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DESARGUES' LAW AND THE REPRESENTATION OF PRIMARY LATTICES

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In an earlier work of B. Jónsson and the author it was shown that an Arguesian primary lattice of geometric dimension at least 3 can be represented as the submodule lattice of a finitely generated module over a completely primary uniserial ring. Inasmuch as the class of primary lattices includes the class of subspace lattices of (nondegenerate) projective geometries, two questions then naturally arise: (1) Is a primary lattice of geometric dimension at least 4 Arguesian? (2) Is an Arguesian primary lattice of geometric dimension 2 representable?

The first question is answered in the affirmative in §1, thus showing that the abovementioned paper subsumes the results of E. Inaba on the representation of primary lattices of geometric dimension at least 4. A counter example is given in §2 showing that an Arguesian lattice of geometric dimension 2 cannot, in general, be represented, but for reasons far deeper than the cardinality arguments given for the representability or nonrepresentability of subspace lattices of 1-dimensional projective geometries.

We will assume that the reader is familiar with the first six sections of [3] (henceforth referred to as PAL) and will adopt the notation and terminology of that work.

It is well known that the class of finite dimensional simple complemented modular lattices of dimension 3 or more coincides with the class of subspace lattices of (nondegenerate) projective geometries of dimension at least 2, and that those lattices which are Arguesian correspond to Arguesian geometries, i.e., those geometries that can be coordinatized by division rings. We will freely use these facts to translate arguments in geometry to arguments in lattice theory. In particular, they will be applied to complemented intervals in primary lattices, which, in view of 6.3 of PAL, are simple.

1. LEMMA 1.1. *Given cycles* $\{w_i\}_{i=0}^{n}$ *in a primary lattice* L *such that* $w_i \not\leqq \sum\{w_j \mid j \neq i\}$, *there are cycles* $\{x_i\}_{i=0}^{n}$ *of* L *and a permutation* φ *of* $\{0, 1, 2, \cdots, n\}$ *such that* $\sum_0^j w_{\varphi(i)} = \dot{\sum}_0^j x_i$ *for* $0 \leqq j \leqq n$, *and* $0 < d[x_j] \leqq d[x_{j-1}]$ *for* $j = 1, 2, \cdots, n$.

Proof. We will show by induction on m that for $0 \leqq m \leqq n$, there is a set of cycles $\{\{w_{ij}\}_{j=i}^{n}\}_{i=0}^{m}$ such that:

$(1)_m$ $\{w_{0i}\}_{i=0}^{n} = \{w_j\}_{j=0}^{n}$,

$(2)_m$ $d[w_{m,m}] \geqq d[w_{m,j}]$, $\qquad j \geqq m$,
$(3)_m$ $0 < d[w_{ii}] \leqq d[w_{i-1,i-1}]$, $\qquad 1 \leqq i \leqq m$,
$(4)_m$ $\sum_0^{m-1} w_{ii} + \dot{w}_{kj} = \sum_0^{m-1} w_{ii} + w_{mj}$, $\qquad k < m \leqq j$,
$(5)_m$ $\sum_0^j w_{0i} = \sum_0^j w_{ii}$, $\qquad j \leqq m$.

For then, taking $m = n$, we will have the desired set of cycles by letting x_i be w_{ii} and φ the permutation given by $w_{0i} = w_{\varphi(i)}$. Inasmuch as such a set of cycles is given for $m = 0$ by taking w_{00} to be the element of $\{w_i\}_{i=0}^n$ of greatest dimension and $\{w_{0i}\}_{i=1}^n$ the remaining elements of $\{w_i\}_{i=0}^n$, we can assume that we have such a set for $0 < m < n$. Then, setting $d[w_{mm}] = r$, we infer from 4.14 of PAL that, for $j > m$, $\sum_0^m w_{ii}[r]$ is geometric in $[0, \sum_0^m w_{ii}[r] + w_{mj}]$, so that, by 5.2 of PAL, it has a complement $w_{m+1,j}$ in this interval. It then follows that

$$(6) \qquad \sum_0^m w_{ii} + w_{m+1,j} = \sum_0^m w_{ii} + w_{mj} ,$$

whence, by a routine argument, we infer that

$$[0, w_{m+1,j}] \cong [(\sum_0^m w_{ii}) w_{mj}, w_{m,j}] .$$

This, together with the assumption that $w_j \nleqq \sum\{w_i \mid i \neq j\}$ yields that $w_{m+1,j}$ is a cycle such that

$$(7) \qquad 0 < d[w_{m+1,j}] \leqq d[w_{m,j}] .$$

Now, if s has the property that $w_{m+1,s}$ is of greatest dimension among the elements $\{w_{m+1,j}\}_{j=m+1}^n$, set

$$w'_{ij} = \begin{cases} w_{ij} & j \neq s, m+1 \\ w_{is} & j = m+1 \\ w_{i,m+1} & j = s . \end{cases}$$

