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ON MEASURES WITH SMALL TRANSFORMS

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G is a locally compact abelian group whose dual Γ is algebraically ordered, i.e., ordered when considered as a discrete group. Every (Radon) complex measure μ on G has a unique Lebesgue decomposition: $d\mu = d\mu_s + g(x)dx$, where $d\mu_s$ is singular and $g \in L^1(G)$. A measure μ on G is of analytic type if $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ.

The main result of the paper is that if $\int_{\gamma<0} |\hat{\mu}(\gamma)|^2 d\gamma < \infty$, or more generally, if, for $\gamma < 0$, $\hat{\mu}(\gamma)$ coincides with the transform $\hat{f}(\gamma)$ of a function f in $L^p(G)$, $1 \leqq p \leqq 2$, then the singular part $d\mu_s$ is of analytic type and $\hat{\mu}_s(0) = 0$.

Throughout the paper the symbol $M(G)$ denotes the Banach algebra under convolution of all regular complex measures on G. Haar measure will be denoted dx on G and $d\gamma$ on Γ. If the singular part $d\mu_s$, of a $\mu \in M(G)$, vanishes, then $d\mu$ is called absolutely continuous.

We first prove that *if $\mu \in M(G)$ and $\hat{\mu} \in L^2(\Gamma)$, then μ is absolutely continuous*. This natural statement must have been proved before, but it does not seem to appear in the literature. It is not implied by the L^1-inversion theorem, which assumes $\mu \in M(G)$ and $\hat{\mu} \in L^1(\Gamma)$, nor by Plancherel's theorem. It is best possible in the sense that $\hat{\mu} \in L^2(\Gamma)$ cannot be replaced by the weaker condition $\hat{\mu} \in L^p(\Gamma)$, $p > 2$; for, as shown by Hewitt and Zuckerman [3], on any nondiscrete locally compact abelian group G, there exists a nonvanishing singular measure μ_s for which $\hat{\mu}_s \in L^p(\Gamma)$, for every $p > 2$.

Next we suppose that the dual Γ is *algebraically ordered*. This means that there exists a semi-group $P \subset \Gamma$ such that $P \cup (-P) = \Gamma$, $P \cap (-P) = \{0\}$. We do not assume that P is closed in Γ, so that, e.g., R^k, $k \geqq 1$, is algebraically ordered. If P is closed in Γ, then Γ is called *ordered* (Rudin [4]). But then R^k is ordered only if $k = 1$. If Γ is discrete, the two notions of ordered and algebraically ordered coincide. A discrete abelian group Γ can be ordered if and only if its (compact) dual G is connected (Rudin [4], 8.1.2 (a) and 2.5.6 (c)). Thus the dual Γ of a locally compact abelian group G can be algebraically ordered if and only if the Bohr compactification $\bar{G}$ of G is connected.

So suppose Γ is algebraically ordered. A measure $\mu \in M(G)$ is said to be of *analytic type* if $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$. Helson and Lowdenslager [2] prove that for a compact abelian group G, with ordered dual Γ, if $\mu \in M(G)$ is of analytic type, then the singular part μ_s is of analytic type and moreover $\hat{\mu}_s(0) = 0$. Our main result is a twofold generalization of this theorem, namely:

Let G be a locally compact abelian group with algebraically ordered dual Γ and let $\mu \in M(G)$. If $\int_{\gamma<0} |\hat{\mu}(\gamma)|^2 d\gamma < \infty$ or more generally if, for $\gamma < 0$, $\hat{\mu}$ coincides with the transform $\hat{f}$ of a function f in $L^p(G)$, $1 \leqq p \leqq 2$, then μ_s is of analytic type and $\hat{\mu}_s(0) = 0$.

This theorem is new even in the case $G = R$. Combined with the F. and M. Riesz theorem it yields the result: if $\mu \in M(R)$ and $\int_{\gamma<0} |\hat{\mu}(\gamma)|^2 d\gamma < \infty$ then μ is absolutely continuous.

