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RATIONAL APPROXIMATION ON CERTAIN PLANE SETS

T. A. McCullough

Let K be a compact subset of the complex plane and let Ω denote its complement. In 1966 Vituskin [11] proved the following generalization of Mergelyan's celebrated theorem on rational approximation [9].

THEOREM. (Vituskin). If each boundary point of K is a boundary point of some component of Ω then $A(K)$, the subset of continuous functions on K which are analytic on the interior of K, is the same as $R(K)$, the uniform closure of the rational functions with poles in Ω.

The complexity of Vituskin's techniques justifies the development of alternate approaches to this problem. For a complete discussion of Vituskin's techniques and results see [14]. The alternate approach we have in mind exploits a recent result of Garnett and Glicksberg [5]. Namely, $R(K) = A(K)$ if they have the same representing measures for each point $\varphi \in K$.

We are unable, at present, to prove Vituskin's result. However, if Ω_i denotes the i^{th} component of Ω, if $A(n, z)$ denotes the annulus $\{(\frac{1}{2})^{n+1} \leqq | \xi - z | \leqq (\frac{1}{2})^n\}$, and if α denotes analytic capacity, then we prove the following

THEOREM. If K is such that (1) $\partial(K)$, the boundary of K, has finitely many components and (2) $\partial K = \{\bigcup \partial\Omega_i\} \cup \{x_1, x_2, \cdots\}$, where

$$\sum_{n=1}^{\infty} 2^n \alpha(A(n, x_k) \cap \Omega) = \infty^1$$

for each x_k, then $R(K) = A(K)$.

We let γ denote logarithmic capacity and we use the associated definitions found in Tsuji [10]. For the definition of analytic capacity and a proof of the fact that $\gamma(E) \geqq \alpha(E)$ see Zalcman [14].

In outline, the proof is as follows. We must show $R(K)$ and $A(K)$ have the same representing measures.

If, for two real measures μ_1 and μ_2,

$$\int \ln \left| \frac{1}{z - \xi} \right| d(\mu_1(\xi) - \mu_2(\xi)) = 0 \qquad \text{a.e. (plane Lebesgue measure)}$$

[1] Ahern has recently shown, among other things, (*A Condition for Peak Points*, to appear) that the hypothesis on the analytic capacity near x_k is unnecessary. See addendum.

then $\mu_1 = \mu_2$ [10]. In § 2 we prove a theorem to aid in evaluating the function

$$P(\mu, z) = \int \ln \left| \frac{1}{z - \xi} \right| d\mu(\xi) ,$$

for z in the support of μ, in terms of its values off the support of μ.

The principal result of § 3 is that if conditions (1) and (2) above are satisfied and if μ is the difference of two representing measures for $R(K)$ and the same $\phi \in K$, then $P(\mu, z)$ is continuous for all z and constant on each component of the boundary of K. This last fact allows us to identify the representing measures for $A(K)$ and $R(K)$. This proves the theorem.

The condition (due to Melnikov) on the inner boundary points x_i is used only to insure that the points x_i are peak points for $R(K)$.

We want to acknowledge observations made by Professor I. Glicksberg (private communication), which (a) simplify our original argument and (b) allow the presence of the exceptional points

$$\{x_n\} \not\subset \{\bigcup \partial\Omega_i\} .$$

2. A theorem on logarithmic potential for plane measures. Let E be a Borel set in the plane and let μ be a real measure supported on E. Define $P(\mu, z)$, the logarithmic potential of μ, by the formula

$$P(\mu, z) = \int_E \ln \left| \frac{1}{z - \xi} \right| d\mu(\xi) .$$

$P(\mu, z)$ is obviously harmonic off E. We will be concerned with its behavior on E if μ is a linear combination of representing measures.

The proof of the following theorem structured after Carleson [3]. The use of the equilibrium distribution measures was suggested by Professor P. C. Curtis, Jr.