Observe that since s and $m + 1$ are both greater than m, the cycles $\{\{w'_{ij}\}_{j=i}^n\}_{i=0}^m$ satisfy the formulas $(1)_m$ through $(5)_m$, and that $\{w'_{ij}\}_{j=i}^n\}_{i=0}^{m+1}$ satisfy (6) and (7) as well. It is clear that $(1)_{m+1}$ and $(2)_{m+1}$ hold, while from (7), $(2)_m$ and $(3)_m$ we conclude that $(3)_{m+1}$ holds. That $(4)_{m+1}$ holds in the case $k = m$ follows from (6). On the other hand, for $k < m$, we infer from $(4)_m$ and (6) that

$$w'_{mm} + \sum_0^{m-1} w'_{ii} + w'_{kj} = w'_{mm} + \sum_0^{m-1} w'_{ii} + w'_{mj} = \sum_0^m w'_{ii} + w'_{m+1,j} ,$$

so that $(4)_{m+1}$ holds. Finally, we infer from $(5)_m$ and $(4)_{m+1}$ that $\sum_0^{m+1} w_{0i} = \sum_0^{m+1} w_{ii}$, which together with $(5)_m$ and the fact that $(\sum_0^m w_{ii}) w_{m+1,m+1} = 0$ yields $(5)_{m+1}$ and completes the proof of the lemma.

Notation. Given a primary lattice L, denote the sum of all atoms of L by $p(L)$.

For any element x of a primary lattice L, we clearly have that $x[1] = xp(L)$.

As an immediate consequence of 4.14 of PAL and 1.1, we have

COROLLARY 1.2. *Given cycles* $\{w_i\}_{i=0}^n$ *in a primary lattice* L *such that* $w_i \nleqq \sum\{w_j \mid j \neq i\}$, *the element* $(\sum_0^n w_i)[1]$ *of* $[0, p(L)]$ *is of dimension* $n + 1$.

LEMMA 1.3. *Given cycles* u, w_0, w_1, w_2 *and* w_3 *in a primary lattice* L *such that* u *is contained in* $\sum_0^3 w_i$ *and is disjoint from* $\sum\{w_i \mid i \neq j\}$ *for* $j = 0, 1, 2, 3$, *there is an integer* $n = 0, 1, 2, 3$ *and a cycle* s *of* L *disjoint from* $\sum\{w_i \mid i \neq j\}$ *for* $j = 0, 1, 2, 3$ *such that* $w_n \leqq s + \sum\{w_i \mid i \neq n\}$.

Proof. Letting $W_i = \sum\{w_j \mid j \neq i\}$, we might as well assume that $w_i \nleqq W_i$ for every i, since otherwise u would suffice. We then infer from 1.1 that there are cycles $\{x_i\}_{i=0}^3$ and a permutation φ of $\{0, 1, 2, 3\}$ such that $\sum_0^j w_{\varphi(i)} = \dot{\sum}_0^j x_i$ and $0 < d[x_j] \leqq d[x_{j-1}]$ for $1 \leqq j \leqq 3$. Thus, setting $k = d[x_3]$ and

$$a = (\dot{\sum}_0^3 x_i)[k] = \dot{\sum}_0^3 x_i[k] ,$$

we have that $[0, a]$ is regular and

$$a \geqq (\sum_0^3 x_i)[1] \geqq (\sum_0^3 w_{\varphi(i)})[1] \geqq u$$

so that, by 5.3 of PAL, u is contained in a point s of $[0, a]$. We infer immediately from 4.14 of PAL that s is disjoint from W_i for every i, since u has this property. Moreover,

$$0 = sW_{\varphi(3)} = s(x_0 + x_1 + x_2) = s(x_0[k] \dotplus x_1[k] \dotplus x_2[k]) ,$$

whence, by a dimension argument,

$$a = s \dotplus x_0[k] \dotplus x_1[k] \dotplus x_2[k] ,$$

and

$$s + W_{\varphi(3)} = s + x_0 + x_1 + x_2 \geqq a \geqq x_3[k] = x_3 .$$

Inasmuch as $s + W_{\varphi(3)}$ also contains $x_0 + x_1 + x_2$, we conclude that $s + W_{\varphi(3)} \geqq w_{\varphi(3)}$ and s is the desired element.