THEOREM 1. *Let μ be a complex measure on the locally compact abelian group G. If $\hat{\mu} \in L^2(\Gamma)$ then μ is absolutely continuous.*

Proof. (Short and due to the referee.) By Plancherel's theorem there is $f \in L^2(G)$ with $\hat{f} = \hat{\mu}$ almost everywhere. Let g be a continuous function, with compact support in G, such that $\hat{g} \in L^1(\Gamma)$. Then

$$\begin{aligned} \int_G f\bar{g} &= \int_\Gamma \hat{f}\bar{\hat{g}} \qquad \text{(Parseval-Plancherel)} \\ &= \int_\Gamma \bar{\hat{g}}(\gamma)d\gamma \int_G \overline{(x, \gamma)}d\mu(x) \qquad (\text{since } \hat{f} = \hat{\mu}) \\ &= \int_G \bar{g}(x)d\mu(x) \qquad (\text{Fubini and } L^1\text{-inversion theorem}). \end{aligned}$$

Now every continuous h with compact support C in G, can be uniformly approximated by g's of the above type, with supports in a fixed compact set C': choose a fixed compact neighborhood V of 0 and a kernel $k \geqq 0$, bounded, with support in V, and put $g = h*k$; then

$$\text{support } g \subset C + V = C', \hat{h}, \hat{k} \in L^2(\Gamma), \hat{g} \in L^1(\Gamma) ,$$

and g may be chosen uniformly close to h. Hence

$$\int_G \bar{h}f = \int_G \bar{h}d\mu$$

for every continuous h with compact support in G. Therefore

$$\int_G |f| \leqq ||\mu|| < \infty, f \in L^1(G) ;$$

since $\hat{\mu} = \hat{f}$, we conclude, by the uniqueness theorem $d\mu(x) = f(x)dx$ and μ is absolutely continuous.

LEMMA 1. *Suppose G is a locally compact abelian group whose dual Γ is algebraically ordered, $\mu \in M(G)$ and μ is of analytic type. Then the singular part of μ is also of analytic type.*

This lemma has been proved in Doss [1] under the assumption that Γ is ordered. But the proof is valid for an algebraically ordered Γ with the following obvious modifications:

The compact interval $[-\delta, \delta]$ is replaced by a compact symmetric neighborhood V of the origin in Γ. The relation $\gamma < -\delta$ is replaced throughout by $\gamma < 0, \gamma \notin V$.

Finally the function k such that

$$(1) \qquad k \in L^1(G) \qquad k(x) \geqq 0$$

$$(2) \qquad \hat{k}(\gamma) \geqq 0 \qquad \hat{k}(\gamma) = 0 \text{ outside } V$$

is obtained as follows:

Choose a symmetric compact neighborhood W of 0 in Γ. Let $u(\gamma) = 1/\text{meas } W$ on W, $u(\gamma) = 0$ outside W. Then $u \in L^1(\Gamma)$, $u \in L^2(\Gamma)$, $\hat{u} \in L^2(G)$. Put $v = u * u$. Then

(2′) $v(\gamma) \geqq 0$, v vanishes outside the compact (symmetric) set

$$V = W + W .$$

Also $v \in L^1(\Gamma)$ and

$$(1') \qquad \hat{v}(x) = |\hat{u}(x)|^2 \geqq 0, \hat{v} \in L^1(G) .$$

By the inversion theorem

$$v(\gamma) = \int_G \hat{v}(x)(x, \gamma)dx .$$

Put $k(x) = \hat{v}(x)$. Then, by (1′)

$$(1) \qquad k \in L^1(G) , \qquad k(x) \geqq 0 .$$

Moreover, $\hat{k}(\gamma) = \int_G k(x)\overline{(x, \gamma)}dx = v(-\gamma)$. Hence, by (2′)

$$(2) \quad \hat{k}(\gamma) \geqq 0 , \qquad \hat{k}(\gamma) = 0 \text{ outside } V .$$

LEMMA 2. *Let G be a locally compact abelian group whose dual Γ is algebraically ordered. Let*

$$d\sigma = ds + w(x)dx$$

be a positive measure on G, where ds is singular and $w \in L^1(G)$. Let K be a compact set in Γ and denote by Ω the set of trigonometric polynomials $p(x)$ of the type

$$p(x) = \sum a(x, \gamma) \quad \gamma < 0 , \quad \gamma \notin K .$$

Let φ be the unique function belonging to the closure of Ω in $L^2(d\sigma)$

and such that

$$\int_G |1 - \varphi|^2 d\sigma = \inf_{p \in \Omega} \int_G |1 - p|^2 d\sigma .$$

