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PROJECTIONS IN $\mathscr{L}_1$ AND $\mathscr{L}_\infty$-SPACES

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This paper is devoted to the study of bounded Boolean algebras of projections in the spaces L_1 and L_∞, or more generally the spaces $\mathscr{L}_1$ and $\mathscr{L}_\infty$, of Lindenstrauss and Petczyński.

Our results in $\mathscr{L}_1$ show that such algebras of projections are in every way analogous to algebras of projections in Hilbert space: theorems concerning the common refinement of two (or more) commuting such algebras, or limits of scalar operators, are just as true as in Hilbert space. It is also shown that every cyclic subspace is a direct summand isomorphic to an L_1-space and under the additional assumption of finite multiplicity the Boolean algebra of projections is isomorphic to a subalgebra of multiplications by characteristic functions on some L_1-space.

For $\mathscr{L}_\infty$-spaces, we show that strongly σ-complete bounded Boolean algebras of projections are isomorphic to rather trivial algebras on c_0 and under the additional assumption that the underlying space is a $\mathscr{P}$-space it is proved that such algebras of projections have a finite number of elements.

1. Preliminaries. We will start by summarizing here some notations, definitions and results which will be useful in what follows. Most of the results in the subsequent sections are concerned with Boolean algebras of projections. A Boolean algebra of projections $\mathscr{E}$ will be called *complete* if for every family $(E_\alpha) \subset \mathscr{E}$ the projections $\bigvee E_\alpha$ and $\bigwedge E_\alpha$ exist in $\mathscr{E}$ and, moreover

$$(\bigvee E_\alpha)X = \operatorname{clm}\{E_\alpha X\}$$
$$(\bigwedge E_\alpha)X = \bigcap E_\alpha X\ .$$

A projection $E \in \mathscr{E}$ will be called *countably decomposable* if every family of disjoint projections in $\mathscr{E}$ bounded by E is at most countable. Bade [2, Lemma 3.1] proved that for every $E \in \mathscr{E}$ there is a family of disjoint countably decomposable projections $E_j \in \mathscr{E}$ such that $E = \bigvee E_j$. For each $x \in X$ the projection $C(x) = \bigwedge \{E \mid E \in \mathscr{E}, Ex = x\}$ will be called the *carrier projection of* x. The *cyclic subspace* $\mathfrak{M}(x)$ spanned by a vector x is $\operatorname{clm}\{Ex \mid E \in \mathscr{E}\}$. Bade introduced in [2] the multiplicity function $m(\cdot)$ for a complete Boolean abgebra (B.A.) of projections as follows: if $E \in \mathscr{E}$ is countably decomposable, the *multiplicity* of E, $m(E)$, is the smallest cardinal of a set A of vectors such that

$$EX = \operatorname{clm}\{\mathfrak{M}(x) \mid x \in A\} .$$

A projection $E \in \mathscr{E}$ is said to have uniform multiplicity n (not necessarily finite) if $m(F) = n$ whenever $0 \neq F \leqq E$.

Next, we shall introduct some definitions concerning the underlying space X which mostly are due to J. Lindenstrauss and A. Pełczyński [16]. Two Banach spaces X and Y are isomorphic if there exists an invertible operator from X onto Y. The distance between two Banach spaces X and Y is defined as follows:

$$d(X, Y) = \inf \| T \| \cdot \| T^{-1} \|$$

where the infimum is taken over all invertible operators T mapping X onto Y (operator means bounded linear operator) if such operators exist; $d(X, Y) = \infty$ if X and Y are not isomorphic. d, which is not a metric, is used instead of $\log d$ which is a metric.

We will often deal $L_p(\Omega, \Sigma, \mu)$ spaces, $1 \leqq p \leqq +\infty$ i.e., the spaces of measurable functions f on some measure space (Ω, Σ, μ) with the norm

$$\| f \|_p = \left(\int_\Omega | f(\omega) |^p \mu(d\omega)\right)^{1/p} , \qquad 1 \leqq p < +\infty ,$$
$$\| f \|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} | f(\omega) | , \qquad p = +\infty ;$$

and with the spaces $C(K)$ of all continuous functions on a compact Hausdorff topological space K. For an abstract set Γ, $l_p(\Gamma)$ will denote the Banach space of all functions f defined on Γ for which

$$\| f \|_p = \left(\sum_\Gamma | f(\gamma) |^p\right)^{1/p} < +\infty ; \qquad 1 \leqq p < +\infty$$
$$\| f \|_\infty = \sup_\Gamma | f(\gamma) | < +\infty ; \qquad p = +\infty .$$

If Γ is countable we denote $l_p(\Gamma)$ by $l_p; 1 \leqq p < +\infty$ and $l_\infty(\Gamma)$ by l_∞ or m. When Γ consists of a finite number n of elements $l_p(\Gamma)$ will be denoted by l_p^n. An important subspace of $l_\infty(\Gamma)$ is $c_0(\Gamma)$ which consists of those $f \in l_\infty(\Gamma)$ for which the set $\{\gamma \mid | f(\gamma) | \geqq \varepsilon\}$ is finite for every $\varepsilon > 0$. When Γ is countable $c_0(\Gamma)$ will be denoted, as usual, by c_0.

A subspace Y of a Banach space X is called complemented if there is a bounded projection from X onto Y. A Banach space is said to be a $\mathscr{P}$ space if it is complemented in every Banach space containing it. The following definition is due to Lindenstrauss and Pełczyński [16].

DEFINITION 1. A Banach space X is called an $\mathscr{L}_{p,\lambda}$ space, $1 \leqq p \leqq +\infty; 1 \leqq \lambda < +\infty$; provided that for every finite-dimensional subspace Y of X there is a finite-dimensional subspace $Z \supset Y$ such that $d(Z, l_p^n) \leqq \lambda$ where $n = \dim Z$. A Banach space X is called an $\mathscr{L}_p$ space $1 \leqq p \leqq \infty$ if it is an $\mathscr{L}_{p,\lambda}$ space for some $\lambda < \infty$.

