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L_p SPACES OVER FINITELY ADDITIVE MEASURES

CHARLES FEFFERMAN

For a space (S, Σ, μ), μ a positive finitely additive set function on a field Σ of subsets of the set S, $L_p(S, \Sigma, \mu)$ is usually not complete. However, if we consider the completion $\dot{L}_p(S, \Sigma, \mu)$ of L_p, we may ask which of the properties of L_p known for the countably additive case, are true in general.

In this paper it is shown that for every (S, Σ, μ) there is a (countably additive) measure space (S', Σ', μ') and a natural injection j from S into S' which induces isometric isomorphisms j_* from $\dot{L}_p(S, \Sigma, \mu)$ onto $L_p(S', \Sigma', \mu')$. j_* also preserves order, and other structures on L_p.

This result shows, roughly, that any theorem valid for L_p over a measure space, applies also to $\dot{L}_p$ over a finitely additive measure. Thus $\dot{L}_p$ and $\dot{L}_q$ are dual $(1 < p < +\infty, 1/p + 1/q = 1)$, $\dot{L}_1$ is weakly complete, and so forth.

Let S be a set, Σ a field of subsets of S, and μ a finitely additive extended real-valued set function on Σ. We call (S, Σ, μ) a triple. If μ is positive or bounded, we call (S, Σ, μ) a positive or bounded triple, respectively.

Let f be a μ-simple function on S. We define the L_p-norm of f, as usual, to be $\left(\int_S |f(s)|^p v(\mu, ds)\right)^{1/p}$ $(1 \leqq p < +\infty)$; and we define the TM-length of f to be arctan $\inf_{\alpha>0}[\alpha + v(\mu, \{s \in S \,||\, f(s)| \geqq \alpha\})]$.

DEFINITION. Let (S, Σ, μ) be a triple. The space $\dot{TM}(S, \Sigma, \mu)$ is defined to be the completion of the space of μ-simple functions under the TM-metric. Define multiplication of elements of $\dot{TM}(S, \Sigma, \mu)$, and an order relation on $\dot{TM}(S, \Sigma, \mu)$ by using Cauchy sequences of simple functions in the obvious way.

Let $\dot{L}_p(S, \Sigma, \mu)$ be the set of limits in $\dot{TM}(S, \Sigma, \mu)$ sequences of μ-simple functions which are Cauchy in the L_p-norm. There is an obvious norm induced on $\dot{L}_p(S, \Sigma, \mu)$ by the μ-simple functions on S.

$\dot{L}_p(S, \Sigma, \mu)$ is canonically isomorphic to the completion of $L_p(S, \Sigma, \mu)$, and thus to S. Leader's space $V_p(S, \Sigma, \mu)$. See [3], which includes equivalents of Theorems 2, 3, and 5.

The purpose of this paper is to prove rhe following.

THEOREM 1. *Let (S, Σ, μ) be a positive triple. There is a positive measure space (S', Σ', μ') and an order-preserving multiplication-preserving isometric isomorphism i from $\dot{TM}(S, \Sigma, \mu)$ onto $TM(S', \Sigma', \mu')$ such that:*

(1) *If* $f \in \dot{T}M(S, \Sigma, \mu)$ *is a characteristic function (simple function), then* $i(f) \in TM(S', \Sigma', \mu')$ *is a characteristic function (simple function).*

(2) i *takes* $\dot{L}_p(S, \Sigma, \mu)$ *onto* $L_p(S', \Sigma', \mu')$ *preserving the* L_p*-norm,* $1 \leqq p < +\infty$.

(3) *If* $f \in \dot{L}_1(S, \Sigma, \mu)$, *then* $\int_S f(s)\mu(ds) = \int_S if(s)\mu'(ds)$.

This leads us to the principle: Let P be any statement about $\dot{T}M(S, \Sigma, \mu)$ which can be formulated in terms of the following concepts:

(1) Multiplication, addition, scalar multiplication, order and length in $TM(S, \Sigma, \mu)$.

(2) The notion $f \in \dot{L}_p(S, \Sigma, \mu)$, and the norm on $\dot{L}_p(S, \Sigma, \mu)$, $1 \leqq p < +\infty$.

(3) The function $f \to \int_S f(s)\mu(ds)$, defined on $\dot{L}_1(S, \Sigma, \mu)$.

If P is true whenever (S, Σ, μ) is a positive measure space, then P is true for any positive triple (S, Σ, μ). Consequences of this principle are listed below.