LEMMA 1.4. *Let* x_0, x_1, x_2 *be elements of a primary lattice* L *such that* x_1 *and* x_2 *are cycles and* $(x_0 + x_1)[1] \neq (x_0 + x_2)[1]$. *Then* $(x_0 + x_1)(x_0 + x_2) = x_0$.

Proof. Observe that $x_0 + x_1$ and $x_0 + x_2$ are cycles in the primary lattice $L' = [x_0, 1]$ such that

$$(x_0 + x_1)p(L') \neq (x_0 + x_2)p(L') .$$

Thus, by 4.14 of PAL, $(x_0 + x_1)(x_0 + x_2) = x_0$, the zero element of L'.

(1.5) THEOREM. *Given cycles $a_0, a_1, a_2, b_0, b_1, b_2, c$, in a primary lattice L, let $a_3 = c = b_3$, and*

$$A_i = \sum\{a_j \mid j \neq i\}, B_i = \sum\{b_j \mid j \neq i\} , \quad i = 0, 1, 2, 3 ,$$
$$p_j = (a_i + a_k)(b_i + b_k) , \quad i \neq j \neq k, i, j, k, = \{0, 1, 2\} .$$

If $a_i + c = a_i + b_i = b_i + c$ for $i = 0, 1, 2$, and there are cycles $\{t_i\}_{i=0}^{3}$ in L such that $d[t_i] \geqq d[c]$ and $t_iA_i = 0$ for $i = 0, 1, 2, 3$, then $a_0 \leqq a_1 + p_2(p_0 + p_1)$.

Proof. Case 1. $\{a_0, a_1, a_2\}\perp$ or $A_0[1] \geqq A_1[1], A_2[1]$. We will first show that there is an atom u of L disjoint from $A_i(1)$ for $i = 0, 1, 2, 3$. There is clearly such an element under any of the conditions:

$$A_0[1] \geqq A_1[1], A_2[1]; \ \{a_0, a_1, a_{2},{}_3a_3\};$$

$a_i \leqq A_i$ for some $i = 0, 1, 2, 3$. Thus we might as well assume that

$$\{a_0, a_1, a_2\}\perp ; a_3A_3 > 0; a_i \nleqq A_i \qquad i = 0, 1, 2, 3 .$$

First observe that according to 1.2, the elements $\{A_i[1]\}_{i=0}^{3}$ are planes in the 4-dimensional simple complemented modular lattice $[0, (\sum_3^0 a_i)[1]]$. Furthermore, inasmuch as a_3 is a cycle and

$$0 = (a_0 + a_1)(a_0 + a_2)(a_1 + a_2) \geqq a_3(a_0 + a_1)(a_0 + a_2)(a_1 + a_2) ,$$

we have that $\{a_3, a_i, a_j\}\perp$ for some i and j, so that for k distinct from i, j and 3, $A_k[1] = a_i[1] \dotplus a_j[1] \dotplus a_3[1]$. Since the assumption that $a_3A_3 > 0$ implies that $a_3[1] \leqq A_3[1]$, we then infer that $A_k[1] \leqq A_3[1]$. But then $A_k[1] = A_3[1]$, because these elements are both planes in $[0, (\sum_0^3 a_i)[1]]$. We can therefore assume that the planes $A_0[1], A_1[1]$ and $A_2[1]$ are distinct, for otherwise we would immediately have the existence of the desired atom u. But then, if $A_i[1]A_k[1] \leqq A_j[1]$ for i, j and k distinct in $\{0, 1, 2\}$, we infer from 1.4 that

$$A_i[1]A_k[1] \leqq A_j[1]A_k[1] = (a_i + a_3)[1] ,$$
$$A_i[1]A_k[1] \leqq A_i[1]A_j[1] = (a_k + a_3)[1] .$$

Inasmuch as $(a_i + a_3)[1]$ and $(a_k + a_3)[1]$ are lines and $A_i[1]$ and $A_k[1]$ are distinct planes in $[0, \sum_0^3 a_i)[1]]$, these formulas are equalities, and the points $a_0[1], a_1[1]$ and $a_2[1]$ are contained on the line $(a_i + a_3)[1]$.