It is known that every $L_p(\Omega, \Sigma, \mu)$ and in particular $l_p(\Gamma)$ and l_p are $\mathscr{L}_{p,\lambda}$ spaces for every $\lambda > 1$ but there exist $\mathscr{L}_p$ spaces which are not isomorphic to $L_p(\mu)$ spaces for $1 \leqq p \leqq +\infty; p \neq 2$ while $\mathscr{L}_2$ spaces coincide with the class of spaces isomorphic with Hilbert spaces. Furthermore, it was proved by A. Lazar and J. Lindenstrauss [14] that every Banach space X whose dual X^* is isometric to a space $L_1(\Omega, \Sigma, \mu)$ is an $\mathscr{L}_{\infty,\lambda}$ space for every $\lambda > 1$. *In particular every* $C(K)$ *space is an* $\mathscr{L}_\infty$*-space.* It is not yet known whether a complemented subspace of an $\mathscr{L}_1$ or $\mathscr{L}_\infty$-space is of the same type.

Finally, we will make use of a result of J. Lindenstrauss and A. Pełczyński [16, Corollary 8 of Th. 6.1]. Stated in the form we need it is:

THEOREM 2. *Let X be a complemented subspace of an $\mathscr{L}_1$-space* (resp. $\mathscr{L}_\infty$) *and $\mathscr{E}$ a bounded* B.A. *projections. Then there exists a constant* M_1 (resp. M_2) *such that for every finite family of disjoint projections* $E_k \in \mathscr{E}$, $k = 1, \cdots, n$,

$$\sum_{k=1}^n \| E_k x \| \leqq M_1 \left\| \left(\sum_{k=1}^n E_k \right) x \right\| ; \qquad x \in X .$$

$$\left(\text{resp. } \left\| \left(\sum_{k=1}^n E_k \right) x \right\| \leqq M_2 \max_{1 \leqq x \leqq n} \| E_k x \| ; \, x \in X \right) .$$

2. Commuting B. A. of projections. A basic problem in Dunford's theory of spectral operators (see [7]) is to find conditions, which being fulfilled, insure that the sum and the product of two commuting spectral operators are also spectral. N. Dunford [7, Th. 19] and S. R. Foguel [9, Th. 7] have proved that if the B.A. of projections generated by the resolutions of the identity of two commuting spectral operators on a weakly complete Banach space is bounded, then the sum and the product of these operators are spectral. Therefore, it is important to determine conditions under which the B.A. of projections generated by two bounded commuting B.A. of projections is bounded. One of us has proved in [19] that it suffices that the underlying space be a subspace of an $L_p(\Omega, \Sigma, \mu)$ space; $2 \leqq p < +\infty$ and W. Littman, C. McCarthy and N. Riviere [17, Th. 6.2 and Th. 6.8] have proved the same for $1 \leqq p < 2$.

The following theorem completes these results.

THEOREM 3. *Let X be a complemented subspace of an $\mathscr{L}_p$-space;*

$1 \leqq p \leqq +\infty$, $\mathscr{E}$ and $\mathfrak{F}$ two bounded commuting B.A. *of projections* ($\|E\| \leqq L_1$ *for* $E \in \mathscr{E}$ *and* $\|F\| \leqq L_2$ *for* $F \in \mathfrak{F}$). *Then, the* B.A. *of projections generated by* $\mathscr{E}$ *and* $\mathfrak{F}$ *is bounded.*

Proof. For $1 \leqq p < +\infty$, J. Lindenstrauss and A. Pełczyński [16], Th. 7.1] have shown that X is isomorphic to a complemented subspace of an $L_p(\Omega, \Sigma, \mu)$ space. Thus, the proof follows from the results mentioned above. In the case $p = 1$ we will give a direct proof. Let us remark that it suffices to prove the existence of a bound for the norm of every expression $\bigvee_{k=1}^{n} E_k F_k$ which is independent of n and the particular projections $E_k \in \mathscr{E}$ and $F_k \in \mathfrak{F}$. With no loss of generality we can assume that $E_k; 1 \leqq k \leqq n$ are disjoint projections and $\sum_{k=1}^{n} E_k = I$. Then, by Theorem 2.

$$\left\|\sum_{k=1}^{n} E_k F_k x\right\| \leqq \sum_{k=1}^{n} \| E_k F_k x \| \leqq L_2 \sum_{k=1}^{n} \| E_k x \| \leqq L_2 M_1 \| x \| ; \qquad x \in X .$$

Finally, when $p = +\infty$ same arguments show that

$$\begin{aligned} \left\|\sum_{k=1}^{n} E_k F_k x\right\| &= \left\|\sum_{j=1}^{n} E_j \Big(\sum_{k=1}^{n} E_k F_k x\Big)\right\| \\ &\leqq M_2 \max_{i \leqq j \leqq n} \| E_j F_j x \| \leqq M_2 L_1 L_2 \| x \| ; \qquad x \in X \end{aligned}$$

which completes the theorem.

Theorem 3 can be improved as follows:

THEOREM 4. *Let X be a complemented subspace of an $\mathscr{L}_1$ or $\mathscr{L}_\infty$ space and $\{\mathscr{E}_n\}$ a sequence of bounded commuting* B.A. *of projections such that $\| E \| \leqq L_n$ for $E \in \mathscr{E}_n$; $n = 1, 2, \cdots$. Suppose the infinite product $\prod_{n=i}^{\infty} L_n$ converges to L. Then, the* B.A. *of projections generated by $\mathscr{E}_n$ is bounded.*

Proof. Assume that X is a complemented subspace of an $\mathscr{L}_1$-space. One can easily see that it is enough to prove the existence of a bound for the norm of every expression

$$\bigvee \{E_{1i_1} E_{2i_2} \cdots E_{ni_n} \mid 1 \leqq i_k \leqq r; 1 \leqq k \leqq n\}$$

the bound independent of r and the choice of $E_{ki_k} \in \mathscr{E}_k$. Again, with no loss of generality we can suppose that $E_{11}, \cdots, E_{1r}$ are disjoint projections whose sum is I. Then by Theorem 2

$$\begin{aligned} \| \textstyle\sum E_{1i_1} \cdots E_{ni_n} x \| &\leqq \textstyle\sum \| E_{1i_1} \cdots E_{ni_n} x \| \\ &\leqq L_2 \cdots L_n \textstyle\sum \| E_{1i_1} x \| \leqq L M_1 \| x \| ; \qquad x \in X . \end{aligned}$$

A similar argument may be used when X is a complemented subspace of an $\mathscr{L}_\infty$-space.

