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HOMOMORPHISMS OF SEMI-SIMPLE ALGEBRAS

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Let $\nu : \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism of a semi-simple Banach algebra $\mathfrak{A}$. The purpose of this paper is to investigate certain topological properties of ν under various assumptions about $\mathfrak{A}$.

Given a Banach algebra homomorphism $\nu : \mathfrak{A} \rightarrow \mathfrak{B}$, let $S(\nu, \mathfrak{B})$ be the set of all $b \in \mathfrak{B}$ such that there is a sequence $\{x_n \in \mathfrak{A} \mid n = 1, 2, \cdots\}$ with $\lim_{n\rightarrow\infty} x_n = 0$, $\lim_{n\rightarrow\infty} \nu(x_n) = b$; and let $S(\nu, \mathfrak{A})$ be the set of all $x \in \mathfrak{A}$ such that there is a sequence $\{x_n \in \mathfrak{A} \mid n = 1, 2, \cdots\}$ with $\lim_{n\rightarrow\infty} x_n = 0$, $\lim_{n\rightarrow\infty} \nu(x_n) = \nu(x)$. Each of these sets is a two-sided closed ideal, in $\mathfrak{A}$ or in the closure of $\nu(\mathfrak{A})$, and the closed graph theorem shows that ν is continuous if and only if $S(\nu, \mathfrak{B}) = (0)$.

This paper is divided into two sections. In the first it is shown that, if $\mathfrak{A}$ is a B*-algebra, then $S(\nu, \mathfrak{A})$ is the closure of the kernel of ν, thus extending a result of Cleveland ([2], p. 1103), and that, if $\mathfrak{A}$ is a commutative regular semi-simple algebra and ν is an isomorphism, then $S(\nu, \mathfrak{A}) = (0)$. The second section is devoted to an analysis of the Badé-Curtis [1] decomposition of homomorphisms of $C(X)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space X.

I. Homomorphisms of B^*-algebras. Let $\nu : \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism of a B^*-algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the closure of $\nu(\mathfrak{A})$ (this latter condition will remain in force throughout the paper). Let K denote the kernel of ν. We recall that a commutative B^*-algebra is either the algebra of all continuous complex-valued functions with supremum norm on some compact Hausdorff space, or those which vanish at infinity on a locally compact Hausdorff space. We let $C(X)$ denote the former, and $C_0(X)$ the latter.

The first lemma is an easy extension of a well-known result for compact Hausdorff spaces ([3], p. 93), and is stated without proof.

LEMMA I.1. *Let X be a locally compact Hausdorff space, I a closed ideal in $C_0(X)$. Then there is a closed set $X_I \subseteqq X$ such that $I = \{f \in C_0(X) \mid f(X_I) = 0\}$.*

The following lemma enables us to locate useful elements in a closed ideal in $C_0(X)$, and is a consequence of Theorem 2.7.23 of [4].

LEMMA I.2. *Let X be locally compact Hausdorff, F a finite subset of X. Let $T(F)$ denote the set of all functions f in $C_0(X)$*

which vanish on some open neighborhood N_f of F, the neighborhood depending on f, and let $M(F) = \{f \in C_0(X) \mid f(F) = 0\}$. Let A be an ideal in $C_0(X)$ such that $\bar{A} = M(F)$. Let $g \in T(F)$, and assume that g vanishes outside a compact set. Then $g \in A$.

If $x \in K$, let $x_n = (1/n)x$ for $n = 1, 2, \cdots$. Clearly $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \nu(x_n) = 0 = \nu(x)$, so $x \in S(\nu, \mathfrak{U})$. Since $S(\nu, \mathfrak{U})$ is a closed ideal, we therefore have $\bar{K} \subseteq S(\nu, \mathfrak{U})$.

THEOREM I.1. *$S(\nu, \mathfrak{U}) = \bar{K}$.*

Proof. Let $S = S(\nu, \mathfrak{U})$, and assume $\bar{K} \neq S$. By [4], Theorem 4.9.2, S is a *-ideal, and is therefore the linear span of its self-adjoint elements. Since $S \neq \bar{K}$, we can therefore find a self-adjoint element y in $S \sim \bar{K}$. Let $\mathfrak{U}_0 = C_0(X)$ be the Banach algebra generated by y. Let $\nu_0 = \nu \mid \mathfrak{U}_0$, and let $K_0 = K \cap \mathfrak{U}_0$. $\bar{K}_0$ is a closed ideal in $\mathfrak{U}_0$, and so there is a closed set F such that $\bar{K}_0 = \{f \in \mathfrak{U} \mid f(F) = 0\}$. We now endeavor to show that $F = \emptyset$; this will show that $\mathfrak{U}_0 \subseteq \bar{K}$, and consequently that $\bar{K} = S$.