THEOREM 1. *Let μ be a real measure supported on a compact plane set E. Let $z_0 \in E$ be such that*

$$\int_E \ln \left| \frac{1}{z_0 - \xi} \right| d\mu(\xi) = P(\mu, z_0)$$

converges absolutely. Let $D(r, z_0)$ be the open disk with radius r and center z_0. If V is an open set such that

$$\limsup_{r \to 0} \frac{\gamma(V \cap D(r, z_0))}{r} > 0 ,$$

then there is a sequence $r_n \to 0$ and probability measures ν_n, independent of μ and supported in $V \cap D(r_n, r_0)$, such that

$$\lim_{n\to\infty} \int P(\mu, z) d\nu_n(z) = P(\mu, z_0) .$$

Proof. Suppose $z_0 = 0$. Choose a sequence $r_n \to 0$ so that for some $a > 0$

$$\gamma(V \cap D(r_n, 0)) > 4ar_n .$$

Now choose compact sets $F_n \subset V \cap D(r_n, 0)$ so that

$$\gamma(F_n) > 2ar_n .$$

Let ν_n be the equilibrium distribution for F_n. We shall show that $\{\nu_n\}$ is the desired sequence of measures.

First we bound $P(\nu_n, \xi)$. If $|\xi| \geqq 2r_n$ then, since ν_n is positive with total mass one,

$$\begin{aligned} \int \ln \left|\frac{1}{z-\xi}\right| d\nu_n(z) &= \ln\left|\frac{1}{\xi}\right| + \int \ln\left|\frac{1}{1 - z/\xi}\right| d\nu_n(z) \\ &\leqq \ln\left|\frac{1}{\xi}\right| + \ln 2 . \end{aligned}$$

If $|\xi| < 2r_n$ then, by Frostman's theorem [10]

$$\int \ln\left|\frac{1}{z-\xi}\right| d\nu_n(z) \leqq \ln \frac{1}{\gamma(F_n)} \leqq \ln\frac{2}{a} + \ln\left|\frac{1}{\xi}\right| .$$

Hence $P(\nu_n, \xi) \leqq c + \ln|1/\xi|$.

Now, for fixed ρ,

$$\left|\int_{|z|\leqq r_n} P(\mu, z) d\nu_n(z) - \int_E \ln\left|\frac{1}{\xi}\right| d\mu(\xi)\right|$$

$$(I_1) \qquad \leqq \left|\int_{|\xi|<\rho}\left(\int_{|z|\leqq r_n} \ln\left|\frac{1}{z-\xi}\right| d\nu_n(z)\right)d\mu(\xi)\right|$$

$$(I_2) \qquad + \left|\int_{|\xi|\geqq\rho}\left(\int_{|\xi|\leqq r_n}\left(\ln\left|\frac{1}{z-\xi}\right| - \ln\left|\frac{1}{\xi}\right|\right)d\nu_n(z)\right)d\mu(\xi)\right|$$

$$(I_3) \qquad + \left|\int_{|\xi|\geqq\rho} \ln\left|\frac{1}{\xi}\right| d\mu(\xi) - \int_E \ln\left|\frac{1}{\xi}\right| d\mu(\xi)\right| .$$

Clearly

$$\begin{aligned} I_1 &\leqq \int_{|\xi|<\rho}\left(c + \ln\left|\frac{1}{\xi}\right|\right)d\mu(\xi) \\ I_2 &\leqq \int_{|\xi|>\rho}\int_{|z|\leqq r_n}\left|\ln\frac{|\xi|}{|z-\xi|}\right| d\nu_n(z)d\mu(\xi) \\ I_3 &\leqq \int_{|\xi|<\rho} \ln\left|\frac{1}{\xi}\right| d\mu(\xi) . \end{aligned}$$

Choose ρ so that $I_1 + I_3 < \varepsilon/2$ and then choose N so that

$$\int_{|z|\leqq r_N} \ln \frac{|\xi|}{|z-\xi|} d\nu_N(z) \leqq \int_{|z|\leqq r_N} \ln \left|\frac{1}{z/\rho - 1}\right| d\nu_N(z) \leqq \frac{\varepsilon}{2\,\|\mu\|} .$$

Then $I_2 \leqq \varepsilon/2$. So, for $r_n \leqq r_N$,

$$\left| \int_{|z|\leqq r_n} P(\mu, z) d\nu_n(z) - \int_E \ln \left|\frac{1}{\xi}\right| d\mu(\xi) \right| < \varepsilon .$$

To apply Theorem 1 we will need the following estimate.