COROLLARY 5. *Let $\{S_n\}$ be a sequence of commuting spectral operators of scalar type on an $L_1(\Omega, \Sigma, \mu)$ space converging weakly to an operator S. If the resolutions of the identity for S_n satisfy $\| E_n(\cdot) \| \leqq L_n$; $n = 1, 2, \cdots$ and $\underline{\lim}_n L_n = 1$, then S is scalar. Furthermore, if S is the strong limit of $\{S_n\}$, $E(\cdot)$ its resolution of the identity and $\prod_{n=1}^{\infty} L_n$ converges then*

$$E(\delta)x = \lim_{n\to\infty} E_n(\delta)x$$

for every Borel set δ in the complex plane for which E (boundary δ) $x = 0$.

Proof. First, observe that we can assume with no loss of generality that $\prod_{n=1}^{\infty} L_n < \infty$ since every sequence converging to 1 contains a subsequence whose infinite product is convergent. Let $\mathfrak{B}$ be the B.A. of projections generated by $\mathscr{E}_n$ which is bounded due to Theorem 4. Obviously S belongs to the algebra of operators generated by $\mathfrak{B}$ in the weak operator topology. Since the underlying space is weakly complete, it follows from Bade [1, Th. 4.5] that S belongs also the algebra of operators generated by $\mathfrak{B}$ in the uniform operator topology, and then by a result of Dunford [7] it is a spectral operator of scalar type. The second assertion follows from Foguel [11].

3. Structure of cyclic subspaces of an $\mathscr{L}_1$-space. Throughout this section X will denote a complemented subspace of an $\mathscr{L}_1$-space, $\|\cdot\|$ its norm and $\mathscr{E}$ a complete B.A. of projections on X. It is well-known that $\mathscr{E}$ can be considered as the range of a spectral measure $E(\cdot)$ defined on the Borel sets of a compact Hausdorff topological space Λ.

LEMMA 6. *Let*

$$| x | = \sup \sum_{i=1}^{n} \| E(\delta_i)x \| \,; \qquad x \in X$$

where the supremum is taken over all finite partitions of Λ into disjoint Borel sets $\delta_1, \cdots, \delta_n$. Then $|\cdot|$ is a norm on X, equivalent to $\|\cdot\|$ and such that:

(a) $| E | \leqq 1$ *for* $E \in \mathscr{E}$,

(b) $| E(\cdot)x |$ *is a σ-additive finite positive Borel measure on Λ for every $x \in X$,*

(c) $\left| \int f(\lambda)E(d\lambda)x \right| = \int | f(\lambda) | \, | E(d\lambda)x |$

whenever at least one of the integrals exists.

Proof. Using Theorem 2 we have

$$\|x\| \leqq |x| \leqq M_1 \|x\|$$

so $|\cdot|$ is an equivalent norm. Furthermore, for every partition $\delta_1, \cdots, \delta_n$ and $E(\sigma) \in \mathscr{E}$ we have

$$\sum_{i=1}^{n} \| E(\delta_i)E(\sigma)x \| \leqq \sum_{i=1}^{n} \{\| E(\delta_i \cap \sigma)x \| + \| E(\delta_i \cap \sigma')x \|\}$$

where $\sigma' = \Lambda - \sigma$. Thus $|E(\sigma)x| \leqq |x|$ and (a) is proved. The finite additivity of $|E(\cdot)x|$ is evident and σ-additivity follows from the fact that if $\sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_n \supset \cdots$ and $\bigcap_{n=1}^{\infty} \sigma_n = \varnothing$ then

$$|E(\sigma_n)x| \leqq M_1 \| E(\sigma_n)x_1 \| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \, .$$

The last assertion (c) can be proved easily for simply functions as follows:

$$\begin{aligned} \left| \sum_{i=1}^{n} \alpha_i E(\sigma_i)x \right| &= \left| \sum_{j=1}^{n} E(\sigma_j) \sum_{i=1}^{n} \alpha_i E(\sigma_i)x \right| \\ &= \sum_{j=1}^{n} |\alpha_j E(\sigma_j)x| = \sum_{j=1}^{n} |\alpha_j| \, |E(\sigma_j)x| \, . \end{aligned}$$

THEOREM 7. *Every cyclic subspace (with respect to $\mathscr{E}$) $\mathfrak{M}(x)$; $x \in X$ is complemented. Moreover, there exists a projection P of X onto $\mathfrak{M}(x)$ which commutes with $\mathscr{E}$ and has norm* 1 *with respect to* $|\cdot|$.

Proof. First, assume that $x_0 \in X$ has carrier projection $C(x_0) = I$. According to Bade [2, Th. 4.5] for every $x \in \mathfrak{M}(x_0)$ there is a Borel function f (not necessarily bounded) such that

$$x = \int f(\lambda)E(d\lambda)x_0$$

and x_0 belongs to the domain of the unbounded operator

$$\int f(\lambda)E(d\lambda) \, .$$

In view of Lemma 6 part (c) define

$$\theta(x) = \int f(\lambda) \, |E(d\lambda)x_0| \, ; \qquad x \in \mathfrak{M}(x_0) \, .$$

It is obvious that θ is a linear functional on $\mathfrak{M}(x_0)$ and

$$|\theta(x)| \leqq \int |f(\lambda)| \, |E(d\lambda)x_0| = \left| \int f(\lambda)E(d\lambda)x_0 \right| = |x| \, ;$$

i.e., $|\theta| \leqq 1$. By Hahn-Banach theorem it can be extended to a bounded linear functional on X, denoted x_0^*, which satisfies $|x_0^*| \leqq 1$. We remark that x_0^* is a Bade functional (see Bade [1, Th. 3.1]), in fact it satisfies:

(i) $x_0^* E(\sigma)x_0 = |E(\sigma)x_0| \geqq 0$ for $E(\sigma) \in \mathscr{E}$,

(ii) if for any $E(\sigma) \in \mathscr{E}$, $x_0^* E(\sigma)x_0 = 0$, then $E(\sigma)x_0 = 0$.