We first show that F is finite. If there is an infinite sequence $\{x_n \mid n = 1, 2, \cdots\}$ contained in F, we can choose sequences

$$\{V_n \mid n = 1, 2, \cdots\} \quad \text{and} \quad \{U_n \mid n = 1, 2, \cdots\}$$

of open sets such that $x_n \in U_n \subseteq \bar{U}_n \subseteq V_n$ and m $\neq n \Rightarrow V_m \cap V_n = \emptyset$. By Urysohn's Lemma, choose functions $f_n \in C_0(X)$ such that $f_n(\bar{U}_n) = 1, f_n(V_n') = 0$, and $0 \leqq f_n \leq 1$. Let $g_n = f_n^{1/3}$. Since $f_n(F) \neq 0$, clearly $\nu_0(f_n) \neq 0$. But since $m \neq n \Rightarrow g_m g_n = 0$, by [2], Theorem 4.9, there is an integer N such that $n \geqq N \Rightarrow \nu_0(f_n) = \nu_0(g_n^3) = 0$. So F must be finite.

Now assume that $F \neq \emptyset$. Since X is locally compact, there is an open set E such that $F \subseteq E$ and $\bar{E}$ is compact. Choose open sets U and V such that $F \subseteq U \subseteq \bar{U} \subseteq V \subseteq \bar{V} \subseteq E$. Define $p \in C_0(X)$ by $p(\bar{U}) = 1, p(V') = 0, 0 \leqq p \leqq 1$. Since $p(F) \neq 0, p \notin \bar{K}_0$, and hence $\nu(p) \neq 0$. We note that $(p^2 - p)(U \cup V') = 0$, so $p^2 - p$ vanishes on a neighborhood of F and outside the compact set $\bar{E}$. So, by Lemma I.2, we see that $p^2 - p \in K_0$, and so $\nu(p^2 - p) = 0 \Rightarrow \nu(p)^2 = \nu(p)$. We have thus found an element $p \in S$ such that $q = \nu(p)$ is a nonzero idempotent in $S(\nu, \mathfrak{B})$, since it is clear that $\nu(S) \subseteq S(\nu, \mathfrak{B})$.

Since $p \in S$, there is a sequence $\{x_n \in \mathfrak{U} \mid n = 1, 2, \cdots\}$ such that $\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} \nu(x_n) = q = \nu(p)$. Since $\lim_{n\to\infty} x_n = 0$, the spectrum of x_n, and consequently the spectrum of $\nu(x_n)$, eventually lies in a small neighborhood of 0. Since q is a nonzero idempotent, the spectrum of q is either $\{0, 1\}$ or $\{1\}$ and so, by a result of Newburgh quoted in [4], p. 37, the spectrum of $\nu(x_n)$ eventually has points arbitrarily close to 1. This contradiction establishes the theorem.

Now let $\nu : \mathfrak{A} \to \mathfrak{B}$ be an isomorphism of a commutative regular semisimple algebra $\mathfrak{A}$. We show that $S(\nu, \mathfrak{A}) = (0)$.

THEOREM I.2. $S(\nu, \mathfrak{A}) = (0)$.

Proof. Assume there is an $s \in S(\nu, \mathfrak{A})$ with $s \neq 0$. Then there is a sequence $\{x_n \in \mathfrak{A} \mid n = 1, 2, \cdots\}$ such that

$$\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} \nu(x_n) = \nu(s) .$$

Let F denote the Badé-Curtis [1] singularity set of ν, and let f be a function in $\mathfrak{A}$ which is zero on a neighborhood of F. If we let $\mathfrak{A}_0$ denote the algebra of all functions in $\mathfrak{A}$ vanishing on that neighborhood, then by [1], Theorem 3.9, ν is continuous on $\mathfrak{A}_0$, and so

$$\lim_{n\to\infty} x_n f = 0 \Rightarrow \lim_{n\to\infty} \nu(x_n f) = 0 .$$

But $\nu(sf) = \lim_{n\to\infty} \nu(x_n f) = 0$, and since ν is an isomorphism, $sf = 0$. Consequently the support of s consists of isolated points. Select one such isolated point p, and multiply s by a function g which is $1/s(p)$ on p and zero elsewhere; the product sg is an idempotent and is in $S(\nu, \mathfrak{A})$ but is nonzero, a contradiction to [2], p. 1102, and the fact that ν is an isomorphism.

