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INVARIANT SUBSPACES OF A DIRECT SUM OF WEIGHTED SHIFTS

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The invariant subspaces of a direct sum of finitely many copies of the adjoint of a monotone 1^2 shift are shown to be spanned by the finite dimensional invariant subspaces that they include. For the case of two copies of such a shift, the invariant subspaces are characterized in terms of a spanning set of vectors, and all infinite dimensional invariant subspaces are shown to be cyclic.

It was shown by Donoghue [2] that if an operator A on H^2 is defined by $Af(z) = zf(z/2)$, then A has a lattice of invariant subspaces anti-isomorphic to $\omega + 1$. (This result has been generalized to a wider class of operators by Nikolskii [4].) Crimmins and Rosenthal [1] have shown that the direct product of two (or even countably many) lattices of invariant subspaces is attainable as the lattice of invariant subspaces of some operator. If B and C are operators on a separable Hilbert space such that their spectra are disjoint and no part of the spectrum of one is surrounded by spectrum of the other, then the lattice of invariant subspaces of $B \oplus C$ is the direct product of the lattice of B by that of C. Thus, for example, their result gives the lattice of $(A + 1) \oplus A$. This prompts the question: What is the lattice of $A \oplus A$? One answer is given by Nikolskii [5] in terms of operators in the commutant of $A \oplus A$. Actually, his results are valid for any operator that is a direct sum of a finite number of copies of a monotone 1^p shift (see [3], p. 97). Adjoints of such operators will be studied in this paper, and the following results will be derived. The invariant subspaces of such an adjoint are spanned by the finite dimensional invariant subspaces that they include, and these are invariant subspaces of finite dimensional nilpotent operators (Theorem 1 and Theorem 2, Corollary 2). The infinite dimensional invariant subspaces are cyclic except possibly for a finite dimensional summand (Theorem 2, Corollary 3 and Theorem 3, Corollary 3). For a sum of two copies of the adjoint of a monotone 1^2 shift, the invariant subspaces can be completely characterized in terms of a spanning set of vectors (Theorem 1, Corollary 1).

We begin by establishing some notation. Although the natural setting for discussing shift operators is a sequence space, it will be somewhat more convenient to deal with functions on the unit circle X in the complex plane C. Let $\mathscr{U}$ be a finite dimensional Hilbert space, μ normalized Lebesgue measure on X, and $\mathfrak{H}$ the Hilbert space of measurable norm square integrable functions from X to $\mathscr{U}$ that are analytic. Thus, to say F is in $\mathfrak{H}$ means F is a measurable function from X to $\mathscr{U}$ such that $\int \| F \|^2 d\mu > \infty$, and if for each integer n, $e_n(z) = z^n$, then $\int F e_n^* d\mu = 0$ whenever n is negative. (The asterisk indicates the complex conjugate, and, as usual, functions that differ only on a set of measure zero are identified.) For each nonnegative integer n, define w_n by $w_n = \int F e_n^* d\mu$ to obtain a sequence of coordinates of F in $\mathscr{U}$, and then $F = \sum_{n=0}^{\infty} w_n e_n$.

A bounded sequence $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ in C induces a weighted shift operator S^* on $\mathfrak{H}$ which is defined by

$$S^* F = \sum_{n=0}^{\infty} \alpha_n w_n e_{n+1} \; . \tag{1.1}$$

We will describe the invariant subspaces of the adjoint S of such an operator for the case of a positive, monotonic and square summable weight sequence.

The connection between shifts as defined here and direct sums of shifts on 1^2 is established in a standard manner. Choose an orthonormal basis $\{u_1, u_2, \cdots, u_m\}$ for $\mathscr{U}$. Then, if F is in $\mathfrak{H}$, let $f_j (1 \leqq j \leqq m)$ be the sequence of Fourier coefficients with nonnegative index of (F, u_j), and identify F with $f_1 \oplus f_2 \oplus \cdots \oplus f_m$. By this means, the shift of multiplicity m, defined by (1.1), is identified with m copies of the shift on 1^2 induced by the sequence $\{\alpha_n\}$.

2. Shifts of arbitrary finite multiplicity. For each nonnegative integer n, let P_n be the projection of $\mathfrak{H}$ onto $\mathscr{U}$ that sends a vector into its n^{th} coordinate; $P_n \sum_{j=0}^{\infty} w_j e_j = w_n$. Define subspaces $\mathfrak{K}_n$ of $\mathfrak{H}$ by $\mathfrak{K}_n = \{F\colon$ if $j \geqq n$, then $P_j F = 0\}$, let $\mathfrak{K}_\infty$ be $\mathfrak{H}$ itself, and define the index of a vector in $\mathfrak{H}$ to be the smallest n such that $\mathfrak{K}_n$ contains the vector. Consider a nontrivial invariant subspace $\mathfrak{M}$ of S, and let N be the largest integer such that $\mathfrak{M}$ includes $\mathfrak{K}_N$. Let $\mathfrak{M}_n$ be $\mathfrak{M} \cap \mathfrak{K}_{N+n+1}$, so $\mathfrak{M}_n$ consists of all vectors in $\mathfrak{M}$ having no nonzezo coordinate beyond the $N + n^{\text{th}}$. Finally, let $\mathscr{V}_n$ be $P_{N+n}\mathfrak{M}_n$. Then $\mathscr{V}_n$ is a subspace of $\mathscr{U}$, and the following assertion is easily verified.