Now suppose $x \in X$. Since the measure $x_0^* E(\cdot)x$ is absolutely continuous with respect to the measure $x_0^* E(\cdot)x_0$ (due to the fact that the carrier projection of x_0 is I), there exists a function $h \in L_1(x_0^* E(\cdot)x_0)$ such that

$$x_0^* E(\sigma)x = \int_\sigma h(\lambda)x_0^* E(d\lambda)x_0 = \int_\sigma h(\lambda)\,|E(d\lambda)x_0|$$

for every Borel set σ. Since $\int h(\lambda)E(d\lambda)x_0$ exists in view of Lemma 6 part (c) one can write

$$x = \int h(\lambda)E(d\lambda)x_0 + \left[x - \int h(\lambda)E(d\lambda)x_0\right].$$

Set $\mathfrak{M}(x_0^*) = \operatorname{clm}\{E^* x_0^* \mid E \in \mathscr{E}\}$ and notice that

$$\int h(\lambda)E(d\lambda)x_0 \in \mathfrak{M}(x_0)$$

and

$$x - \int h(\lambda)E(d\lambda)x_0 \in \mathfrak{M}(x_0^*)^\perp ,$$

since for every $E(\sigma) \in \mathscr{E}$ we have

$$[E^*(\sigma)x_0^*]\left[x - \int h(\lambda)E(d\lambda)x_0\right] = x_0^* E(\sigma)x - \int_\sigma h(\lambda)x_0^* E(\sigma)x_0 = 0 .$$

Thus

$$X = \mathfrak{M}(x_0) \oplus \mathfrak{M}(x_0^*)^\perp$$

which proves the existence of a projection P of X onto $\mathfrak{M}(x_0)$ commuting with $\mathscr{E}$. Using again Lemma 6 part (a) and (c) we have

$$\begin{aligned}\left|\int h(\lambda)E(d\lambda)x_0\right| = \int |h(\lambda)|\,|E(d\lambda)x_0| &= \text{tot. var. } x_0^* E(\cdot)x \\ &\leqq \text{tot. var. } |E(\cdot)x| \leqq |x|\end{aligned}$$

i.e., $|P| \leqq 1$.

If x_0 has carrier projection $C(x_0) \neq I$ then, the first part of the proof insures the existence of a projection Q of $C(x_0)X$ onto $\mathfrak{M}(x_0)$ and, hence $QC(x_0)$ will be a projection of X onto $\mathfrak{M}(x_0)$ having all wanted properties.

REMARK. $I - P$ has norm at most 2 and may have $|\cdot|$-norm as great as 2. For instance, consider the two-dimensional space l_1^2, $\mathscr{E} = \{0, I\}$, $x_0 = (1, 0)$ and $x_0^* = (1, 1)$. Then the projection $I - P$ has norm exactly 2.

COROLLARY 8. *The second commutant* $(\mathscr{E}^c)^c$ *of* $\mathscr{E}$ *coincides with the algebra* $\mathfrak{A}(\mathscr{E})$ *generated by* $\mathscr{E}$ *in the uniform operator topology.*

Proof. Let $T \in (\mathscr{E}^c)^c$. Since every cyclic subspace $\mathfrak{M}(x)$ admits a projection $P \in \mathscr{E}^c$ it follows that $T\mathfrak{M}(x) \subseteqq \mathfrak{M}(x)$; $x \in X$ and further $TY \subseteqq Y$ for every subspace Y of X which is invariant under E. Hence, by a well-known result to Bade [1, Th. 4.3] $T \in \mathfrak{A}(\mathscr{E})$.

REMARK. This is not true in general. J. Dieudonné [6] has constructed an example of a B.A. of projections $\mathfrak{F}$ of finite uniform multiplicity ($n = 2$) for which $\mathfrak{A}(\mathfrak{F})$ is a proper subalgebra of $(\mathfrak{F}^c)^c$.

COROLLARY 9. *If* A *is a scalar operator in a separable complemented subspace of an* $\mathscr{L}_1$*-space, every operator* T *which commutes with every operator commuting with* A *is a Borel function of* A.

Proof. If the underlying space is separable the resolution of the identity for A is a complete B.A. of projections and the proof follows from Corollary 8.

THEOREM 10. *For every cyclic subspace* $\mathfrak{M}(x)$; $x \in X$ *there exists a positive finite Borel measure space* $(\Lambda, \mathfrak{B}, \mu)$ *such that* $\mathfrak{M}(x)$ *is isomorphic to* $L_1(\Lambda, \mathfrak{B}, \mu)$. *Moreover, the image of the restriction of* $\mathscr{E}$ *to* $\mathfrak{M}(x)$ *under this isomorphism is the* B.A. *of projections consisting of multiplications by characteristic functions in* $L_1(\Lambda, \mathfrak{B}, \mu)$.

Proof. As we have already mentioned, for every $x \in \mathfrak{M}(x_0)$ there is an Borel function f such that

$$x = \int f(\lambda) E(d\lambda) x_0 \,.$$

By Lemma 6 part (c) it follows that $f \in L_1(\Lambda, \mathfrak{B}, |E(\cdot)x_0|)$ and

$$|x| = \left|\int f(\lambda) E(d\lambda) x_0\right| = \int |f(\lambda)|\,|E(d\lambda)x_0|$$

which implies that the correspondence $x \leftrightarrow f$ is an isometry between $\mathfrak{M}(x_0)$ with the norm $|\cdot|$ and $L_1(\Lambda, \mathfrak{B}, |E(\cdot)x_0|)$. Thus $\mathfrak{M}(x_0)$ with the original norm $\|\cdot\|$ is isomorphic to $L_1(\Lambda, \mathfrak{B}, |E(\cdot)x_0)$. The last assertion is obvious.

From Theorems 7 and 10 immediately follows:

COROLLARY 11. *Assume $\mathscr{E}$ is a countably decomposable* B.A. *of projections having finite uniform multiplicity N. Then there exist N vectors $x_k \in X$, $k = 1, \cdots, N$, such that*

$$X = \mathfrak{M}(x_1) \oplus \cdots \oplus \mathfrak{M}(x_N) .$$

Furthermore, there is a positive finite Borel measure space $(\Lambda, \mathfrak{B}, \mu)$ such that X is isomorphic to $L_1(\Lambda, \mathfrak{B}, \mu)$ and under this isomorphism every $E \in \mathscr{E}$ corresponds to a multiplication by a characteristic function in $L_1(\Lambda, \mathfrak{B}, \mu)$.