THEOREM 2. *Let* (S, Σ, μ) *be a positive triple. The dual of* $\dot{L}_p(S, \Sigma, \mu)$ *is canonically isomorphic to* $\dot{L}_q(S, \Sigma, \mu)$ *by the duality*

$$\langle f, g\rangle = \int_S (f \cdot g)(s)\mu(ds) \ (f \in \dot{L}_p, g \in \dot{L}_q) ,$$

wherever $1 < p < +\infty$, $1/p + 1/q = 1$.

COROLLARY 1. $\dot{L}_p(S, \Sigma, \mu)$ *is reflexive,* $1 < p < +\infty$.

COROLLARY 2. $\dot{L}_p(S, \Sigma, \mu)$ *is weakly complete,* $1 < p < +\infty$.

COROLLARY 3. *A bounded subset of* $\dot{L}_p(S, \Sigma, \mu)$ *is weakly sequentially compact.*

THEOREM 3. $\dot{L}_1(S, \Sigma, \mu)$ *is weakly complete.*

THEOREM 4. *Let* (S, Σ, μ) *and* (S', Σ', μ') *be positive triples, let* L_0 *be the space of all complex-valued* μ*-integrable simple functions on* S, *and let* T *be a linear map from* L_0 *to* $\dot{T}M(S', \Sigma', \mu')$.

If for a given pair (p, q), T *has an extension to a bounded linear mapping of* $\dot{L}_p(S, \Sigma, \mu)$ *into* $\dot{L}_q(S', \Sigma', \mu')$, *let* $| T |_{p,q}$ *denote the norm of this extension; if no such extension exists, let* $| T |_{p,q} = +\infty$. *Then* $\log | T |_{1/a,1/b}$ *is a convex function of* (a, b) *in the rectangle* $0 < a, b \leqq 1$.

Theorem 4 generalizes the Riesz Convexity Theorem.

THEOREM 5. *Assume that* (S, Σ, μ) *is a bounded triple. Let* (f_n)

be a sequence in $\dot{L}_p(S, \Sigma, \mu)$ *converging weakly to* $f \in \dot{L}_p(S, \Sigma, \mu)$. *Then* (f_n) *converges strongly to* f *if and only if* (f_n) *converges to* f *in* $\dot{T}M(S, \Sigma, \mu)$.

COROLLARY 1. *Let* (S, Σ, μ) *be a bounded triple. If* (f_n) *is a sequence in* $L_p(S, \Sigma, \mu)$, *converging wealy to* $f \in L_p(S, \Sigma, \mu)$, *then* f *is the strong limit of* (f_n) *if and only if* (f_n) *converges in measure to* f.

Theorems 2, 3, and 5 are obvious from the above principle. The usual proof (see [2]) of the Riesz Convexity Theorem uses countable additivity only through use of the result that L_q is dual to L_p. Since we know Theorem 2, the proof of the Riesz Convexity Theorem may be easily adapted to the finitely additive case.

So in order to establish Theorems 2 through 5, we need only prove Theorem 1.

2. **Proof of Theorem 1.** Let B_0 be the set of characteristic functions of sets of Σ, and let B be the closure of B_0 in $\dot{T}M(S, \Sigma, \mu)$. B is a closed subset of $\dot{T}M(S, \Sigma, \mu)$ and so is a complete metric space. The function $\bigcup_0: B_0 \times B_0 \to B$ defined by $\bigcup_0(x_E, x_F) = x_{E \cup F} \in B_0 \subseteqq B$ is easily seen to be uniformly continuous on $B_0 \times B_0$ and therefore $\bigcup_0$ extends to a uniformly continuous $\bigcup: B \times B \to B$. If $F, G \in B$ abbreviate $\bigcup(F, G)$ by $F \cup G$. Similarly, the function $N_0: B_0 \to B$ defined by $N_0(x_E) = x_{S-E} \in B_0 \subseteqq B$ is uniformly continuous on B_0 and therefore N_0 extends to a uniformly continuous $N: B \to B$. If $F \in B$, abbreviate $N(F)$ by $\sim F$. Define $F \cap G$ to be $\sim(\sim F \cup \sim G)$, $F, G \in B$. Observe that $\bigcap: B \times B \to B$ is a composite of uniformly continuous functions and so is uniformly continuous. Define a function μ_1, on B as follows: For $F \in B$, there is a sequence $\{x_{E_n}\}E_n \in \Sigma$, converging to F in $\dot{T}M(S, \Sigma, \mu)$. Let $\mu_1(F) = \lim_{n\to\infty} \mu(E_n)$. It is easily verified that μ_1 is well-defined and continuous, from B to the positive reals and $+\infty$, the latter given its usual topology.

