subseteqq Y$ and $\bar{S}_\varphi = Y$.

Given (C), there is a bounded generating set Γ for Y and $\nu = \nu_\Gamma$ is the required continuous norm on Y^*. Conversely, given (E), let $B = \{\eta \in Y^* : \nu(\eta) \leqq 1\}$; the strong continuity of ν means B is a neighborhood of 0 in Y^* and so there exists a bounded set $\Gamma \subset Y$ such that $\Gamma^0 \subseteqq B$. Then ν_Γ is an everywhere finite continuous seminorm on Y^* and, as $\nu_\Gamma \geqq \nu$ and ν is a norm, ν_Γ is a norm on Y^* and so $[\Gamma] = Y$.

For convenience we state separately the result above in the case $X = Y$.

COROLLARY. *The following conditions on a space X are equivalent*:

(A) *There exists $\varphi \in \Phi(X)$ such that S_φ is dense.*

(B) *There is a bounded generating set for X (i.e., X is a BG space).*

(C) *There exists a continuous norm ν on X^*.*

REMARK. Clearly, if X is a Fréchet space the φ of condition (A) may be taken to be countable.

LEMMA. *A Fréchet space X has a pre-compact generating set K if and only if it is separable.*

Proof. Suppose X is separable and $\{x_1, x_2, \cdots\}$ is a countable dense subset. Letting ρ be any metric giving the topology on X, set $y_n = x_n/[1 + \rho(0, x_n)]$ and $K = \{y_n\}$. Since $y_n \to 0$, K is pre-compact. Since each x_n is in the linear hull of K and $\{x_1, x_2, \cdots\}$ is dense, K is a generating set. Conversely, if K is a pre-compact generating set then $\bar{K}$ (being a compact metric space) contains a countable dense subset $\{y_1, y_2, \cdots\}$; let S be the set of all finite linear combinations of the $\{y_n\}$ with (complex) rational coefficients. Then S is countable and, as $\{y_1, y_2, \cdots\}$ is dense generating set $\bar{K}$, S is dense in X which is thus separable.

Since a pre-compact set must be bounded, any separable Fréchet space is a BG space and a positive partial answer to the question raised above is that, for a separable Fréchet space X, one can always choose $\varphi \in \Phi(X)$ so that S_φ is dense; from the construction one can clearly arrange that S_φ contain any specified countable set. We now collect some conditions under which a positive answer may be given to the question.

THEOREM 2. *Any of the following is sufficient to ensure that X is a BG space:*

(A) *X is a Banach space.*
(B) *X is a separable Fréchet space.*
(C) *X is a product of BG spaces.*
(D) *X is the image, under a continuous linear map, of a BG space.*
(E) *X is a quotient of a product of Banach spaces.*

Proof. (A) is trivial, (B) follows from the lemma above. (C) holds because the product of bounded subsets of the factors is bounded in a product space and the product of generating sets is a generating set. (D) holds because the image, under a continuous map, of a bounded set is bounded, of a dense set is dense. (E) follows immediately from (A), (C) and (D).

We now note that the final answer to the question: Does there always exist $\varphi \in \Phi$ such that S_φ is dense? is negative even when restricted to Fréchet spaces.

EXAMPLE 2. Let $\Lambda = \{\lambda = [\lambda_1, \lambda_2, \cdots]: 1 = \lambda_1 < \lambda_2 < \cdots; \lambda_n$ integers$\}$ and let $\boldsymbol{H}$ be a Hilbert space big enough to contain an orthonormal set $\{a_\lambda: \lambda \in \Lambda\}$. Now let $\boldsymbol{X} = \prod_1^\infty \boldsymbol{H}_n$ with each $\boldsymbol{H}_n = \boldsymbol{H}$; the topology on $\boldsymbol{X}$ is determined by the sequence of semi-norms $\varphi = \{\varphi_1, \varphi_2, \cdots\}$; $\varphi_n(\boldsymbol{x}) = \|\pi_n \boldsymbol{x}\|$ where $\pi_n: \boldsymbol{X} \to \boldsymbol{H}_n = \boldsymbol{H}$ is the canonical projection taking $\boldsymbol{x} = [x_1, x_2, \cdots] \in \boldsymbol{X}$ into $x_n \in \boldsymbol{H}$, the norm being that of $\boldsymbol{H}$. Then $\boldsymbol{X}$ is a Fréchet space; we observe that, by conditions (A) and (C) of Theorem 2, it is a BG space. For each $\lambda \in \Lambda$, set $\boldsymbol{b}_\lambda = [a_\lambda, \lambda_2 a_\lambda, \lambda_3 a_\lambda, \cdots] \in \boldsymbol{X}$ and let $\boldsymbol{Y}$ be the closed linear hull of $\{\boldsymbol{b}_\lambda: \lambda \in \Lambda\}$. We assert that the Fréchet space $\boldsymbol{Y}$ (we give $\boldsymbol{Y}$ the induced topology, determined by φ) contains no bounded generating sets, i.e., $\boldsymbol{Y}$ is not a BG space.