THEOREM 12. *A complemented subspace X of an $\mathscr{L}_1$-space is isomorphic to an $L_1(\nu)$-space if and only if there exists on X a complete* B.A. *of projections $\mathfrak{F}$ having multiplicity* 1.

Proof. If X is isomorphic to an $L_1(\nu)$-space then the image under this isomorphism of the B.A. of projections in $L_1(\nu)$ consisting of multiplications by characteristic functions will be a complete B.A. of projection in X having multiplicity 1.

Conversely, assume that $\mathfrak{F}$ is a complete B.A. of projections on X having multiplicity 1 and let $|\cdot|$ be again the equivalent norm whose existence is insured by Lemma 6. By Bade [2, Lemma 3.1],

$$I = \bigvee_{\gamma} F(\sigma_\gamma)$$

where $F(\sigma_\gamma)$ are disjoint countably decomposable projections belong to $\mathfrak{F}$. Further, by Bade [2, Th. 3.4],

$$F(\sigma_\gamma)X = \mathfrak{M}(x_\gamma) , \qquad x_\gamma \in F(\sigma_\gamma)X .$$

Then for every $x \in X$ we have $x = \sum_\gamma F(\sigma_\gamma)x$ where at most countably many of the vectors $F(\sigma_\gamma)x$ are different from zero. Observe that

$$F(\sigma_\gamma)x = \int_{\sigma_\gamma} f_\gamma(\lambda) F(d\lambda) x_\gamma$$

for some Borel function $f_\gamma \in L_1(\Lambda, \mathfrak{B}, |F(\cdot)x_\gamma|)$ (see Theorem 10). Now define

$$f(\lambda) = f_\gamma(\lambda) \qquad \lambda \in \sigma_\gamma .$$

Since $I = \bigvee_\gamma F(\sigma_\gamma)$, f is defined everywhere in Λ. For any Borel set δ for which $\sum_\gamma |F(\delta)x_\gamma|$ converges let us put

$$\nu(\delta) = \sum_\gamma |F(\delta)x_\gamma| = \sum_\gamma |F(\delta \cap \sigma_\gamma)x_\gamma| ;$$

otherwise, $\nu(\delta) = +\infty$. Then

$$\begin{aligned} |x| &= \sum_{\gamma} |F(\sigma_\gamma)x| = \sum_{\gamma} \left| \int_{\sigma_\gamma} f_\gamma(\lambda) F(d\lambda) x_\gamma \right| \\ &= \sum_{\gamma} \int_{\sigma_\gamma} |f(\lambda)| \, |F(d\lambda)x_\gamma| = \int |f(\lambda)| \, \nu(d\lambda) , \qquad x \in X , \end{aligned}$$

since $\nu(\delta) = |F(\delta)x_\gamma|$ for $\delta \subset \sigma_\gamma$ and the infinite sum $\sum_\gamma |F(\sigma_\gamma)x|$ is in fact countable. It follows that the correspondence $x \leftrightarrow f$ is an isometry from X with the norm $|\cdot|$ and $L_1(\Lambda, \mathfrak{B}, \nu)$. Thus X is isomorphic to $L_1(\Lambda, \mathfrak{B}, \nu)$.

Grothendieck [12] has proved that if a projection in an L_1-space is contractive, i.e., it has norm $\leqq 1$, then its range is isometric to an L_1-space. The following result is a necessary and sufficient condition for the range to be *isomorphic* to an L_1-space.

COROLLARY 13. *Let P be a projection in X. Then PX is isomorphic to an L_1-space if aud only if P can be imbedded in a complete* B.A. *of projections in X with respect to which P has multiplicity* 1.

Proof. This corollary follows immediately from the preceding theorem when applied to PX.

4. Structure of cyclic subspaces of an $\mathscr{L}_\infty$-space.

Throughout this section X will denote a complemented subspace of an $\mathscr{L}_\infty$-space; $\|\cdot\|$ its norm and $\mathscr{E}$ a B.A. of projections in X. As in the previous section $\mathscr{E}$ will be considered as the range of a spectral measure $E(\cdot)$ defined on the Borel sets of a compact Hausdorff topological space Λ.

THEOERM 14. *For every σ-complete* B.A. *of projections $\mathscr{E}$ in X and $x \in X$ the vector valued measure $E(\cdot)x$ is atomic, i.e., if Γ is the set of all points λ_γ in Λ for which $E(\{\lambda_\gamma\}) \neq 0$ then for every Borel set δ we have*

$$E(\delta)x = \sum_{\lambda_\gamma \in \delta \cap \Gamma} E(\{\lambda_\gamma\})x$$

where at most countably many of the vectors $E(\{\lambda_\gamma\})x$ are different from zero.

Proof. First, notice that by Bade [1, Lemma 2.6] the restriction of $\mathscr{E}$ to $\mathfrak{M}(x)$ is a complete countably decomposable B.A. of projections so at most countably many of the vectors $E(\{\lambda_\gamma\})x$ are different from zero. If $\{\lambda_n\}$ is the sequence of all λ_γ for which $E(\{\lambda_\gamma\})x \neq 0$ then

$$E\Big(\bigcup_{n=1}^{\infty} \{\lambda_n\}\Big)x = \sum_{n=1}^{\infty} E(\{\lambda_n\})x .$$

Now in order to prove the theorem observe that it is enough to show that $E(\bigcup_{n=1}^{\infty}\{\lambda_n\})x = x$. Set $E(\Lambda_0) = I - E(\bigcup_{n=1}^{\infty}\{\lambda_n\})$ and assume that $E(\Lambda_0)x \neq 0$. Let x_0^* be the Bade functional for $E(\Lambda_0)x$ whose existence has been proved in [1, Th. 3.1]. Obviously, the measure $E(\cdot)E(\Lambda_0)x$ is nonatomic and consequently the positive measure $\mu(\cdot) = x_0^*E(\cdot)E(\Lambda_0)x$ is nonatomic and its support is Λ_0. By a well-known result (see for instance Halmos [13, Exercise 2 p. 174]), Λ_0 can be decomposed for each n into disjoint sets $\Lambda_k^{(n)}k = 1, \cdots, 2^n$ such that

$$\mu(\Lambda_k^{(n)}) = 2^{-n}\mu(\Lambda_0) , \qquad \Lambda_{2k-1}^{(n)} \cup \Lambda_{2k}^{(n)} = \Lambda_k^{(n-1)} .$$