LEMMA 1. *Let $C(r, z_0)$ denote the circle with center z_0 and radius r. Let V be an open set such that $z_0 \in \partial V$. If for all small r the Lebesgue measure of $\{0 \leqq x \leqq r: C(x, z_0) \cap V \neq \emptyset\} = r$, then*

$$\limsup_{r\to 0} \frac{\gamma(D(r, z_0) \cap V)}{r} > 0 .$$

Proof. Tsuji [10, Corollary 6, p. 85].

3. The potential generated by representing measures for $R(K)$. Let $\varphi \in K$. Whenever it is convenient we will think of φ as a multiplicative linear functional on $R(K)$. A positive measure of mass one supported on ∂K is said to be a representing measure for $R(K)(A(K))$ and the functional (point) φ if

$$f(\varphi) = \int_{\partial K} f d\mu \quad \text{for all} \quad f \in R(K)(A(K)) .$$

We let $M_{\varphi,R}$ denote the collection of all representing measures for $R(K)$ and the point φ.

There is a distinguished member of $M_{\varphi,R}$ if φ is an interior point of K. Let E be the component of K^0, the interior of K, which contains φ. We have in mind the unique measure, λ_φ, supported on ∂E with the property that for all $f \in C(K)$ which are harmonic on K^0

$$f(\varphi) = \int_{\partial K} f d\lambda_\varphi .$$

We call λ_φ the harmonic measure for φ. It is not difficult, using hypothesis (2) and the fact that two plane measures with the same logarithmic potential are equal, to see that λ_φ is unique. Also observe that (2) guarantees that $P(\lambda_\varphi, z)$ is continuous for all z. To see this, note that each $x \in \partial E$ is a peak point for $R(K)$ and hence is a regular point for E. Now use the formula (Tsuji [10], p. 88)

$$g(z, \varphi) = \ln \left|\frac{1}{z-\varphi}\right| - \int_{\partial E} \ln \left|\frac{1}{z-\xi}\right| d\lambda_\varphi(\xi)$$

and recall that $g(z, \varphi)$ (Green's function) vanishes at regular points.

Let $S_{\varphi,R}$ denote the real linear span of $\{\nu - \lambda_\varphi : \nu \in M_{\varphi,R}\}$. The main result of this section is that hypothesis (2) implies $P(\mu, z)$ is constant on each component of ∂K for each $\mu \in S_{\varphi,R}$. We begin with some technical lemmas.

LEMMA 2. *If* $\varphi \in K^0$ *and* $\nu \in M_{\varphi,R}$,

$$P(\mu, z_0) = \int \ln \left| \frac{1}{z - \xi} \right| d\mu(\xi)$$

converges absolutely for each z_0 *in the boundary of some component of the complement of* K.

Proof. Let Ω_i denote a component of Ω for which $z_0 \in \partial\Omega_i$. If z_1 and z_2 belong to Ω_i,

$$\int_{\partial K} \left(\ln \left| \frac{1}{z_1 - \xi} \right| - \ln \left| \frac{1}{z_2 - \xi} \right| \right) d(\mu - \lambda_\varphi) = 0 ,$$

i.e., $P(\mu - \lambda_\varphi, z)$ is constant on Ω_1. Let $z_n \in \Omega_i$ and $z_n \to z_0 \in \partial\Omega_i$. If δ is the diameter of K then we may assume

$$\ln \frac{1}{3\delta} < P(\mu, z_n) = P(\mu - \lambda_\varphi, z_n) + P(\lambda_\varphi, z_n) .$$

Now $P(\mu - \lambda_\varphi, z_n) = C$ and

$$(*) \qquad | P(\lambda_\varphi, z_n) | = \left| \ln \left| \frac{1}{\varphi - z_n} \right| \right| \leqq M$$

imply

$$\liminf_{z_n \to z_0} P(\mu, z_n) < \infty .$$

By the lower continuity,

$$P(\mu, z_0) \leqq \liminf_{z \to z_0} P(\mu, z) \leqq C + M ,$$

and the lemma is proved.