LEMMA 1. $\mathscr{U} \neq \mathscr{V}_0 \supset \mathscr{V}_1 \supset \mathscr{V}_2 \supset \cdots$.

It will be shown that every invariant subspace of S is the span of the finite dimensional invariant subspaces that it includes (Theorem 2, Corollary). Theorem 1 below implies that every finite dimensional invariant subspace of S is included in K_n for some integer n. The restriction of S to $\mathfrak{K}_n$ is a nilpotent operator of index n on a space of dimension mn. Thus, all invariant subspaces of S can be produced by forming spans of invariant subspaces of finite dimensional nilpotent operators.

THEOREM 1. *Every invariant subspace of S which is infinite dimensional or contains a vector of infinite index includes an infinite orthonormal set of vectors of finite index.*

Proof. If $\mathfrak{M}$ is an invariant subspace of S and $\mathfrak{M}$ contains vectors of arbitrarily large finite index, then the Gram Schmidt process may be used to complete the proof. Suppose therefore that $\mathfrak{M}$ contains a vector $F = \sum_{n=0}^{\infty} w_n e_n$ of infinite index. Induction will be used to establish the existence of an infinite orthonormal sequence $\{G_0, G_1, G_2, \cdots\}$ in $\mathfrak{M}$ such that the index of G_n is no greater than $n + 1$. Let G_{-1} be 0. Suppose an orthonormal sequence $\{G_0, G_1, \cdots, G_{m-1}\}$ has been found, which is in $\mathfrak{M}$, is empty if $m = 0$, and has the asserted index property. Let Q_m be the projection on $\mathfrak{K}_m$, and let R_m be the projection on the orthogonal complement of $\{G_{-1}, G_0, \cdots, G_{m-1}\}$. Choose a sequence of integers $n(k)$ such that (1) $\| w_{n(k)+m} \| \geqq \| w_j \|$ for all $j \geqq n(k) + m$, and (2) if $H_{n(k)} = Q_{m+1}R_mS^{n(k)}F$ then $\{H_{n(k)}/\| H_{n(k)} \|\}$ converges in the finite dimensional subspace $\mathfrak{K}_{m+1}$ to a unit vector G_m. Then G_m is orthogonal to $\{G_{-1}, \cdots, G_{m+1}\}$ and of index no greater than $m + 1$. The proof will be completed by showing that G_m is in $\mathfrak{M}$.

Since $\{G_0, G_1, \cdots, G_{m-1}\}$ is included in $\mathfrak{K}_m$, it follows that the projection R_m does not change the coefficient of e_m. Thus,

$$\| H_{n(k)} \| \geqq \alpha_m \alpha_{m+1} \cdots \alpha_{m+n(k)-1} \| w_{n(k)+m} \| . \tag{2.1}$$

The vector $R_m S^{n(k)} F$ is in M for each k, and

$$\begin{aligned} \| \| H_{n(k)} \|^{-1} R_m S^{n(k)} F - G_m \|^2 &= \| \| H_n(k) \|^{-1} H_{n(k)} - G_m \|^2 \\ &+ \| \| H_{n(k)} \|^{-1} \sum_{j=m+1}^{\infty} \alpha_j \alpha_{j+1} \cdots \alpha_{j+n(k)-1} w_{j+n(k)-1} e_j \|^2 . \end{aligned} \tag{2.2}$$

By the definition of G_m, the first term on the right hand side of (2.2) converges to zero as k tends to infinity. The inequality (2.1), the first condition on $n(k)$, and the hypothesis that $\{\alpha_j\}$ is monotonically decreasing, imply that the second term on the right hand side of (2.2) is no greater than $\alpha_m^{-2} \sum_{j=m+1}^{\infty} \alpha_{j+n(k)-1}^2$. This also converges to zero as k tends to infinity, since $\{\alpha_j\}$ is square summable. Thus, G_m is the limit of a sequence in $\mathfrak{M}$; hence, it is in $\mathfrak{M}$; and the theorem is proved.

REMARK. The proof above is a modification of a technique devised by S. Parrott to give an alternative proof of the result of Donoghue mentioned in the introduction. See [3], Problem 151.