By Theorem 2, for each n there exists at least one $k(n)$ for which

$$\| E(\Lambda_n)x \| \geqq \frac{1}{M_2} \| E(\Lambda_0)x \| \qquad (\Lambda_n = \Lambda_{k(n)}^{(n)}) .$$

Let $\sigma_n = \bigcup_{k=n+1}^{\infty}\Lambda_k$; then $\Lambda_0 \supset \sigma_1 \supset \sigma_2 \cdots$ and $\mu(\sigma_n) \leqq 2^{-n}\mu(\Lambda_0)$ and consequently $\mu(\bigcap_{n=1}^{\infty}\sigma_n) = 0$. This implies, in view of the properties of Bade functionals that $E(\bigcap_{n=1}^{\infty}\sigma_n)x = 0$. But denoting by M the bound of $\| E \|$, $E \in \mathscr{E}$, we have

$$\frac{1}{M_2} \| E(\Lambda_0)x \| \leqq \| E(\Lambda_n)x \| = \| E(\Lambda_n)E(\sigma_{n-1})x \| \leqq M \| E(\sigma_{n-1})x \| ;$$

but by the σ-additivity of the vector valued measure $E(\cdot)x$,

$$0 = \left\| E\left(\bigcap_{n=1}^{\infty}\sigma_n\right)x \right\| \geqq \frac{1}{MM_2} \| E(\Lambda_0)x \| \gneqq 0 .$$

which provides a contradiction.

COROLLARY 15. *Let S be a spectral operator of scalar type on X and $\{\lambda_\gamma\}; \gamma \in \Gamma$ the set of its eigenvalues. Then*

$$Sx = \sum_{\gamma \in \Gamma} \lambda_\gamma E(\{\lambda_\gamma\})x ; \qquad x \in X$$

where at most countably many of the vectors $E(\{\lambda_\gamma\})x$ are different from zero for every $x \in X$.

Proof. It suffices to apply the preceding theorem to the resolution of the identity for S, taking into account that by Foguel [10, Th. 1], $E(\{\lambda\}) \neq 0$ if and only if λ is an eigenvalue of S.

A particular $\mathscr{L}_\infty$-space is c_0. In this space one can consider the natural B.A. of projections generated by the following projections

$$E(n)(x_1, x_2, \cdots, x_n, \cdots) = (0, \cdots, 0, x_n 0, \cdots) ; \qquad n = 1, 2, \cdots .$$

Obviously, this is a complete B.A. of projections bounded by 1 and having multiplicity 1, i.e.,

$$c_0 = \mathfrak{M}(u)$$

where u can be, for example, the sequence $(1, 1/2, \cdots, 1/n, \cdots)$.

We shall now that this is essentially the general case.

THEOREM 16. *Every cyclic subspace $\mathfrak{M}(x)$; $x \in X$ is either finite-dimensional or isomorphic to c_0. Moreover, under this isomorphism the restriction of $\mathscr{E}$ to $\mathfrak{M}(x)$ corresponds to the above mentioned natural* B.A. *of projections in c_0.*

Proof. Let $x_0 \in X$ and $x = \int f(\lambda)E(d\lambda)x_0 \in \mathfrak{M}(x_0)$. Using Theorem 15 we have

$$E(\{\lambda_\gamma\})x = f(\lambda_\gamma)E(\{\lambda_\gamma\})x_0 \; ; \qquad \gamma \in \Gamma \; ,$$

and again denote by $\{\lambda_n\}$ the sequence of all $\lambda_\gamma, \gamma \in \Gamma$, for which $E(\{\lambda_\gamma\})x_0 \neq 0$. Then

$$x = \sum_n f(\lambda_n)E(\{\lambda_n\})x_0$$

which implies that

$$\lim_{n\to\infty} f(\lambda_n) \, || E(\{\lambda_n\})x_0 || = 0 \; .$$

Now define

$$\tau x = (f(\lambda_1) \, || E(\{\lambda_n\})x_0 ||, \cdots, f(\lambda_n) \, || E(\{\lambda_n\})x_0 ||, \cdots) \in c_0 \; ; \qquad x \in \mathfrak{M}(x_0)$$

(if this sequence is finite for every f one can see that $\mathfrak{M}(x_0)$ is finite dimensional). In view of Theorem 2 we have

$$|| x || \leqq M_2 \sup_n || E(\{\lambda_n\})x || = M_2 \sup_n | f(\lambda_n) | \, || E(\{\lambda_n\})x_0 || \; ,$$

and if M denotes the bound of $|| E ||$ for $E \in \mathscr{E}$

$$M || x || \geqq || E(\{\lambda_n\})x || = | f(\lambda_n) | \, || E(\{\lambda_n\})x_0 || \; .$$

Thus

$$\frac{1}{M} \sup_n | f(\lambda_n) | \, || E(\{\lambda_n\})x_0 || \leqq || x || \leqq M_2 \sup_n | f(\lambda_n) | \, || E(\{\lambda_n\})x_0 ||$$

and τ is an isomorphism from $\mathfrak{M}(x_0)$ into c_0 under which the restriction of $\mathscr{E}$ to $\mathfrak{M}(x_0)$ corresponds to the natural B.A. of projections in c_0. To complete the proof we have to show that τ is onto c_0. We consider a sequence $(a_1, \cdots, a_n, \cdots) \in c_0$ and prove that

$$\sum_n a_n \frac{E(\{\lambda_n\})x_0}{\|E(\{\lambda_n\})x_0\|}$$

is a convergent series. Indeed

$$\left\|E(\{\lambda_k\})\left[\sum_{n=p}^{q} a_n \frac{E(\{\lambda_n\})x_0}{\|E(\{\lambda_n\})x_0\|}\right]\right\| = |a_k|, \qquad p \leqq k \leqq q ,$$

and then by Theorem 2

$$\left\|\sum_{n=p}^{q} a_n \frac{E(\{\lambda_n\})x_0}{\|E(\{\lambda_n\})x_0\|}\right\| \leqq M_2 \max_{p \leqq k \leqq q} |a_k| \to 0 \quad \text{as} \quad p \to \infty .$$

If y is the sum of this series, evidently $y \in \mathfrak{M}(x_0)$ and $\tau y = (a_1, \cdots, a_n, \cdots)$ which completes the proof.