LEMMA 1. $(B, \bigcup, \bigcap, \sim)$ *is a Boolean algebra, and* μ_1 *is positive and finitely additive on* B. *If* $F \in B$ *and* $\mu_1(F) = 0$, *then* $F = \emptyset$, *the null element of the Boolean algebra.*

Proof. The set

$$R = \{(F, G, H) \in B \times B \times B ! ((F \cup G) \cup H) = (F \cup (G \cup H))\}$$

is closed in $B \times B \times B$ since $\bigcup$ is continuous. On the other hand, it is clear from the definitions of $\bigcup_0$ and $\bigcup$ that $B_0 \times B_0 \times B_0 \subseteqq R$. Since B_0 is dense in B, $R = B \times B \times B$ and therefore $F \cup (G \cup H) = (F \cup G) \cup H$ when $F, G, H \in B$. The other laws of Boolean algebra are verified similarly. The function

$$r(F, G) = [\arctan \mu_1(F \cup G)] - [\arctan (\mu_1(F - G) + \mu_1(G))]$$

taking $B \times B$ to the reals is obviously continuous. Moreover, $r(F, G) = 0$ when $F, G \in B_0$. Since B_0 is dense in B, r is identically zero. So μ_1 is finitely additive on B.

Finally, suppose that $\mu_1(F) = 0$. This means that F is the limit in $\dot{T}M(S, \Sigma, \mu)$ of a sequence $\{x_{E_i}\}$, $E_i \in \Sigma$, with $\lim_{i\to\infty} \mu(E_i) = 0$. But then $\{x_{E_i}\}$ converges to zero in measure, i.e. in $\dot{T}M(S, \Sigma, \mu)$. Therefore, $F = 0$, which acts as $\varnothing$ in $(B, \bigcup, \bigcap, \sim)$.

To simplify notation, identify a set $E \in \Sigma$ with its characteristic function $x_E \in B_0$.

LEMMA 2. *Let $G_1, G_2, \cdots \in B$, and suppose that $G_i \cap G_j = \varnothing$, $i \neq j$. Then there is a double sequence $\{E_i^n\}$, $E_i^n \in \Sigma$, such that*

(1) *$\lim_{n\to\infty} E_i^n = G_i$ in B, for each i.*

(2) *$E_i^n \cap E_j^n = \varnothing$ $(i \neq j)$.*

(3) *If $m \geqq n \geqq j$, then $\mu_1(E_j^n \Delta E_j^m) < 1/n \cdot 2^n$ where Δ denotes the symmetric difference.*

Proof. Since $G_i \in B$, we can find a sequence $\{A_i^k\}$, $A_i^k \in \Sigma$, such that $\lim_{k\to\infty} A_i^k = G_i$ in B. Let $R_i^k = A_i^k - \bigcup_{j<i} A_j^k$. Obviously $R_i^k \cap R_j^k = \varnothing$ $(i \neq j)$. By continuity of $-$ and $\bigcup$,

$$\lim_{k\to\infty} R_i^k = \lim_{k\to\infty} A_i^k - \bigcup_{j<i} \lim_{k\to\infty} A_j^k = G_i - \bigcup_{j<i} G_j = G_i .$$

Pick a subsequence $\{R_i^{k_n}\}$ of R_i^k inductively, as follows: For each n and j we can pick a k_{nj} so large that for $k, k' \geqq k_{nj}$, $\mu_1(R_j^k \Delta R_j^{k'}) < 1/n \cdot 2^n$ (this follows from $\lim_{k\to\infty} R_j^k = G_j$.) For fixed n, take k_n to be any integer which is simultaneously greater than k_{n-1}, and greater than k_{nj}, $j \leqq n$.

For $m \geqq n \geqq j$, $k_m \geqq k_n$ and $j \leqq n$ so that by definition of k_n and k_{nj}, $\mu_1(R_j^{k_n} \Delta R_j^{k_m}) < 1/n \cdot 2^n$. Therefore, letting $E_i^n = R_i^{k_n}$, we have verified conclusion (3). Since $\lim_{k\to\infty} R_i^k = G_i$, conclusion (1) follows from the fact that (E_i^n) is a subsequence of (R_i^k). Since $R_i^k \cap R_j^k = \varnothing$ $(i \neq j)$ holds for all k, it holds for $k = k_n$. So $E_i^n \cap E_j^n = \varnothing$ $(i \neq j)$, verifying conclusion (2).