From now on it will be assumed $\mathfrak{M}$ is an infinite dimensional invariant subspace of S. By Theorem 1 and Lemma 1, if $\mathscr{V} = \bigcap_{n=0}^{\infty} \mathscr{V}_n$, then $\mathscr{V} \neq \{0\}$. The next task is to describe a convenient basis (not in general orthonormal) for $\mathfrak{M}_n \ominus \mathfrak{K}_N (= \mathfrak{M}_n \cap \mathfrak{K}_N^{\perp})$, and it will suffice to do this for the special case in which $\mathscr{V} = \mathscr{V}_0$. Let $\{v_1, v_2, \cdots, v_p\}$ be an orthonormal basis for $\mathscr{V}$, and for each $j (1 \leqq j \leqq p)$, let G_j be a vector of index $N + n + 1$ in $\mathfrak{M}_n$ that has v_j as leading coefficient, i.e.,

$$G_j = v_j e_{N+n} + H_j \ ,$$

where H_j is a vector in $\mathfrak{K}_{N+n}$. The set $\{S^n G_j, S^{n-1} G_j, \cdots, G_j\}$ is included in $\mathfrak{M}_n$ for each j, and the projection of an appropriate multiple of $S^n G_j$ on the complement of $\mathfrak{K}_N$ is a vector $F_j(0)$ in $\mathfrak{M}$ such that

$$F_j(0) = v_j e_N \ . \tag{2.3}$$

Suppose $\{F_j(0), F_j(1), \cdots, F_j(k-1)\}$ has been defined for $k \leqq n$ and for each j. Define $F_j(k)$ by

$$\begin{aligned} F_j(k) = (\alpha_{N+n-1}\alpha_{N+n-2} \cdots \alpha_{N+k})^{-1}[(1 - Q_N)S^{n-k}G_j \\ - \sum_{i=1}^{p} \sum_{m=0}^{k-1} (S^{n-k}G_j, v_i e_{N+m}) F_i(m)] \ . \end{aligned}$$

Thus $F_j(k)$ is a vector of index $N + k + 1$ in $\mathfrak{M} \ominus \mathfrak{K}_N$ such that its $N + k^{\text{th}}$ coordinate is v_j, and all its other coordinates are orthogonal to $\mathscr{V}$, i.e.,

$$F_j(k) = v_j e_{N+k} + \sum_{i=0}^{k-1} w(i, j, k) e_{N+i} \ ,$$

where $w(i, j, k) \perp \mathscr{V}$. The vectors $F_j(k)$ make up the desired basis.

For each $j (1 \leqq j \leqq k)$ and each $k (k \geqq 1)$, let $u_j(k) = w(0, j, k)$. It will be shown that

$$F_j(k) = v_j e_{N+k} + \sum_{i=0}^{k-1} \alpha(N, k, i) u_j(k - i) e_{N+i} \ , \tag{2.4}$$

where $\alpha(N, k, 0) = 1$ and

$$\alpha(N, k, i) = (\alpha_{N+k-1}\alpha_{N+k-2} \cdots \alpha_{N+k-i})/(\alpha_N \alpha_{N+1} \cdots \alpha_{N+i-1}) \ .$$

Since the $F_j(k)$, $(1 \leqq j \leqq p,\ 0 \leqq k \leqq n)$ form a basis for $\mathfrak{M}_n \theta \mathfrak{K}_N$,

$$SF_j(k) = \alpha_{N+k-1} F_j(k - 1) + \alpha_{N-1} u_j(k) e_{N-1} \ . \tag{2.5}$$

It follows by induction, on equating coefficients of both sides of (2.5), that

$$\begin{aligned} w(i, j, k) &= \alpha_{N+k-1} w(i-1, j, k-1)/\alpha_{N+i-1} \\ &= \alpha_{N+k-1}\alpha(N, k-1, i-1)u_j(k-i)/\alpha_{N+i-1} \\ &= \alpha(N, k, i)u_j(k-i) \; . \end{aligned}$$

For later use we insert here a fact concerning the coefficients $\alpha(N, k, i)$.

LEMMA 2. *$\sum_{i=1}^{k-1}\alpha(N, k, i)^2 \leqq \alpha_{N+k-1}^2 C_N$ for $k \geqq 2$, where $C_N = \alpha_N^{-2} + (\alpha_N\alpha_{N+1})^{-2}\sum_{i=N+1}^{\infty}\alpha_i^2$.*

Proof. Define c_k as the quotient of the term on the left hand side of the asserted inequality by $(\alpha_{N+k-1}/\alpha_N)^2$. We claim that $c_{k+1} \leqq c_k + (\alpha_{N+k-1}/\alpha_{N+1})^2$.

For,

$$\begin{aligned} &c_{k+1} - (\alpha_{N+k-1}/\alpha_{N+1})^2 \\ &\quad = 1 + \sum_{i=3}^{k}(\alpha_{N+k-1}/\alpha_{N+i-1})^2(\alpha_{N+k-2}\cdots\alpha_{N+k+1-i}/\alpha_{N+1}\cdots\alpha_{N+i-2})^2 \leqq c_k \; . \end{aligned}$$

Since $c_2 = 1$, it follows $c_k \leqq 1 + \alpha_{N+1}^{-2}\sum_{i=N+1}^{\infty}\alpha_i^2$, and this implies the assertion.