LEMMA 3. *Fix a* $\varphi \in K^0$ *and a* $\nu \in M_{\varphi,R}$. *For each* $z \in \bigcup \partial\Omega_i$, *where* Ω_i *is a component of* Ω, *let the set* $W(z)$ *be the union of all connected subsets of* $\bar{\Omega}$ *containing* z *on which* $P(\nu - \lambda_\varphi, z)$ *is a constant. We assert that*

$$P(\nu, t) = \int \ln \left| \frac{1}{t - \xi} \right| d\upsilon(\xi)$$

converges absolutely for $t \in \overline{W(z)}$.

Proof. We need only consider $t \in \partial W(z)$. For such t use the proof of Lemma 2 (beginning with line 4) with Ω_i replaced by $W(z)$.

LEMMA 4. *For* $\varphi \in K^0$ *and* $\mu \in S_{\varphi,R}$, $P(\mu, z)$ *is constant on* $\bar{\Omega}_i$ *for each component* Ω_j *of* Ω.

Proof. By definition $\mu = \Sigma \alpha_i \mu_i$, where the summation is finite, $\mu_i = \nu_i - \lambda_\varphi$, and $\nu_i \in M_{\varphi,R}$. Then

$$P(\mu, z) = \Sigma \alpha_i P(\mu_i, z) = \Sigma \alpha_i P(\nu_i - \lambda_\varphi, z)$$

and

$$P(\nu_i - \lambda_\varphi, z)\,|_{\Omega_j} = C_{ij} \;.$$

By Lemma 2, $P(\nu_i - \lambda_\varphi, z)$ converges absolutely for each $z \in \partial\Omega_j$. Taking Ω_j to be the open set in the hypothesis of Lemma 1, we conclude from Theorem 1 and Lemma 1 that for $z \in \partial\Omega_j$,

$$C_{ij} = P(\nu_i - \lambda_\varphi, z) = P(\mu_i, z) \;.$$

Thus $P(\mu, z) = \Sigma \alpha_i C_{ij}$ is a constant on $\bar{\Omega}_j$.

THEOREM 2. *If* ∂K *satisfies* (2) *and* $\varphi \in K^0$ *then, for each* $\nu \in M_{\varphi,R}$, $P(\nu - \lambda_\varphi, z)$ *is constant on each component of* ∂K.

Proof. Let $W(z)$ be as in Lemma 3. If $x_n \in \overline{W(z)}$ for some $z \in \bigcup \partial\Omega_i$, then by Lemma 3, $P(\nu - \lambda_\varphi, x_n)$ converges absolutely. If $x_n \notin \bigcup \{\overline{W(z)}: z \in \partial\Omega_i\}$, then set $W(x_n) = \{x_n\}$.

Assert that each $W(z)$ is a closed set. To prove this we verify the hypothesis of Lemma 1 so that we may use Theorem 1. Fix $z \in \bigcup \partial\Omega_i$, let $z_1 \in \partial W(z)$, and pick $r_1 > 0$ so that $C(r, z_1) \cap W(z) \neq \varnothing$ for all $0 < r \leqq r_1$ (recall that $W(z)$ is connected). Let

$$E = \{0 < r \leqq r_1 : C(r, z_1) \cap \Omega \cap W(z) = \varnothing\} \cup \{0\} \;.$$

Evidently the complement of E is open. We assert that E is countable. First observe that for each component Ω_i of Ω there can be at most two distinct $r \in E$ with $C(r, z) \cap \bar{\Omega}_i \neq \varnothing$. Now if $r \in E$ there is a $y \in C(r, z_1) \cap W(z) \cap \bar{\Omega}$ and either $y = x_n$, for some n, or $y \in \partial\Omega_i$ for some i. Hence E is countable. Since E is closed and countable, we have, for small r, the Lebesgue measure of

$$\{x \leqq r;\, C(x, z_1) \cap W(z) \cap \Omega \neq \varnothing\} = r \;.$$

By Lemma 1

$$\limsup_{r \to 0} \frac{\gamma(W(z) \cap D(r, z_1) \cap \Omega)}{r} \geqq c > 0 .$$

By Theorem 1, with $V = W(z) \cap \Omega$, we have

$$P(\nu - \lambda_\varphi, z_1) = P(\nu - \lambda_\varphi, z)$$

and hence $W(z)$ is closed.