LEMMA 3. *Let $G_1, G_2, \cdots \in B$ and suppose that $G_i \cap G_j = \varnothing$ $(i \neq j)$. Assume $\sum_{i=1}^\infty \mu_1(G_i) < +\infty$. Then there is a $G \in B$ such that $G_i \subseteqq G$ and $\mu_1(G) = \sum_{i=1}^\infty \mu_1(G_i)$.*

Proof. Pick a double sequence $\{E_i^n\}$ as in Lemma 2. Observe that by (1) and (3) of Lemma 2, $\mu_1(E_j^n \Delta G_j) \leqq 1/n\cdot 2^n$ $(j \leqq n)$. Let $A^n = \bigcup_{j=1}^n E_j^n$. By (2) of Lemma 2, $A^n = \sum_{j=1}^n x_{E_j}$. So

$$\begin{aligned}\mu_1(A^{n+1}\Delta A^n) &= \int_S \left| \left(\sum_{j=1}^{n+1} x_{E_j^{n+1}}(s)\right) - \left(\sum_{j=1}^{n} x_{E_j^n}(s)\right) \right| \mu_1(ds) \\ &\leqq \sum_{j=1}^n \int_S | x_{E_j^{n+1}}(s) - x_{E_j^n}(s) | \mu_1(ds) + \int_S x_{E_{n+1}^{n+1}}(s)\mu_1(ds) \\ &= \sum_{j=1}^n \mu_1(E_j^{n+1}\Delta E_j^n) + \mu_1(E_{n+1}^{n+1}) \leqq \left(\sum_{j=1}^n \frac{1}{n\cdot 2^n}\right) + \mu_1(E_{n+1}^{n+1}) \\ &\leqq \sum_{j=1}^n \frac{1}{n\cdot 2^n} + \mu_1(G_{n+1}) + \frac{1}{(n+1)\cdot 2^{(n+1)}} < \frac{2}{2^n} \\ &\quad + \mu_1(G_{n+1}) \,.\end{aligned}$$

Since the series $\sum_{n=1}^\infty (2/2^n + \mu_1(G_{n+1}))$ converges, $\{A^n\}$ is Cauchy in measure. Let $G = \lim_{n\to\infty} A^n$, $G \in B$. Now

$$\mu_1(G_i - G) = \lim_{n\to\infty} \mu_1(E_i^n - A^n) = 0 \,,$$

since $E_i^n \subseteqq A^n$ for $n > i$. So by Lemma 1, $G - G_i = \varnothing$, and therefore $G_i \subseteqq G$. It remains to show that $\mu_1(G) = \sum_{i=1}^\infty \mu_1(G_i)$. By virtue of $G_i \subseteqq G$, we have $\sum_{i=1}^n \mu_1(G_i) = \mu_1(\bigcup_{i=1}^n G_i) \leqq \mu_1(G)$. Since n is arbitrary, $\sum_{i=1}^\infty \mu_1(G_i) \leqq \mu_1(G)$. On the other hand,

$$\begin{aligned}\mu_1(G) = \lim_{n\to\infty} \mu_1(A^n) &= \lim_{n\to\infty} \sum_{j=1}^n \mu_1(E_j^n) \\ &\leqq \lim_{n\to\infty} \left(\sum_{j=1}^n \left(\mu_1(G_j) + \frac{1}{n\cdot 2^n}\right)\right) \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^n \mu_1(G) + \frac{1}{2^n}\right) = \sum_{i=1}^\infty \mu_1(G_j) \,.\end{aligned}$$

By the Stone Representation Theorem, there is a set S' and a field Σ_0' of subsets of S' such that Σ_0' is isomorphic as a Boolean algebra with B. Let $j\colon B \to \Sigma_0'$ denote the isomorphism. j induces a positive finitely additive set function μ_0' on Σ_0' defined in the obvious way using j and μ_1. Lemmas 1 through 3 carry over from (B, μ_1) to (S', Σ_0', μ_0') by virtue of the isomorphism. Σ_0' need not be a sigma-field. However,

LEMMA 4. *μ_0' is countably additive on Σ_0'.*

Proof. Let $A_1, A_2, \cdots \in \Sigma_0'$ be pairwise disjoint, and let $A = \bigcup_{i=1}^\infty A_i$, $A \in \Sigma_0'$. We must show that $\mu_0'(A) = \sum_{i=1}^\infty \mu_0'(A_i)$.