THEOREM 2. *Let $\mathscr{V}$ be a nontrivial subspace of $\mathscr{U}$, let $\{v_1 v_2, \cdots, v_p\}$ be an orthonormal basis for $\mathscr{V}$, let $\{u_j(k)\}_{k=1}^{\infty}$ for $j = 1, 2, \cdots, p$ be norm square summable sequences in $\mathscr{V}^{\perp}$, and let N be a nonnegative integer. Define $F_j(k)$ for $j = 1, 2, \cdots, p$ and $k = 0, 1, 2, \cdots$ according to (2.3) and (2.4). Then the (closed) span $\mathfrak{M}$ of $\mathfrak{K}_N$ and all the vectors $F_j(k)$ is an invariant subspace of S such that $P_{N+n}(\mathfrak{M} \cap \mathfrak{K}_{N+n+1}) = \mathscr{V}$ for $n = 0, 1, 2, \cdots$. Conversely, every invariant subspace $\mathfrak{M}$ of S such that $P_j(\mathfrak{M} \cap \mathfrak{K}_{j+1})$ is $\mathscr{U}$ for $j < N$ and $\mathscr{V}$ for $j > N$ is obtained in this manner.*

Proof. Given $F_j(k)$ as above, the relation (2.5) holds, and therefore the span $\mathfrak{M}$ of the $F_j(k)$ and K_N is invariant under S. It will be shown that $P_{N+n}(\mathfrak{M} \cap \mathfrak{K}_{N+n+1}) = \mathscr{V}$ at the end of the proof.

Let $\mathfrak{M}$ be an invariant subspace of S such that $P_j(\mathfrak{M} \cap K_{j+1})$ is $\mathscr{U}$ if j $< N$ and is a nontrivial subspace $\mathscr{V}$ of $\mathscr{U}$ if $j \geqq N$. Then, as above, each $\mathfrak{M}_n$ is spanned by $\mathfrak{K}_N$ and vectors $F_j(k)$ for $1 \leqq j \leqq p$ and $0 \leqq k \leqq n$ which satisfy (2.3) or (2.4). It must be shown that the sequences $\{u_j(k)\}_{k=1}^{\infty}$ for $1 \leqq j \leqq p$ are norm square summable and that the $F_j(k)$ span $\mathfrak{M} \ominus \mathfrak{K}_N$.

First, it will be shown that each sequence $\{u_j(k)\}_{k=1}^{\infty}$ is bounded. If one of them, which we denote simply $\{u(k)\}_{k=1}^{\infty}$, is not bounded, then it has a subsequence $\{(u(k')\}$ with the properties $\|u(i)\| \leqq \|u(k')\|$ if

$i \leqq k'$, and $\{u(k')/\| u(k')\|\}$ converges to a unit vector u in $\mathscr{Y}^{\perp}$. Extending the convention of dropping subscripts to indicate all vectors with the same subscript as the unbounded sequence $\{u(k)\}$, we then have that $\| u(k')\|^{-1}F(k')$ is in $\mathfrak{M}$ for each k, and

$$\| u(k')\|^{-1}F(k') - ue_N = (\| u(k')^{-1}u(k') - u)e_N + \sum_{i=1}^{k'-1} \alpha(N, k', i)\| u(k')\|^{-1}u(k' - i)e_{N+i} + \| u(k')\|^{-1}ve_{N+k'} .$$

As will be shown, the right hand side converges to zero as k tends to infinity. It follows that ue_N is in $\mathfrak{M}$ which is impossible since ue_N is orthogonal to $\mathfrak{M}_0$, and this contradiction implies boundedness. To return to the above equation, note that the first term on the right hand side converges to zero by the choice of the subsequence, and the third term also does since $\{\| u(k')\|^{-1}\}$ converges to zero. As for the second term, the summands are orthogonal and of norm less than or equal $\alpha(N, k', i)$; therefore, the norm squared of the second term is no greater than $\sum_{i=1}^{k'-1} \alpha(N, k', i)^2$, which tends to zero as k tends to infinity by Lemma 2. The proof of boundedness is complete, so there is a constant β such that $\| u_j(k)\| \leqq \beta$ for all j and k.

Suppose next that one of the sequences $\{u_j(k)\}_{k=1}^{\infty}$ is not norm square summable, and denote this sequence $\{u(k)\}$. To derive a contratiction, the first step will be to produce a square summable sequence of complex numbers σ_k together with a sequence of integers $n(j)$ such that the vectors w_j in $\mathscr{Y}^{\perp}$, defined by

$$w_j = \sum_{k=j}^{n(j)} \sigma_k u(k) ,$$

are all of norm at least one. This may be accomplished by taking an orthonormal basis $\{x_1, x_2, \cdots, x_q\}$ for $\mathscr{Y}^{\perp}$ and considering the q sequences $\{(u(k), x_i)\}_{k=1}^{\infty}$. By the Parsevaal identity at least one of these sequences, which we denote $\{(u(k)), x)\}$, is not square summable. Choose a square summable sequence $\{\sigma_k\}$ such that $\sigma_k(u(k), x) \geqq 0$ and $\sum_{k=1}^{\infty} \sigma_k(u(k), x) = \infty$ (see [3], p. 14), and corresponding to each j choose $n(j)$ such that