Since there exist discontinuous isomorphisms of commutative regular semi-simple algebras ([1], pp. 597–598), we see that having $S(v, \mathfrak{A}) = (0)$ for an isomorphism is not enough to insure continuity of that isomorphism.

II. Homomorphisms of $C(X)$. Throughout this section we shall be concerned with a Banach algebra homomorphism $\nu : C(X) \to \mathfrak{B}$, X a compact Hausdorff space. Using the Badé-Curtis [1] decomposition of ν, it is possible to obtain further information about ν. We write $\nu = \mu + \lambda$, where μ is the continuous, and λ the singular, part of ν. Let R denote the Jacobson radical of $\mathfrak{B} = \overline{\nu(C(X))}$. By construction ν and μ agree on a dense subalgebra of $C(X)$.

In general, if $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a Banach algebra homomorphism such that $\mathfrak{B} = \overline{\varphi(\mathfrak{A})}$ and $\mathfrak{A}$ is commutative, then $S(\varphi, \mathfrak{B})$ is contained in the Jacobson radical of $\mathfrak{B}$. If for each $b \in \mathfrak{B}$ we define

$$\Delta(b) = \inf_{x \in \mathfrak{A}} (\|x\| + \|b - \nu(x)\|)$$

then by [2], p. 1102, we must have the spectral radius of $b \leqq \Delta(b)$ for all $b \in \mathfrak{B}$. In [2] it is shown that $S(\varphi, \mathfrak{B}) = \{b \in \mathfrak{B} \mid \Delta(b) = 0\}$, and since $\mathfrak{B}$ is commutative, it is clear that $S(\varphi, \mathfrak{B})$ must be contained in the Jacobson radical of $\mathfrak{B}$. If $\mathfrak{B}$ is $C(X)$ for some compact Hausdorff X, then equality holds, as seen by the following proposition.

PROPOSITION II. 1. $S(\nu, \mathfrak{B}) = R$.

Proof. We need merely show that $R \subseteqq S(\nu, \mathfrak{B})$. Let $r \in R$. By [1], (Th. 4.3 b), there is a sequence $\{x_n \in C(X) \mid n = 1, 2, \cdots\}$ such that $\lim_{n\to\infty} \lambda(x_n) = r$. Letting $R(F)$ denote the dense subalgebra of $C(X)$ consisting of functions constant in some neighborhood of each point of F, by construction $\lambda(R(F)) = 0$. Since $R(F)$ is dense, choose $y_n \in R(F)$ such that $\lim_{n\to\infty} \| x_n - y_n \| = 0$. Then

$$\lim_{n\to\infty} \nu(x_n - y_n) = \lim_{n\to\infty} \mu(x_n - y_n) + \lim_{n\to\infty} \lambda(x_n - y_n) = \lim_{n\to\infty} \lambda(x_n) = r ,$$

and so $r \in S(\nu, \mathfrak{B})$.

Since μ and ν agree on a dense subalgebra, it is reasonable to suspect that their kernels are closely related. We have the following proposition.

PROPOSITION II.2. $\text{Ker}\ (\mu) = \overline{\text{Ker}\ (\nu)}$.

Proof. If $\mu(x) = 0$, then $\nu(x) = \lambda(x) \in R$, and if $\nu(x) \in R$, then $\mu(x) = \nu(x) - \lambda(x) \in R$, and so $\mu(x) = 0$ by [1], Theorem 4.3 a. But by Proposition II.1, $R = S(\nu, \mathfrak{B})$, and so $\mu(x) = 0$ if and only if $\nu(x) \in S(\nu, \mathfrak{B})$, that is, if and only if $x \in S(\nu, \mathfrak{U})$. By Theorem I.1, however, $S(\nu, \mathfrak{U}) = \overline{\text{Ker}\ (\nu)}$.

A Banach algebra homormorphism $\nu : C(X) \to \mathfrak{B}$ determines two sets that are of interest—the Badé-Curtis finite singularity set F, and the closed set X_0 that determines closure of the kernel of ν, in the sense of Lemma I.1. We define $T(F)$ to be the algebra of all functions vanishing on some neighborhood of F, the neighborhood varying with the function.