Finally note that, by Lemma 4 there are only countably many distinct sets $W(z)$ for $z \in \bigcup \partial\Omega_i \bigcup \{x_1, x_2, \cdots\}$.

Let Γ be a component of ∂K. If $\Gamma \not\subset W(z)$ for some z, then a countable union of the $W(z)$ cover Γ. However it is standard fact [8] that a connected set cannot be the disjoint union of countably may closed sets. Hence $\Gamma \subset W(z)$ for some $z \in \bigcup \partial\Omega_i \bigcup \{x_1, x_2, \cdots\}$ (indeed for some $z \in \bigcup \partial\Omega_i$, if ∂K contains no singletons) and $P(\nu - \lambda_\varphi, z)$ is constant on Γ.

COROLLARY. *If, in addition to the above hypothesis, ∂K has a finite number of components then, for $\mu \in S_{\varphi,R}$, $P(\mu, z)$ is a continuous function of z and is harmonic except on ∂K.*

Proof. Write $P(\mu, z) = \sum \alpha_i P(\mu_i, z)$ where $\mu_i + \lambda_\varphi = \nu_i \in M_{\varphi,R}$. Thus $P(\nu_i, z)|_{\partial K}$ is continuous. Hence, by Tsuji III. 2. [10], $P(\mu_i, z) = P(\nu_i, z) - P(\lambda_\varphi, z)$ is continuous for all z.

4. Representing measures for $R(K)$ and $A(K)$. $A(K)$ is the Banach algebra of all functions on K and analytic on K^0. Arens [2] shows that multiplicative linear functionals on $A(K)$ can be identified with the points of K, so that $A(K)$ and $R(K)$ have the same maximal ideal space. In this section we show that $R(K)$ and $A(K)$ have the same representing measures for each $\varphi \in K$ provided that hypothesis (1) and (2) hold.

As Glicksberg observed, it is sufficient to show that for each $\varphi \in K$ any $\mu \in S_{\varphi,R}$ annihilates $A(K)$. For if $\nu \in M_{\varphi,R}$ then $\nu - \lambda_\varphi \in S_{\varphi,R}$ so that ν is a representing measure for $A(K)$. Hence by Garnett and Glicksberg [5] we are done. Finally note (i) by Silov's Idempotent theorem we can assume K is connected and then (ii) there are no isolated points in ∂K since K is compact.

LEMMA 5. *If ∂K has $n + 1$ components and $\varphi \in K^0$ then dimension of $S_{\varphi,R} \leqq n$.*

Proof. First suppose $\nu_1, \cdots, \nu_{n+2} \in S_{\varphi,R}$. For each ν_j, let $C_{jk} = P(\nu_j, z)|_{\Gamma_k}$, where Γ_k is the k^{th} component of ∂K. By Theorem 2 the C_{jk}'s are constant. The matrix (C_{jk}) is obviously singular and hence there are real scalars $\alpha_1, \cdots, \alpha_{n+2}$ such that

$$(*) \qquad \Sigma\alpha_j P(\nu_j, z)\,|_{\partial K} \equiv 0 \qquad j \in \{1, \cdots, n+2\}\ .$$

However, by the corollary to Theorem 2 the potential generated by the measure

$$\Sigma\alpha_j\nu_j \in S_{\varphi,R}$$

is a continuous function and is harmonic except on ∂K where, by $(*)$, it is zero. Hence by the maximum principle for harmonic functions

$$P(\Sigma\alpha_j\nu_j, z) = 0 \qquad \text{all } z\ .$$

Since the zero measure is the only measure with zero potential we conclude that the dimension of $S_{\varphi,R} \leqq n+1$.

Finally, if Ω_∞ is the unbounded component of Ω then $P(\nu, z) = 0$ on $\bar{\Omega}_\infty$ for all $\nu \in S_{\varphi,R}$. Hence dimension of $S_{\varphi,R} \leqq n$.