$$\sum_{k=j}^{n(j)} \sigma_k(u(k), x) > 1 .$$

With these choices,

$$\| w_j \| \geqq | (\sum_{k=j}^{n(j)} \sigma_k u(k), x) | > 1 ,$$

and the first step is complete. Next, take a subsequence $\{w_{j'}\}$ such that $\{w_{j'}/\| w_{j'}\|\}$ converges to a unit vector w in $\mathscr{Y}^{\perp}$. The contradiction now arises because the sequence $\{\| w_{j'} \|^{-1} \sum_{k=j'}^{n(j')} \sigma_k F(k)\}$ in $\mathfrak{M}$ converges to we_N, which is orthogonal to $\mathfrak{M}_0$. (As above, $F(k)$ denotes that $F_j(k)$ with the same subscript as the sequence $\{u(k)\}$.) To see this, write $F(k) = ve_{N+k} + G(k) + u(k)e_N$, where

$$G(k) = \sum_{i=1}^{k-1} \alpha(N, k, i)u(k-i)e_{N+i} ,$$

and consider the difference

$$\| w_{j'} \|^{-1} \sum_{k=i'}^{n(j')} \sigma_k F(k) - we_N = \| w_{j'} \|^{-1} \sum_{k=j'}^{n(j')} \sigma_k ve_{N+k}$$
$$+ \| w_{j'} \|^{-1} \sum_{k=j'}^{n(j')} \sigma_k G(k) + (\| w_{j'} \|^{-1} \sum_{k=j'}^{n(j')} \sigma_k u(k) - w)e_N .$$

On the right hand side of this equation, the first term tends to zero as j tends to infinity because the sequence $\{\sigma_k\}$ is square summable, and the third term also does so by the choice of the subsequence $\{w_{j'}\}$. By Lemma 2, $\| G(k) \| \leqq \beta C_N^{1/2} \alpha_{N+k-1}$. Thus, by the triangle inequality and the fact that $\| w_{j'} \|^{-1} < 1$, the second term is no greater in norm than $\beta C_N^{1/2} \sum_{k=j'}^{n(j')} | \sigma_k | \alpha_{N+k-1}$, which tends to zero as j tends to infinity since both $\{\sigma_k\}$ and $\{\alpha_k\}$ are square summable. Hence each of the sequences $\{u_j(k)\}_{k=1}^{\infty}$ is norm square summable.

To complete the proof we will need the fact that for each j, $\{\sigma_k F_j(k)\}_{k=0}^{\infty}$ is summable in $\mathfrak{H}$ whenever $\{\sigma_k\}$ is a square summable sequence of complex numbers. If m and n are integers, then, dropping the subscript j and using the notation introduced in the preceding paragraph, we may write

$$\sum_{k=m}^{n} \sigma_k F(k) = \sum_{k=m}^{n} \sigma_k ve_{N+k} + \sum_{k=m}^{n} \sigma_k G(k) + \sum_{k=m}^{n} \sigma_k u(k)e_N .$$

As above, the first two terms on the right hand side can be made small by taking m and n sufficiently large, but in addition the third term has norm no greater than $\sum_{k=m}^{n} | \sigma_k | \| u(k) \|$, which can also be made small by taking m and n large since $\{\| u(k) \|\}$ is now also known to be square summable. This implies that $\{\sigma_k F_j(k)\}_{k=0}^{\infty}$ is summable in $\mathfrak{H}$.

To see that the $F_j(k)$ span $\mathfrak{M} \ominus \mathfrak{K}_N$ suppose F in $\mathfrak{M}$ is orthogonal to $\mathfrak{K}_N$; define $\sigma_j(k)$ by $\sigma_j(k) = (F, v_j e_{N+k})$ for $k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, p$; and define G by $G = \sum_{j=1}^{p} \sum_{k=0}^{\infty} \sigma_j(k) F_j(k)$. By the remarks of the preceding paragraph the definition of G is permissible, and thus $F - G$ is a vector in $\mathfrak{M} \ominus \mathfrak{K}_N$ such that $P_{N+n}(F - G)$ is orthogonal to $\mathscr{V}$ for all n. If $F - G$ were not zero, then the technique of Theorem 1 could be employed to produce a vector in $\mathfrak{M}_0$ orthogonal to $\{F_1(0), F_2(0), \cdots, F_p(0)\}$, which is impossible. Thus $F = G$ and the vectors $F_j(k)$ span $\mathfrak{M} \ominus \mathfrak{K}_N$.

The final step is to supply the proof that if $\mathfrak{M}$ is the span of $\mathfrak{K}_N$ and the vectors $F_j(k)$, then $P_{N+n}(\mathfrak{M} \cap \mathfrak{K}_{N+n+1}) = \mathscr{V}$ for $n = 0, 1, 2, \cdots$. That the set on the left includes the one on the right is clear. To obtain the opposite inclusion, by Lemma 1, it will suffice to prove that if w is any nonzero vector in $\mathscr{V}^{\perp}$, then we_N is at a positive distance from the span of the $F_j(k)$. For each j $(1 \leqq j \leqq p)$, let $\{\sigma_j(k)\}_{k=0}^{\infty}$ be any eventually null sequence of complex numbers. Then

$$\begin{aligned} \| we_N - \sum_{j=1}^{p} \sum_{k=0}^{\infty} \sigma_j(k) F_j(k) \|^2 &\geqq \| \sum_{j=1}^{n} \sum_{k=0}^{\infty} \sigma_j(k) v_j e_{N+k} \|^2 \\ &+ \| (w - \sum_{j=1}^{p} \sum_{k=1}^{\infty} \sigma_j(k) u_j(k)) e_N \|^2 \\ &= \sum_{j=1}^{p} \sum_{k=0}^{\infty} | \sigma_j(k) |^2 + \| w - \sum_{j=1}^{p} \sum_{k=1}^{\infty} \sigma_j(k) u_j(k) \|^2 , \end{aligned}$$

and the sum on the right may be shown to be bounded away from zero independently of the choice of the $\sigma_j(k)$. This completes the proof of the theorem.