Then

$$\int_G |1 - \varphi|^2 d\sigma \leqq \int_G w dx .$$

Proof. φ is the unique function belonging to the closure of Ω in $L^2(d\sigma)$, for which

$$\int_G \overline{(x, \gamma)}(1 - \varphi) d\sigma = 0 \qquad \text{for} \quad \gamma < 0, \gamma \notin K . \tag{1}$$

We can easily find, by means of an appropriate kernel, an $f \in L^1(G)$ whose transform $\hat{f}$ is equal to the transform of the measure $(1 - \varphi)d\sigma$, for $\gamma < 0$. But then the measure $(1 - \varphi)d\sigma - f(x)dx$ is of analytic type. By Lemma 1, the singular part $(1 - \varphi)ds$ is of analytic type:

$$\int_G \overline{(x, \gamma)}(1 - \varphi) ds = 0 \qquad \text{for} \quad \gamma < 0 .$$

By continuity (or by the Helson-Lowdenslager theorem, in case Γ is discrete), the same relation holds for $\gamma = 0$. We conclude

$$\int_G \overline{(x, \gamma)}\overline{(1 - \varphi)}(1 - \varphi) ds = 0 \qquad \text{for} \quad \gamma \leqq 0 ,$$

and since $|1 - \varphi|^2 ds$ is real, the above relation is true for $\gamma \geqq 0$. Hence, by the uniqueness theorem:

$$|1 - \varphi|^2 ds = 0 \qquad (1 - \varphi) ds = 0 . \tag{2}$$

Hence (1) reduces to

$$\int_G \overline{(x, \gamma)}(1 - \varphi) w dx = 0 \qquad \text{for} \quad \gamma < 0, \gamma \notin K .$$

Since φ belongs to the closure of Ω in $L^2(w)$ we conclude

$$\int_G |1 - \varphi|^2 w dx = \inf_{p \in \Omega} \int_G |1 - p|^2 w dx \leqq \int_G w dx .$$

Hence, by (2)

$$\int_G |1 - \varphi|^2 d\sigma \leqq \int_G w dx$$

and the lemma is proved.

MAIN THEOREM. *Let G be a locally compact abelian group* *whose dual Γ is algebraically ordered. Let*

$$d\mu = d\mu_s + g(x)dx$$

be a complex measure on G, where $d\mu_s$ is singular and $g \in L^1(G)$. If $\int_{\gamma<0} |\hat{u}(\gamma)|^2 d\gamma < \infty$, or more generally if, for $\gamma < 0, \hat{u}(\gamma)$ coincides with the transform $\hat{f}(\gamma)$ of some function $f \in L^r(G), 1 \leqq r \leqq 2$, then $d\mu_s$ is of analytic type and $\hat{u}_s(0) = 0$.

Proof. It is sufficient to prove $\hat{u}_s(0) = 0$, for by translation, we get $\hat{u}_s(\gamma) = 0$ for $\gamma < 0$. By hypothesis there is $f \in L^r(G)$, such that

$$\hat{f}(\gamma) = \hat{u}(\gamma) \quad \text{a.e. for } \gamma < 0 \, .$$

Let $\varepsilon > 0$ be given. There is $k_1 \in L^1(G)$ such that $\hat{k}_1$ has compact support K_1 and such that

$$\| g - g * k_1 \|_1 < \varepsilon \, .$$

(see e.g. [4], 2.6.6). Also there is $h_1 \in L^1(G)$ such that $\hat{h}_1$ has compact support H_1 and such that

$$\| f - f * h_1 \|_r < \varepsilon^{1/r}$$

(the proof of 2.6.6 in [4] works unchanged). Put

$$g_1 = g - g * k_1 \, , \quad f_1 = f - f * h_1 \, .$$

Then

$$\| g_1 \|_1 < \varepsilon \, , \quad \| f_1 \|_r < \varepsilon^{1/r} \, .$$

Put

$$\begin{aligned} d\lambda &= d\mu_s + g_1(x)dx \\ d\sigma &= d|\mu_s| + |g_1(x)|\,dx + |f_1(x)|^r dx \, . \end{aligned}$$

Let V be a symmetric compact neighborhood of the origin independent of ε and the subsequent choice of k_1, h_1, K_1, H_1. Put

$$K = K_1 + H_1 + V$$

so that K is compact.