THEOREM 17. *Every cyclic subspace $\mathfrak{M}(x)$; $x \in X$ is complemented. Moreover, there exists a projection P of X onto $\mathfrak{M}(x)$ which commutes with $\mathscr{E}$.*

Proof. We shall keep the notation introduced in the proof of Theorem 16. Obviously, $\mathfrak{M}(E(\{\lambda_n\})x_0)$ is an one-dimensional subspace of $E(\{\lambda_n\})X$. Hence it is complemented, i.e., there exists a subspace $\mathfrak{M}_n$ of $E(\{\lambda_n\})X$ such that

$$E(\{\lambda_n\})X = \mathfrak{M}(E(\{\lambda_n\})x_0) \oplus \mathfrak{M}_n , \qquad n = 1, 2, \cdots ,$$

and the projection of $E(\{\lambda_n\})X$ onto $\mathfrak{M}(E(\{\lambda_n\})x_0)$ has norm one. Then for every $x \in X$ we have a decomposition

$$E(\{\lambda_n\})x = a_n E(\{\lambda_n\})x_0 + y_n , \qquad n = 1, 2, \cdots ,$$

where $y_n \in \mathfrak{M}_n$ and

$$|a_n| \, \|E(\{\lambda_n\})x_0\| \leqq \|E(\{\lambda_n\})x\| , \qquad n = 1, 2, \cdots .$$

Consequently,

$$\lim_{n \to \infty} |a_n| \, \|E(\{\lambda_n\})x_0\| = 0$$

and by arguments already used in the proof of the previous theorem we see that $\sum_n a_n E(\{\lambda_n\})x_0$ is a convergent series. It follows that $\sum_n y_n$ also converges and

$$x = \sum_n a_n E(\{\lambda_n\})x_0 + \sum_n y_n .$$

Thus

$$X = \mathfrak{M}(x_0) \oplus \mathfrak{M}$$

where $\mathfrak{M} = (\sum_n \oplus \mathfrak{M}_n)_0$ is the direct sum of these spaces in the c_0-sense. Since $E(\{\lambda_n\})\mathfrak{M} = \mathfrak{M}_n \subseteqq \mathfrak{M}$, $\mathfrak{M}(x_0)$ and $\mathfrak{M}$ are invariant under every projection $E \in \mathscr{E}$, which completes the proof.

REMARK. Sobczyk [20] has proved that a Banach space Y which is isomorphic to c_0 is complemented in every separable Banach space containing it. This implies that a cyclic subspace $\mathfrak{M}(x), x \in X$, is complemented in X provided the carrier projection of $x, C(x)$, has separable range. But even in this case we do not know if the corresponding projection whose existence is insured by Sobczyk's result commutes with $\mathscr{E}$.

Using the same arguments as in § 3 we are able to prove the following three corollaries.

COROLLARY 18. *The second commutant $(\mathscr{E}^c)^c$ of $\mathscr{E}$ coincides with the algebra $\mathfrak{A}(\mathscr{E})$ generated by $\mathscr{E}$ in the uniform operator topology.*

COROLLARY 19. *If A is a scalar operator in a separable complemented subspace of an $\mathscr{L}_\infty$-space, every operator T which commutes with every operator commuting with A is a Borel function of A.*

COROLLARY 20. *Assume $\mathscr{E}$ is a complete countably decomposable* B.A. *of projections having finite uniform multiplicity N. Then, there exist N vectors $x_k \in X; k = 1, \cdots, N$ such that*

$$X = \mathfrak{M}(x_1) \oplus \cdots \oplus \mathfrak{M}(x_N)\ .$$

Furthermore, X is isomorphic to c_0 and under this isomorphism every $E \in \mathscr{E}$ corresponds to a natural projection in c_0.

THEOREM 21. *A complemented subspace X of an $\mathscr{L}_\infty$-space is isomorphic to $c_0(\Gamma)$ for a suitable set Γ if and only if there exists on X a complete* B.A. *of projections $\mathfrak{F}$ having multiplicity one.*

Proof. For any $\sigma \subset \Gamma$ and $f(\cdot) \in c_0(\Gamma)$ define the projections $F(\sigma)f(\cdot) = g(\cdot)$ where $g(\gamma) = f(\gamma)$ whenever $\gamma \in \sigma$ otherwise $g(\gamma) = 0$. Obviously $F(\cdot)$ form a complete B.A. of projections and $\| F(\sigma) \| \leqq 1$ for every subset $\sigma \subset \Gamma$. If σ is a countable subset of $\Gamma, F(\sigma)$ is countably decomposable and has multiplicity one. That proves that if X is isomorphic to $c_0(\Gamma)$ it admits a complete B.A. of projections having multiplicity one. Conversely, suppose $\mathfrak{F}$ is a complete B.A. of projections in X having multiplicity one and let $\lambda_\gamma, \gamma \in \Gamma$, be the atoms of $\mathfrak{F}$ whose existence and properties were discussed in Theorem

14. Evidently, $F(\{\lambda_\gamma\})$, $\gamma \in \Gamma$, is countably decomposable and therefore, by Bade [2, Th. 3.4], there exists $x_\gamma \in X$ such that $\|x_\gamma\| = 1$ and

$$F(\{\lambda_\gamma\})X = \mathfrak{M}(x_\gamma) .$$

Moreover, for any $x \in X$ we have

$$F(\{\lambda_\gamma\})x = a_\gamma x_\gamma$$

since $\mathfrak{M}(x_\gamma)$ is an one-dimensional subspace. Then

$$x = \sum_{\gamma \in \Gamma} F(\{\lambda_\gamma\})x = \sum_{\gamma \in \Gamma} a_\gamma x_\gamma$$

which implies that $\{x_\gamma\}_{\gamma \in \Gamma}$ is an unconditional basis for X. Thus, by J. Linderstrauss and A. Pełczyński [16, Corollary 2 of Th. 6.1], X is isomorphic to $c_0(\Gamma')$ for a suitable set Γ'. One can easily see that $\Gamma' = \Gamma$.