COROLLARY 1. *If $\mathscr{U}$ has dimension two, then every nontrivial invariant subspace of S is either finite dimensional or else consists of the span of $\mathfrak{K}_N$ for some $N(\geqq 0)$ and a sequence $\{F_n\}_{n=0}^{\infty}$ in which each F_n is of index $N + n + 1$. The vectors F_n may be defined by means of an orthonormal basis $\{v, u\}$ for $\mathscr{U}$ and a square summable sequence $\{\rho_k\}_{k=1}^{\infty}$ in C: $F_0 = ve_N$; and if $n > 0$, then*

$$F_n = ve_{N+n} + \sum_{j=0}^{n-1} \alpha(N, n, j) \rho_{n-j} u e_{N+j} .$$

REMARK. A complete description of the finite dimensional invariant subspaces of S in the above terms may also be given in this case. For a finite dimensional invariant subspace, the sequence $\{F_n\}$ is merely finite or nonexistent.

Proof. If $\mathscr{U}$ has dimension two, then every nontrivial infinite dimensional invariant subspace of S satisfies the conditions of the theorem with $p = 1$.

COROLLARY 2. *Every invariant subspace of S is spanned by the finite dimensional ones which it includes, and each of these consists of vectors of finite index.*

Proof. Since $\mathfrak{H}$ itself is spanned by the finite dimensional invariant subspaces $\mathfrak{K}_n$, it is sufficient to consider the case of a nontrivial infinite dimensional invariant subspace $\mathfrak{M}$ of S. Define the sequence of subspaces $\mathscr{V}_n^\frown$ of $\mathscr{U}$ as in Lemma 1. In general the intersection $\mathscr{V}^\frown$ of these subspaces will be smaller than $\mathscr{V}_0^\frown$, but Lemma 1 and Theorem 1 imply it will be nontrivial. Let $\mathscr{V}_q^\frown$ be the first subspace in the sequence which is equal to $\mathscr{V}^\frown$. If $M = N + q$, then define $\mathfrak{N}$ as $\mathfrak{M} + \mathfrak{K}_M$ to obtain a (closed) invariant subspace of S which satisfies the conditions of Theorem 2. Clearly,

$$\mathfrak{N} = \mathfrak{M} + (\mathfrak{K}_M \ominus \mathfrak{M}_{q-1}) ,$$

and this is a direct sum decomposition of $\mathfrak{N}$. Since $\mathfrak{K}_M \ominus \mathfrak{M}_{q-1}$ is finite dimensional, the projection of $\mathfrak{N}$ onto $\mathfrak{M}$ along $\mathfrak{K}_M \ominus \mathfrak{M}_{q-1}$ is continuous. Thus, if F is in $\mathfrak{M}$, then it is the limit of a sequence of vectors of finite index in $\mathfrak{N}$. The image of this sequence under the projection on $\mathfrak{M}$ is a sequence of vectors of finite index in $\mathfrak{M}$, and it also converges to F. This proves the corollary.

COROLLARY 3. *Every invariant subspace of S is the sum of a cyclic subspace and a finite dimensional invariant subspace of S.*

Proof. It may be shown that $\mathfrak{H}$ itself is cyclic (see [3], p. 282, for an analogous situation). Suppose $\mathfrak{M}$ is an invariant subspace of S that has the form required for an application of Theorem 2, and let $F_j(k)$ be the set of vectors that spans $\mathfrak{M} \ominus \mathfrak{K}_N$. Define F by

$$F = \sum_{k=0}^{\infty} \sum_{j=1}^{p} (pk + j)^{-1} F_j(pk + j) ,$$

and consider the sum $\mathfrak{M}'$ of the cyclic subspace generated by F and the finite dimensional invariant subspace $\mathfrak{K}_N$. Since the projection of $S_n F_j(k)$ on the orthogonal complement of $\mathfrak{M}_N$ is

$$\alpha_{N+K-1} \cdots \alpha_{N+K-1} F_j(k - n)$$

if $k \geqq n$, an induction argument may be used to show that $\mathfrak{M}'$ contains all the $F_j(k)$, and thus $\mathfrak{M}'$ includes $\mathfrak{M}$. The opposite inclusion is trivial, and the proof for the special case is complete.