Proof. We suppose, to the contrary, that Γ is a bounded generating set for $\boldsymbol{Y}$ (we may, and do, assume that Γ is also closed, balanced, and convex) and proceed to show a contradiction. Boundedness of Γ implies that each φ_n is bounded on Γ so $c_n = \sup\{\varphi_n(x): x \in \Gamma\} < \infty$ $(n = 1, 2, \cdots)$. Choose $\mu \in \Lambda$ such that $\sup_n\{\mu_n/(1 + c_n)\} = \infty$; we show that $b_\mu \notin [\Gamma]$ so Γ cannot be a generating set. Suppose, now, that $\boldsymbol{b}_\mu \in [\Gamma]$; then, as $[\Gamma] = \overline{\cup_k k\Gamma}$, there would exist, for some $k = k^*$, an $\boldsymbol{x}^{(0)} \in k^*\Gamma$ such that $\varphi_1(\boldsymbol{x}^{(0)} - \boldsymbol{b}_\mu) < 1/4$. By the definition of $\boldsymbol{Y}$, there is, for each n, a linear combination $\boldsymbol{x}^{(n)} = \Sigma_{\lambda \in \Lambda} \beta_\lambda b_\lambda$ (a finite sum: $\beta_\lambda = \beta_\lambda(n)$ vanishes for λ not in some finite set, in general depending on n) such that both $\varphi_1(\boldsymbol{x}^{(0)} - \boldsymbol{x}^{(n)}) < 1/4$ and $\varphi_n(\boldsymbol{x}^{(0)} - \boldsymbol{x}^{(n)}) < 1$. Then $\varphi_1(\boldsymbol{x}^{(n)} - \boldsymbol{b}_\mu) < 1/2$ so $1/4 > \|\pi_1(\boldsymbol{x}^{(n)} - \boldsymbol{b}_\mu)\|^2 = \Sigma_{\lambda \neq \mu} |\beta_\lambda|^2 + |\beta_\mu - 1|^2$ whence $|\beta_\mu| = |\beta_\mu(n)| > 1/2$. Now $\varphi_n(\boldsymbol{x}^{(0)} - \boldsymbol{x}^{(n)}) < 1$ implies that

$$\begin{aligned} 1 &> \|\Sigma_\lambda \beta_\lambda \lambda_n a_\lambda - \pi_n \boldsymbol{x}^{(0)}\| \\ &\geqq \|\Sigma_\lambda \beta_\lambda \lambda_n a_\lambda\| - \|\boldsymbol{x}_n^{(0)}\| \geqq |\beta_\mu(n)| \mu_n - \varphi_n(\boldsymbol{x}^{(0)}) \geqq \mu_n/2 - k^* c_n \,. \end{aligned}$$

This, however, would imply that $\mu_n/2 \leqq 1 + k^* c_n$ so $\mu_n/(1 + c_n) \leqq 2k^*$ for $n = 2, 3, \cdots$ which contradicts the choice of the sequence μ.

It is known that there may exist in Banach spaces, closed subspaces which are not the range of any continuous projection; considering $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the example above in the light of condition (D) of Theorem 2 gives the following analogue.

COROLLARY. *There exists Fréchet spaces (indeed, countable products of Banach spaces) containing closed subspaces which are not the range of any continuous linear operator on the space.*

A number of open questions may be mentioned here.

(1) Ordering Φ by inclusion, let Φ_0 be the set of all *minimal*

families of semi-norms determining the topology on X. In Example 1 we have $\psi \in \Phi_0$ and S_ψ dense while $\varphi \notin \Phi_0$ and S_φ not dense. If X is a BG space and $\varphi \in \Phi_0$, must S_φ be dense?

(2) Call a space X a 'hereditary BG space' (HBG space) if *every* closed subspace is a BG space. Every Banach space and every separable Fréchet space is HBG. Are there any other HBG spaces? Is the product of two HBG spaces necessarily HBG? In particular, is the product of the spaces X of Example 1 and $\boldsymbol{H}$ of Example 2 an HBG space?

(3) Is every BG space a quotient of a product of Banach spaces? In particular, is every Fréchet BG space a quotient of a countable product of Banach spaces?

Reference

1. L. Maté, *On a semi-group of operators in Fréchet space*, Doklady Akad. Nauk SSSR **142** (1962), 1247-1250.
(English translation in: Sov. Math. Doklady **3** (1962), 288-292.)

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