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INVARIANT SUBSPACES OF $C(G)$

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The purpose of this paper is to show the ease with which certain methods from the theory of locally compact abelian groups carry over to general compact groups. The principal tool is a generalized Fourier transform which is a faithful representation of the group algebra $L'(G)$ (G compact) into a direct sum of finite dimentional matrix algebras.

Only the space $C(G)$ of continuous complex-valued functions of G will be considered here, although the methods are also applicable to $L^p(G)$.

The mapping $R_x[L_x]: C(G) \to C(G)$ by $R_x(f)(y) = f(yx)[L_x(f)(y) = f(xy)]$ for each $x \in G$ gives an action of G on $C(G)$. A closed subspace of $C(G)$ is called right [left] invariant if it is invariant under all $R_x[L_x]$ for $x \in G$. Proposition 1 states that a closed subspace of $C(G)$ is right (left) invariant if it is a right (left) ideal of the convolution algebra $C(G)$. This fact is used in Theorem 2 to give another description of a closed right (left) invariant subspace in terms of the Fourier transform. This description is the analog of the Spectral Synthesis Theorem. Finally the notion of a Sidon set is used to describe certain two-sided (both right and left) invariant subspaces of $C(G)$.

The notation and definitions of [1] are used throughout the paper.

PROPOSITION 1. A closed subspace A of $C(G)$ is right (left) invariant if and only if it is a right (left) ideal of the convolution algebra $C(G)$.

REMARK. Proposition 1 seems to be known. A proof may be constructed along the lines of Theorem 7.12 of [6], or by using the Fourier transform.

For the next theorem we recall from [1] the definition of the Fourier transform and the algebra $M = \bigoplus_{r \in \Delta} B(H_r)$. Theorem 2 is an exact analong of the Spectral Synthesis Theorem for locally compact abelian groups. The "right" theorem is proved, but the "left" theorem is also true by a similar proof.

THEOREM 2. *If A is a closed right invariant subspace of $C(G)$, then there exists a unique self-adjoint projection $p \in M$ such that $A = \{f \in C(G): p\hat{f} = \hat{f}\}$.*

Proof. For each r, $\Lambda(A)_r$ is a right ideal of $B(H_r)$, thus (since H_r is finite-dimensional) has the form $p_r B(H_r)$, for some self-adjoint projection p_r in $B(H_r)$. Set $p = \{p_r\}_{r\in\Delta}$. Let $\{u_\alpha\}_{\alpha\in I}$ be a bounded, central approximate identity as constructed in the proof of Theorem 3.4 of [3]. Since each u_α is a trigonometric polynomial, $(\hat{u}_\alpha)_r = 0$ for all but a finite set of $r \in \Delta$. Thus $\Lambda^{-1}(p\hat{u}_\alpha) \in A$ for each $\alpha \in I$. Clearly $\{f: p\hat{f} = \hat{f}\}$ is a closed right ideal which contains A, so we need only prove the reverse inclusion. Suppose $p\hat{f} = \hat{f}$. Then $\Lambda^{-1}(p\hat{u}_\alpha)^*f \in A$, and $(\Lambda^{-1}(p\hat{u}_\alpha)^*f)^\wedge = p\hat{u}_\alpha\hat{f} = p\hat{f}\hat{u}_\alpha = \hat{f}\hat{u}_\alpha = (f^*u_\alpha)^\wedge$. Thus by uniqueness of Fourier transform, $\Lambda^{-1}(p\hat{u}_\alpha)^*f = f^*u_\alpha \in A$. Taking limits over $\alpha \in I$, we get $f \in A$, using the fact that $\{u_\alpha\}$ is a norm approximate identity for $C(G)$.

REMARK. If the subspace A of Theorem 2 is two-sided invariant, then $\Lambda(A)_r$ is a two-sided ideal of $B(H_r)$ for each $r \in \Delta$. Thus either $\Lambda(A)_r = B(H_r)$ or $\Lambda(A)_r = \{0\}$. Thus the projection p given by the theorem would be a central projection in M.

It is to be emphasized that this paper is more an illustration of technique than anything else. There are many technical problems relating to noncommutativity, some of which are formidable. We are indebted to the referee for pointing out one of them which necessitated a correction in an earlier version of this paper.

We now turn to the case of two-sided invariance. The aim is to give a version of Theorem 2.7 of [5].

DEFINITION. If A is a closed two-sided invariant subspace of $C(G)$ and $A_0 \subset A$ is a closed right invariant subspace, a projection $T: A \to A_0$ of A onto A_0 is called locally self-adjoint if the linear functional $\Phi(f) = T(f)(e)$ is self-adjoint on A, where $e \in G$ is the group identity.

PROPOSITION 3. Suppose A is a closed two-sided invariant subspaces of $C(G)$, and A_0 is a closed right invariant subspace with $A_0 \subset A$. If $T: A \to A_0$ is a bounded *locally self-adjoint* projection of A onto A_0 which commutes with right translations, then there exists $m \in M(G)$ self-adjoint such that $T(f) = m^*f$ for all $f \in A$.