By Lemma 2 there is a

$$p(x) = \sum a_n (x, \gamma_n) \quad \gamma_n < 0, \gamma_n \notin K$$

such that

$$\int_G |1 - p|^2 d\sigma \leqq \varepsilon + \| g_1 \|_1 + \| f_1^r \|_1 \leqq 3\varepsilon \, . \tag{1}$$

Put $p_1 = \frac{2}{r} \frac{1}{p_1} + \frac{1}{q_1} = 1$. By Hölder's inequality and (1)

$$\int_G \overline{(1-p)} f_1 |^r dx \leqq \int_G |1 - p|^r d\sigma$$
$$\leqq \left[\int_G |1 - p|^{rp_1} d\sigma\right]^{1/p_1} \left[\int_G d\sigma\right]^{1/p_1} \leqq (3\varepsilon)^{1/q_1} \sigma(G)^{1/q_1} .$$

This, combined with $\| f_1 \|_r < \varepsilon^{1/r}$ gives

$$\| \bar{p} f_1 \|_r \leqq \varepsilon^{1/r} + [(3\varepsilon)^{1/p_1} \sigma(G)^{1/q_1}]^{1/r} . \tag{2}$$

By the Schwarz inequality and (1)

$$\int_G |1 - p| \, d\sigma \leqq (3\varepsilon)^{1/2} \sigma(G)^{1/2} .$$

Hence

$$\left| \int_G \overline{(x, \gamma)} \overline{(1 - p)} d\lambda \right| \leqq (3\varepsilon)^{1/2} (\sigma(G))^{1/2}$$

i.e.,

$$| \hat{\lambda}(\gamma) - (\bar{p} d\lambda)^\wedge(\gamma) | \leqq (3\varepsilon)^{1/2} \sigma(G)^{1/2} . \tag{3}$$

Now from the definition of $d\lambda$ and from $\hat{f}(\gamma) = \hat{u}(\gamma)$ a.e. for $\gamma < 0$ we see that

$$\hat{\lambda}(\delta) = \hat{u}(\delta) = \hat{f}(\delta) = \hat{f}_1(\delta) \text{ a.e. for } \delta < 0, \delta \notin K_1 \cup H_1 . \tag{4}$$

But $\gamma_n < 0, \gamma_n \notin (K_1 \cup H_1) - V$. Hence, if $\gamma \leqq 0, \gamma \in V$ we have

$$\gamma + \gamma_n < 0, \gamma + \gamma_n \notin K_1 \cup H_1 .$$

Whence, by (4)

$$\int_G \overline{(x, \gamma)} \overline{(x, \gamma_n)} d\lambda = \hat{f}_1(\gamma + \gamma_n) \text{ a.e. for } \gamma \leqq 0, \gamma \in V .$$

Therefore,

$$(\bar{p} d\lambda)^\wedge(\gamma) = (\bar{p} f_1)^\wedge(\gamma) \quad \text{a.e. for } \gamma \leqq 0, \gamma \in V .$$

We deduce, by (3)

$$| \hat{\lambda}(\gamma) - (\bar{p} f_1)^\wedge(\gamma) | \leqq (3\varepsilon)^{1/2} \sigma(G)^{1/2} \quad \text{a.e. for } \gamma \leqq 0, \gamma \in V .$$

Finally

$$| \hat{u}_s(\gamma) - (\bar{p} f_1)^\wedge(\gamma) | < \varepsilon + (3\varepsilon)^{1/2} \sigma(G)^{1/2} \quad \text{a.e. for } \gamma \leqq 0, \gamma \in V . \tag{5}$$

Now $\varepsilon > 0$ is arbitrary. By (2) and (5) there exists a sequence $\varphi_n \in L^r(G)$ such that

$$(6) \qquad \begin{aligned} &\| \varphi_n \|_r \to 0 \\ &\hat{\varphi}_n(\gamma) \to \hat{u}_s(\gamma) \qquad \text{a.e. for } \gamma \leqq 0, \gamma \in V\,. \end{aligned}$$

We deduce from (6)

$$\| \hat{\varphi}_n \|_{r'} \to 0 \qquad \left(\frac{1}{r} + \frac{1}{r'} = 1 \right).$$

This shows that $\hat{\mu}_s(\gamma) = 0$ a.e. for $\gamma \leqq 0, \gamma \in V$.

In particular, by continuity, $\hat{u}_s(0) = 0$ and the theorem is proved.

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