LEMMA 6. *If K satisfies* (1) *and* (2) *and* $\varphi \in K^0$ *then* $S_{\varphi,R}$ *annihilates* $A(K)$.

Proof. Essentially the proof is the identification of a basis for $S_{\varphi,A}$. We construct measures μ_i on ∂K as suggested by Ahern and Sarason [1] (see also Garnet and Glicksberg [5]).

The hypothesis on K implies $\bar{\Omega}$ has a finite number of components. Each component, Γ_i^*, of $\bar{\Omega}$ may separated from the other components by a finite number of simple smooth oriented contours whose union we denote by Λ_i. For $f \in C(\partial K)$, let $\tilde{f}$ be its harmonic extension to K^0 and for each Γ_i^*, except the one containing ∞, let

$$\int_{\partial K} f d\mu_i = \frac{1}{2\pi}\int_{\Lambda_i} \frac{\partial}{\partial n}\tilde{f} ds\ .$$

($\partial/\partial n$ is the normal derivative). The following facts about μ_i are easily established:

(1) if $f \in A(K)$, $\int_{\partial K} f d\mu_i = 0$

(2) $\int_{\partial K} \ln\left|\frac{1}{z-a}\right| d\mu_i = \begin{cases} 1 \text{ if } a \in \Gamma_i^* \\ 0 \text{ if } a \in \bar{\Omega}\backslash\Gamma_i^* \end{cases}$

By Theorem 2, for $\nu \in S_{\phi,R}$, $P(\nu, z)$ is constant on each component Γ_i^* of $\bar{\Omega}$ hence, for all z,

$$P(\nu, z) = \Sigma\alpha_i P(\mu_i, z) \qquad i = 1, \cdots, n-1\ .$$

Thus $\nu = \Sigma\alpha_i\mu_i$, i.e., $\nu \perp A(K)$.

COROLLARY. *$R(K)$ and $A(K)$ have the same representing measures.*

Proof. Now we need only concern ourselves with points $z \in \partial K$. If $z \in \bigcup \partial \Omega_i$ then it is easy to see that

$$\Sigma 2^n \alpha(A(n, z) \cap \Omega) = \infty .$$

If $z \in \partial K - \bigcup \partial \Omega_i$ then, by assumption,

$$\Sigma 2^n \alpha(A(n, z) \cap \Omega) = \infty .$$

In either case by [4, Th. 3.5], z is a peak point for $R(K)$ so that the only representing measure is the unit mass at z. Hence $A(K)$ and $R(K)$ have the same representing measures for each $z \in K$.

The desired generalization of Mergelyan's theorem now follows from Garnett and Glicksberg [5, Th. 1.7].

5. **Added August 19, 1968.** Since this paper was written Ahern (*A condition for Peak Points*, to appear in the Duke Math. Journal) has proven, among other things, that each $x_n \in \partial K - \{\bigcup \partial \Omega_i\}$ is a peak point provided that $\partial K - \{\bigcup \partial \Omega_i\}$ is countable. Ahern's argument can be simplified as follows. First, as Ahern observes, because ∂K has finitely many components each x_n is a regular point for K, we can apply Theorem 2. Suppose x_n is not a peak point. By Wilkin's theorem, the part, P, containing x_n has positive planar measure. Since $P \cap (\bigcup \partial \Omega_i) = \phi$, P contains a point $\phi \in K^0$. Let $\mu \in M_{x_n, R}$, $\mu(\{x_n\}) = 0$. By a theorem of Bishop there exists $0 < c < 1$ and $\mu_\phi \in M_{\phi, R}$ such that $\mu_\phi - c\mu \geqq 0$. Hence $\nu_\phi = (\mu_\phi - c\mu) + c\delta_{x_n} \in M_{\phi, R}$ and $P(\nu_\phi, x_n) = \infty$. This contradicts Theorem 2. (An argument along these lines was suggested to me independently by A. M. Davie and J. Garnett.)

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UNIVERSITY OF CALIFORNIA, LOS ANGELES, AND
CALIFORNIA STATE COLLEGE, LONG BEACH