If $\mathfrak{M}$ is an arbitary nontrivial infinite dimensional invariant subspace of S, then define $\mathfrak{N}$ as in the proof of the preceding corollary. Take a vector F in $\mathfrak{N}$ such that the sum of the cyclic subspace it generates and $\mathfrak{K}_M$ is $\mathfrak{N}$. There is a vector G in $\mathfrak{M}$ such that the difference $F -$ G has index at most $\mathfrak{M}$. Consider the sum $\mathfrak{M}'$ of the cyclic subspace generated by G and the finite dimensional invariant subspace $\mathfrak{M}_{q-1}$. It is clear that $\mathfrak{M}'$ is included in $\mathfrak{M}$. If H is in $\mathfrak{M}$, then $H = F_1 + F_2$, where F_1 is in the cyclic subspace generated by F_1 and F_2 is in $\mathfrak{K}_M$. Further, $F_1 = G_1 + G_2$, where G_1 is in the cyclic subspace generated by G, and G_2 is in $\mathfrak{K}_M$. Then $H - G_1 = G_2 + F_2$

is in $\mathfrak{M} \cap \mathfrak{K}_M$, i.e., in $\mathfrak{M}_{q-1}$, and it follows that H is in $\mathfrak{M}'$. This establishes that $\mathfrak{M}$ is included in $\mathfrak{M}'$, which completes the proof.

REMARK. In case $\mathscr{U}$ is two dimensional every infinite dimensional invariant subspace of S is cyclic (Theorem 3, Corollary 3). This is not true for higher dimensions, as may be seen by considering the case in which $\mathscr{U}$ is three dimensional and the invariant subspace is the sum of $\mathfrak{K}_1$ and a slice through a one-dimensional subspace of $\mathscr{U}$.

3. Shifts of multiplicity 2. In the special case under consideration a complete characterization of the invariant subspaces of S has been obtained (Theorem 2, Corollary 1). An infinite dimensional invariant subspace $\mathfrak{M}(N, v, u, \{\rho_k\})$ is determined by a nonnegative integer N, an orthonormal basis $\{v, u\}$ for $\mathscr{U}$, and a square summable sequence $\{\rho_k\}$ in C. It is easy to see that $\mathfrak{M}(N, v, u, \{\rho_k\}) = M(N', v', u', \{\rho'_k\})$ if $N = N'$ and there exist complex constants α and β of modulus one such that $v = \alpha v'$, $u = \beta u'$ and $\rho_k = \alpha\beta^*\rho'_k$. The converse of this statement is contained in the following theorem.

THEOREM 3. *If $\{v, u\}$ and $\{v', u'\}$ are bases for $\mathscr{U}$, and if $\{\rho_k\}$ and $\{\sigma_k\}$ are square summable sequences in C, then*

$$\mathfrak{M}(M, v, u, \{\rho_k\}) \subset \mathfrak{M}(N, v', u', \{\sigma_k\})$$

if and only if

(1) $M \leqq N$,

(2) *there exist constants α and β of unit modulus such that $v = \alpha v'$, $u = \beta u'$ and*

(3) $\rho_k = \sigma_k \alpha(M, N - M + k, N - M)^{-1}\alpha\beta^*$.

The inclusion is proper if and only if $M < N$.

Proof. Suppose the three conditions are satisfied for invariant subspaces $\mathfrak{M}$ and $\mathfrak{N}$, where $\mathfrak{M} = \mathfrak{M}(M, v, u, \{\rho_k\})$ and $\mathfrak{N} = \mathfrak{M}(N, v', u', \{\sigma_k\})$. Let F_n be the sequence in $\mathfrak{M}$ determined by v, u and the sequence $\{\rho_k\}$, and let $\{G_n\}$ be the analogous sequence in $\mathfrak{N}$. Since condition (1) implies that $\mathfrak{K}_M$ is included in $\mathfrak{N}$, it suffices to show that F_n is in $\mathfrak{N}$ for each n. This is immediate if $n \leqq N - M$, for then F_n is in the span of $\mathfrak{K}_N$ and $v'e_N$. If $k > 0$, then a calculation using the third assumption shows that

$$\alpha(M, N - M + k, N - M + j)\rho_{k-j} = \alpha(N, k, j)\sigma_{k-j}\alpha\beta^*$$

for each $j (0 \leqq j \leqq k)$, and it follows from this that

$$F_{N-M+k} = \alpha G_k + H_k \ ,$$

where H_k is a vector in $\mathfrak{K}_N$. Thus $\mathfrak{M}$ is included in $\mathfrak{N}$.

Conversely, suppose $\mathfrak{M}$ and $\mathfrak{N}$ are two invariant subspaces of S, as in the preceding paragraph, such that $\mathfrak{N}$ includes $\mathfrak{M}$. Trivially, $\mathfrak{K}_M$ is included in $\mathfrak{N}$, and this implies $M \leqq N$. Since $F_{N-M} = ve_N + H_1$, where H_1 is in $\mathfrak{K}_N$, it follows $ve_N = \alpha G_0 = \alpha v' e_N$, where α is a complex number of unit modulus. Hence, $v = \alpha v'$. Since

$$F_{N-M+1} = ve_{N+1} + \alpha(M, N-M+1, N-M)\rho_1 u e_N + H_2 ,$$

where H_2 is a vector in $\mathfrak{K}_N$, it follows that

$$F_{N-M+1} - H_2 = \alpha G_1 = \alpha v' e_{N+1} + \alpha \sigma_1 u' e_N ,$$

and hence, $u = \beta u'$ for some β of unit modulus, and

$$\rho_1 = \sigma_1 \alpha(M, N-M+1, N-M)^{-1} \alpha \beta^* .$$

Similarly, F_{N-M+k} is in $\mathfrak{N}$; its projection on the orthogonal complement of $\mathfrak{K}_N$ is αG_k; and a comparison of the coefficient of e_N in this projection with the corresponding coefficient in G_k yields the third condition.