From the fact that μ_0' is posititive and finitely additive, we have immediately that $\sum_{i=1}^\infty \mu_0'(A_i) \leqq \mu_0'(A)$. In case $\sum_{i=1}^\infty \mu_0'(A_i) = +\infty$, we are already finished. We may therefore suppose that $\sum_{i=1}^\infty \mu_0'(A_i) < +\infty$. Since Lemma 3 carries over to (S', Σ', μ_0'), there is a set $A' \in \Sigma_0'$ such that $A_i \subseteqq A'$ and $\mu_0'(A') = \sum_{i=1}^{\infty} \mu_0'(A_i)$. From $A_i \subseteqq A'$ we conclude that $A \subset A'$, and therefore $\mu_0'(A) \leqq \mu_0'(A') = \sum_{i=1}^{\infty} \mu_0'(A_i)$.

Since μ_0' is countably additive on Σ_0', we can extend μ_0' to a positive measure μ' on Σ', the sigma field generated by Σ_0'.

We shall show that (S', Σ', μ') is the measure space asserted to exist in the statement of Theorem 1. Thus, for instance, $\dot{L}_p(S, \Sigma, \mu)$ is isomorphic to $L_p(S', \Sigma', \mu')$.

Since $B \subseteqq \dot{T}M(S, \Sigma, \mu)$ is total, we can extend $j: B \to \Sigma_0'$ to i: $\dot{T}M(S, \Sigma, \mu) \to TM(S', \Sigma', \mu')$ by extending first to μ-simple functions, setting $i_0(\sum_{i=1}^{n} \alpha_{iE_i}) = \sum_{i=1}^{n} \alpha_i x_j(E_i)$, and then extending i_0 from the space of simple functions to $\dot{T}M(S, \Sigma, \mu)$ (in which the μ-simple functions are dense). One must, of course, show that i_0 is well-defined, but that is easy.

From the definition of i, it is immediate that i is an order preserving multiplication-preserving isometric isomorphism into, taking characteristic functions $(f \in B)$ to characteristic functions $(x_{j(f)})$.

For $A \in \Sigma_0'$, $\chi_A = i(j^{-1}(A))$, so that $\chi_A \in \operatorname{im} i$. Since $\{\chi_A \mid A \in \Sigma_0'\}$ is total in $TM(S', \Sigma', \mu')$, i is onto.

If $G \in B$ and $\mu_1(G) < +\infty$ then $G \in \dot{L}_p(S, \Sigma, \mu)$, and $|G| = \mu_1(G)$ where the norm is taken in $\dot{L}_p$. Therefore, i_0^{-1} takes μ_0'-integrable simple functions to elements of $\dot{L}_p(S, \Sigma, \mu)$ and preserves the L_p-norm. Therefore i^{-1} takes $L_p(S', \Sigma_0', \mu_0')$ into $\dot{L}_p(S, \Sigma, \mu)$ preserving norms. But $L_p(S', \Sigma_0', \mu_0') = L_p(S', \Sigma', \mu')$, so i^{-1} takes $L_p(S', \Sigma', \mu')$ isometrically into $\dot{L}_p(S, \Sigma, \mu)$.

If $E \in \Sigma$ and $\mu(E) < +\infty$, then $\mu_0'(j(E)) < +\infty$ so that $x_{j(E)} \in L_p(S', \Sigma', \mu')$. Since $\chi_A = i^{-1}(x_{j(E)})$, we have $\chi_A \in \operatorname{im} i^{-1}$. On the other hand, $\{\chi_A \mid \mu(E) < +\infty\}$ is total in $\dot{L}_p(S, \Sigma, \mu)$. So i^{-1} is onto. This verifies (2) in the statement of the theorem.

By what we have already shown,

$$K = \left\{ f \in \dot{L}_1(S, \Sigma, \mu) \,\middle|\, \int_S f(s)\mu(ds) = \int_{S'} (if)(s')\mu'(ds') \right\}$$

is a closed subspace of $\dot{L}_1(S, \Sigma, \mu)$. But clearly, every μ-simple function on S is in K. Therefore $K = \dot{L}_1(S, \Sigma, \mu)$. This verifies (3) in the statement of the theorem.

Theorem 1 could also have been proved with the assumption that μ is bounded, replacing the assumption that μ is positive. In order to effect the change, we repeat the above proof, replacing μ by its total variation u. μ as well as u can obviously be extended from Σ to B. Minor changes then convert the proof for μ positive, to a proof for μ bounded.

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