Proof. Let $e \in G$ be the group identity. Then $f \to T(f)(e)$ defines a bounded self-adjoint linear functional on A, which thus extends by the Hahn-Banach theorem to a self-adjoint bounded linear functional $m \in M(G) = C(G)^*$. Here the duality is given by

$$m(f) = \int_G f(x^{-1})dm(x) .$$

Now take $f \in A$ and $x \in G$ and we get

$$\begin{aligned}(Tf)(x) &= (R_xTf)(e) = (T(R_xf))(e) = \int_G (R_xf)(y^{-1})dm(y)\\ &= \int_G f(y^{-1}x)dm(y) = m^*f(x) .\end{aligned}$$

DEFINITION. Let A be a closed 2-sided invariant subspace of $C(G)$, and let $z \in M$ be the central projection given by Theorem 2 such that $A = \{f \in C(G): z\hat{f} = \hat{f}\}$. Define the *spectrum of* $A = \mathrm{sp}(A)$ to be (using notation of [1]) $\{r \in \Delta: z_r \neq 0\}$.

Proposition 3 and Theorem 4 are patterned after Theorem 2.7 of [5].

THEOREM 4. *Let A be a closed two-sided invariant subspace of $C(G)$. Then the following are equivalent.*

(1) *If A_0 is a closed right invariant subspace of $C(G)$ and $A_0 \subset A$, then there exists a bounded right-invariant locally self-adjoint projection of A onto A_0.*

(2) *The spectrum of A is a Sidon set (as defined in* [1]).

(3) $\sum_{r \in \Delta} d_r tr(|f_r g_r|) < \infty$ *for all* $f \in A$, $g \in L^1(G)$.

Proof. The fact that (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are immediate consequences of [1], Theorem 2.

Assume (3) is true. Fix $f \in A$. Then $\{r: \hat{f}_r \neq 0\}$ is countable. Enumerate the set as $\{r_k\}$. For each positive integer n, define $T_n: L^1(G) \rightarrow F$ (the pre-dual of M as defined in [1]) by

$$T_n(g) = \sum_{k=1}^{n} d_{r_k} \hat{f}_{r_k} \hat{g}_{r_k} .$$

By assumption (3), T_n is a pointwise convergent sequence of bounded operators and hence is uniformly bounded and by the uniform boundedness theorem. Thus there exists a constant K with $\| T_n | \leqq K$ for all $n = 1, 2, \cdots$.

In $\{u_\alpha\}$ is the approximate identity introduced in the proof of Theorem 2, we have for any $n = 1, 2, \cdots$,

$$\sum_{k=1}^{n} d_{r_k} tr\, |\hat{f}_{r_k} (\hat{u}_\alpha)_{r_k}|_{\alpha \in I} \sum_{k=1}^{n} d_{r_k} tr\, |\hat{f}_{r_k}| \leqq K .$$

Thus letting $n \rightarrow \infty$ we have

$$\sum_{k=1}^{n} d_{r_k} tr\, |\hat{f}_{r_k}| \longrightarrow \sum_{r \in \Delta} d_{r_k} tr\, |\hat{f}_r| < \infty .$$

By Theorem 2 of [1], $\mathrm{sp}(A)$ is a Sidon set.

Now assume (1) holds. Let z be the central projection in M such that $A = \{f \in C(G): z\hat{f} = \hat{f}\}$. According to Theorem 2 of [1], we need to show that for each unitary operator $u \in M$ there exists $m \in M(G)$ such that $\| z(\hat{m} - u) \| < 1$. Choose $u \in M$ unitary, and by the spectral theorem choose self-adjoint projections $p_1, \cdots, p_n \in M$ and scalars $\alpha_1, \cdots, \alpha_n$ such that $\| u - \sum_{k=1}^{n} \alpha_k p_k \| < 1$. For each k, define

$$A_k = \{f \in A: p_k \bar{f} = \hat{f}\}\ .$$

Then A_k is a closed right invariant subspace of A, so there is a bounded right-invariant locally self-adjoint projection $T_k: A \to A_k$. By Proposition 3, there exists $m \in M(G)$ such that $m_k^* f = T(f)$ for all $f \in A$, and m_k is self-adjoint. Now $\Lambda(A)$ is dense in the weak* topology of zM (since it contains all finite dimensional operators). Thus since $m_k^* m_k^* f = m_k^* f$ for all $f \in A$, $\hat{m}_k \hat{m}_k \hat{f} = \hat{m}_k \hat{f}$, and hence $\hat{m}_k \hat{m}_k z = \hat{m}_k z$. Since m_k is self-adjoint, so is $\hat{m}_k$, so $\hat{m}_k z$ is a self-adjoint projection.

Since $\Lambda(A)_k$ is weak* dense in $zp_k M$, and since $m_k^* f = f$ for $f \in A_k$, we get $z\hat{m}_k \geqq zp_k$. If $z\hat{m}_k$ is strictly greater than zp_k, then there is some $g \in A$ such that $\hat{g} = (z\hat{m}_k - zp_k)\hat{g}$. Thus $m_k^* g \notin A_k$, a contradiction. Thus $z\hat{m}_k = zp_k$. Set $m = \sum_{k=1}^n \alpha_k m_k$. Then

$$|| z(u - \hat{m}) || = \left\| z(u) - z\Big(\sum_{k=1}^n \alpha_k p_k\Big) \right\| = \left\| z\Big(u - \sum_{k=1}^n \alpha_k p_k\Big) \right\| < 1\ .$$

Thus (1) implies (2).

We remark that the proof of Theorem 1 of [7] can be used to eliminate the need for rigt invariance in the projection of condition (1) of the last theorem.

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