Finally, it is clear that the inclusion is proper if $M < N$. If $\mathfrak{M}$ is included in $\mathfrak{N}$ and $M = N$, then since $\mathfrak{N} \cap \mathfrak{K}_{N+k}$ has dimension $2N + k$, and includes $\{F_0, F_1, \cdots, F_{k-1}\}$ and $\mathfrak{K}_N$, it follows that $\mathfrak{N} \cap \mathfrak{K}_{N+k} = \mathfrak{M} \cap \mathfrak{K}_{N+k}$. Hence $\mathfrak{M} = \mathfrak{N}$ and the theorem is proved.

COROLLARY 1. *An infinite dimensional invariant subspace* $\mathfrak{M}(N, v, u, \{\sigma_k\})$ *of* S *properly includes another infinite dimensional invariant subspace of* S *if and only if* $N > 0$ *and* $\sum_{k=1}^{\infty} |\sigma_k/\alpha_{N+k-1}|^2 < \infty$.

Proof. If the condition holds, then $\mathfrak{M}(N-1, u, v, \{\alpha_{N-1}\sigma_k/\alpha_{N+k-1}\})$ is an invariant subspace of S which is properly included in the given one. Conversely, if $\mathfrak{M}(M, v, u, \{\rho_k\})$ is properly included in $\mathfrak{M}(N, v, u, \{\sigma_k\})$, then $0 \leqq M < N$, and

$$|\alpha_M \sigma_k/\alpha_{N+k-1}| \leqq \alpha(M, N-M+k, N-M)^{-1} |\sigma_k| = |\rho_k| .$$

Hence, square summability of $\{\sigma_k/\alpha_{N+k-1}\}$ follows from that of $\{\rho_k\}$, which completes the proof.

COROLLARY 2. *Every infinite dimensional invariant subspace of* S *includes at most finitely many infinite dimensional invariant subspaces of* S, *and these are linearly ordered.*

Proof. This follows directly from the theorem and preceding corollary.

COROLLARY 3. *Every infinite dimensional invariant subspace of S is cyclic.*

Proof. Let $\mathfrak{N}$ be an infinite dimensional invariant subspace of S. The case of $\mathfrak{N}$ trivial was considered in Corollary 3 of Theorem 2, so we suppose $\mathfrak{N} = \mathfrak{M}(N, v, u, \{\sigma\})$. Let $\mathfrak{M}$ be the unique minimal infinite dimensional invariant subspace of S included in $\mathfrak{N}$, and let F be a vector of infinite index in $\mathfrak{M}$. If $\mathfrak{N} = \mathfrak{M}$, then F is cyclic for $\mathfrak{N}$, and we are done. If $\mathfrak{M}$ is properly included in $\mathfrak{N}$, then define G by $G = F + ve_{N-1}$, and let $\mathfrak{N}'$ be the cyclic subspace determined by G. Since $\mathfrak{N}'$ is included in $\mathfrak{N}$ and since $\mathfrak{M}$ is the unique minimal infinite dimensional invariant subspace of S included in $\mathfrak{N}$, it follows that $\mathfrak{N}'$ includes $\mathfrak{M}$. Thus F is in $\mathfrak{N}'$; ve_{N-1} is in $\mathfrak{N}'$; and it follows easily that $\mathfrak{K}_N$ is included in $\mathfrak{N}'$. But this implies that $\mathfrak{N} = \mathfrak{N}'$, and hence $\mathfrak{N}$ is cyclic.

REMARKS. 1. The dimension condition in Corollary 3 is clearly necessary since $\mathfrak{K}_1$, for example, is not cyclic.

2. Throughout this paper it has been assumed that the sequence $\{\alpha_n\}$ which determines S is monotonically decreasing and square summable. In fact, it is possible to get by with a somewhat weaker hopothesis. If the sequence $\{\alpha_n\}$ consists of positive terms, is eventually monotonically decreasing and belongs to some 1^p class $(0 < p < \infty)$, then all the above proofs may be modified to yield the same results.

REFERENCES

1. T. Crimmins and P. Rosenthal, *On the decomposition of invariant subspaces*, Bull. Amer. Math. Soc. **73** (1967), 97-99.
2. W. F. Donoghue, *The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation*, Pacific J. Math. **7** (1957), 1031-1035.
3. P. R. Halmos, *A Hilbert Space problem Book*, Van Nostrand, Princeton, New Jersey, 1967.
4. N. K. Nikolskii, *Invariant subspaces of certain completely continuous operators*, Vestnik Leningrad University **7** (1965), 68-77 (Russian).
5. ———, *The unicellularity and nonunicellularity of weighted shift operators*, Dokl. Akad. Nauk SSSR (2) **172** (1967), Soviet Math. Dokl **8** (1967), 91-94.

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