PROPOSITION II.3. $\overline{\text{Ker}\ (\nu)} \cap T(F) = \text{Ker}(\nu) \cap T(F)$.

Proof. If $x \in \overline{\text{Ker}\ (\nu)} \cap T(F)$, by Proposition II.2,

$$x \in \text{Ker}(\mu) \cap T(F) \subseteqq \text{Ker}\ (\mu) \cap R(F) .$$

Since λ is zero on $R(F)$, we have $\mu(x) = \lambda(x) = 0$, and consequently $\nu(x) = 0$.

We are now naturally led to inquire whether the singularity set

F is a subset of X_0. This is indeed the case.

PROPOSITION II.4. $F \subseteqq X_0$.

Proof. Let $F_1 = F \cap X_0$, and let $F_2 = F \sim F_1$. In order to show that $F_2 = \emptyset$, it suffices to show that ν is continuous on $R(F_1)$. Since $F_2 \cap X_0 = \emptyset$, there exist open sets N_1 and N_2 with disjoint closures such that $F_2 \subseteqq N_1$, $X_0 \subseteqq N_2$. Let $f \in R(F_1)$ be arbitrary, and choose $g \in C(X)$ such that $g(\bar{N}_1) = 1$, $g(\bar{N}_2) = 0$. Since $g(N_2) = 0$, by Lemma I.2 $g \in \mathrm{Ker}(\nu)$. Let $h = f - gf$. Now $h(N_1) = f(N_1) - g(N_1)f(N_1) = f(N_1) - f(N_1) = 0$, and since $f \in R(F_1)$ we see that $h \in R(F)$. Since ν is continuous on $R(F)$, there is a constant M such that

$$t \in R(F) \Rightarrow \| \nu(t) \| \leqq M \| t \|,$$

and so $\| \nu(h) \| \leqq M \| h \|$. Since $\nu(g) = 0$, $\nu(h) = \nu(f) - \nu(g)\nu(f) = \nu(f)$, and we also have $\| h \| \leqq (1 + \| g \|) \| f \|$. Therefore

$$\| \nu(f) \| \leqq M(1 + \| g \|) \| f \|$$

for all $f \in R(F_1)$, and so $F_2 = \emptyset$.

One of the most immediate consequences of the continuity of a given homomorphism is that its kernel is closed. P. Curtis has observed to the author that, if every kernel of a homomorphism of $C(X)$ is closed, then every homomorphism of $C(X)$ is continuous. If every such kernel were closed, so would every kernel of a homomorphism of $C_0(Y)$ be closed, Y locally compact Hausdorff. By [1], Theorem 4.3 c, $\lambda \mid M(F)$ is a homomorphism; closure of its kernel (which we know contains $R(F)$) would therefore contain $M(F)$, and so $\lambda(M(F)) = 0$. Given $f \in C(X)$, let $F = \{\omega_i \mid 1 \leqq i \leqq n\}$ be the singularity set of ν. Choose $\{e_i \in C(X) \mid 1 \leqq i \leqq n\}$ such that $i \neq j \Rightarrow e_i e_j = 0$, $0 \leqq e_i \leqq 1$, and $e_i(\omega) \equiv 1$ in a neighborhood of $\omega_i \in F$. Then $f - \sum_{i=1}^n f(\omega_i)e_i \in M(F)$. Since μ is continuous on $C(X)$, there is a constant M such that $g \in C(X) \Rightarrow \| \mu(g) \| \leqq M \| g \|$. Since

$$f = \textstyle\sum_{i=1}^n f(\omega_i)e_i + (f - \sum_{i=1}^n f(\omega_i)e_i) ,$$

we have $\nu(f) = \sum_{i=1}^n f(\omega_i)\nu(e_i) + \mu(f - \sum_{i=1}^n f(\omega_i)e_i)$ and so

$$\begin{aligned} \| \nu((f) &\leqq \| f \| \sum_{i=1}^n \| \nu(e_i) \| + M \| f - \sum_{i=1}^n f(\omega_i)e_i \| \\ &\leqq \left[\sum_{i=1}^n \| \nu(e_i) \| + M(n+1) \right] \| f \| , \end{aligned}$$

thus demonstrating the continuity of ν.

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