COROLLARY 22. *Let P be a projection in X. Then PX is isomorphic to a $c_0(\Gamma)$ space if and only if P can be imbedded in a complete* B.A. *of projections in X with respect to which it has multiplicity one.*

C. Bessaga and A. Pełczyński [4] have proved that for K an infinite compact metric space, $C(K)$ is isomorphic to c_0 if and only if K is homeomorphic to the space $[\alpha]$ of all ordinals $\leqq \alpha$ with the order topology for some ordinal α satisfying $\omega \leqq \alpha < \omega^\omega$ where ω denotes the first infinite ordinal number. From this result and Theorem 21 immediately follows:

COROLLARY 23. *There are no complete* B.A. *of projections having multiplicity one in the space $C(0, 1)$ and $C([\omega^\omega])$.*

Using a theorem of Sobczyk mentioned previously we can conclude that $C(0, 1)$ contains an isometric image of c_0 which is complemented. It follows that there are *nonfinite* complete B.A. of projections in $C(0, 1)$. We shall see now that this property does not hold in complemented subspaces of an $\mathscr{L}_\infty$-space which are $\mathscr{P}$-spaces or in particular conjugate spaces.

THEOREM 24. *A σ-complete* B.A. *of projections in X has a finite number of elements provided X is a $\mathscr{P}$-space. Then any spectral operator of scalar type in X can be written as a finite combination*

$$S = \sum_{i=1}^{p} \lambda_i E_i$$

for some disjoint projections E_i, $1 \leqq i \leqq p$ whose sum is the identity I.

Proof. Let $\mathfrak{G}$ be a σ-complete B.A. of proiections in X. According to Bade [1, Th. 2.7], $\mathfrak{G}$ can be imbedded in a complete B.A. of projections in X, so with no loss of generality we may suppose that $\mathfrak{G}$ is complete. By Bade [2, Lemma 3.1],

$$I = \bigvee_{\gamma \in \Gamma} G_\gamma ,$$

where $G_\gamma \in \mathfrak{G}$ are disjoint countably decomposable projections each of then being the carrier projection of a vector $0 \neq x_\gamma \in X$. Assume $\{G_{\gamma_n}\}$ is an infinite subsequence of the projections G_γ and set

$$x = \sum_{n=1}^{\infty} \frac{x_{\gamma_n}}{2^n \, || x_{\gamma_n} ||} .$$

Obviously $\mathfrak{M}(x)$ contains the vectors $\{x_{\gamma_n}\}$ which are linearly independent i.e., $\mathfrak{M}(x)$ is infinite dimensional. Hence by Theorems 16 and 17, $\mathfrak{M}(x)$ is a complemented subspace of X which is isomorphic to c_0, which contradicts the fact that X is a $\mathscr{P}$-space. Thus Γ is a finite set and I must be countably decomposable; consequently, there exists a vector $x_0 \in X$ whose carrier projections is $C(x_0) = I$. By arguments already used in this proof one can easily show that $\mathfrak{M}(x_0)$ must be a finite dimensional subspace. It follows that the family of the vectors $\{Gx_0 \mid G \in \mathfrak{G}\}$ contains a maximal linearly independent system

$$\{G_1x_0, G_2x_0, \cdots, G_sx_0\} .$$

By taking all possible intersections of projections G_i, $1 \leqq i \leqq s$, we can construct a new system of vectors (not necessarily linearly independent)

$$\{E_1x_0, E_2x_0, \cdots, E_px_0\}$$

where $0 \neq E_j \in \mathfrak{G}$, $1 \leqq j \leqq p$, are disjoint projections and

$$\{Gx_0 \mid G \in \mathfrak{G}\} \subset \operatorname{span}\{E_1x_0, \cdots, E_px_0\} .$$

Then every $G \in \mathfrak{G}$ yields

$$Gx_0 = \sum_{i=1}^{p} \alpha_i E_i x_0 ,$$

and further

$$GE_ix_0 = \alpha_i E_i x_0 , \qquad i = 1, \cdots, p .$$

Now let us multiply both members by $E_i - GE_i$. We have

$$0 = \alpha_i(E_i - GE_i)x_0\,, \qquad i = 1, \cdots, p\,.$$

If $\alpha_i \neq 0$ for some i, $(E_i - GE_i)x_0 = 0$ which implies $\alpha_i = 1$ since $E_i x_0 \neq 0$ (the carrier projection of x_0 is I and $E_i \neq 0$). Suppose $\alpha_1 = \alpha_2 = \cdots = \alpha_q = 1$ and $\alpha_{q+1} = \cdots = \alpha_p = 0$. Then

$$Gx_0 = \sum_{i=1}^{q} E_i x_0$$

and $E_i = GE_i$, $i = 1, \cdots, q$. Since the carrier projection of x_0 is I we obtain

$$G = \sum_{i=1}^{q} E_i\;;$$

i.e., every $G \in \mathfrak{G}$ is a partial sum of $\{E_1, \cdots, E_p\}$ which completes the proof.

REMARK. Theorem 24 is a particular case of a result proved by Dean [5] by quite different methods.

COROLLARY 27. *Every σ-complete* B.A. *of projections in $L_\infty(\mu)$ for some measure μ (and in particular in l_∞) has only a finite number of elements and any spectral operator of scalar type in such a space can be written as a finite combination*

$$S = \sum_{i=1}^{p} \lambda_i E_i\,.$$

Moreover, every scalar operator in $L_\infty(0, 1)$ is similar to an operator of the form $Tf = fg$, $f \in L_\infty(0, 1)$ where g is a simple function.

Proof. The first part follows from Theorem 24 since $L_\infty(\mu)$ is an $\mathscr{P}$-space. The second part is a consequence of a recent result of J. Lindenstrauss [15] asserting that every complemented subspace of l_∞ (which is isomorphic to $L_\infty(0, 1)$) is isomorphic to l_∞.

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