This contradicts, by 4.14 of PAL, the independence of the cycles a_0, a_1 and a_2, and we conclude that $A_i[1]A_k[1] \not\leqq A_j[1]$ for i, j and k distinct in $\{0, 1, 2\}$. From this we see that the desired point u of $[0, (\sum_0^3 a_i)[1]]$ exists by considering the dual situation in which there are four points in a 4-dimensional simple complemented modular lattice of which exactly three are distinct and are not collinear. It is clear that in such a case there is a plane not containing any of the points.

Thus we can apply Lemma 1.3 to obtain a cycle s of L disjoint from A_i for every i and such that $s + A_n \geqq a_n$ for some n. Then, letting

$$a_i' = (s + a_i)A_n\ , \qquad b_i' = (s + b_i)A_n\ , \qquad i = 0, 1, 2$$
$$c' = (s + c)A_n\ ,$$

it is easily seen that $d[c'] \leqq d[c]$, and $a_i' + b_i' = a_i' + c' = b_i' + c'$ for $i = 0, 1, 2$. Choosing a cycle $c_1 \leqq t_n$ of the same dimension as c', and a cycle c_2 such that $c_1 \dotplus c' = c_2 \dotplus c' = c_1 \dotplus c_2$, let $d_i = (c_1 + a_i')(c_2 + b_i')$ for $i = 0, 1, 2$, and

$$g_i = (c_1 + d_j + d_k)(c_2 + d_j + d_k)A_n\ , \qquad i \neq j \neq k \neq i, i = 0, 1, 2\ ,$$

and cyclically. It can then be shown that
$a_i + d_i = c_1 + a_i' = e_1 + d_i$, and $b_i' + d_i = c_2 + b_i' = c_2 + d_i$ for $i = 0, 1, 2$. Clearly

$$\begin{aligned} g_2(g_0 + g_1) &\geqq (d_0 + d_1)A_n[(d_1 + d_2)A_n + (d_0 + d_2)A_n] \\ &= (d_0 + d_1)A_n[d_1 + d_2 + (d_0 + d_2)A_n] \\ &= (d_0 + d_1)A_n\ , \end{aligned}$$

since

$$d_2 + A_n \geqq d_2 + a_2' + a_0' = d_2 + c_1 + a_0' \geqq d_0\ .$$

Thus

$$(8) \qquad a_1' + g_2(g_0 + g_1) \geqq (a_1' + d_0 + d_1)A_n = (a_1' + d_0 + c_1)A_n \geqq a_0'\ .$$

On the other hand, since s is disjoint from every A_i, we have that for $j, k = 0, 1, 2$

$$s(a_j + a_k + b_j + b_k) = s(c + a_j + a_k) = 0\ ,$$
$$s(p_0 + p_1 + p_2) \leqq s(a_0 + a_1 + a_2) = 0\ ,$$

so that $\{s, a_j + a_k, b_j + b_k\}D$, and $\{s, p_2, p_0 + p_1\}D$. Consequently

$$\begin{aligned} g_0 &= (c_1 + a_1' + a_2')(c_2 + b_1' + b_2')A_n = (a_1' + a_2')(b_1' + b_2') \\ &= (s + a_1 + a_2)(s + b_1 + b_2)A_n = (s + p_0)A_n \end{aligned}$$

and cyclically, whence

$$\begin{aligned} g_2(g_0 + g_1) &= (s + p_2)A_n[(s + p_0)A_n + (s + p_1)A_n] \\ &= [s + p_2(p_0 + p_1)]A_n\ , \end{aligned}$$

and

$$(9) \qquad a_1' + g_2(g_0 + g_1) = [a_1' + s + p_2(p_0 + p_1)]An \ .$$

Combining (8) and (9), we have

$$a_0' \leqq a_1' + s + p_2(p_0 + p_1) \ ,$$

and

$$s + a_0 \leqq s + a_1 + p_2(p_0 + p_1) \ .$$

Multiplying both sides of this formula by $a_0 + a_1$ and observing that $s(a_0 + a_1) \leqq sA_3 = 0$, we conclude that $a_0 \leqq a_1 + p_2(p_0 + p_1)$.

Case 2. $\{a_0, a_1, a_2\} \not\perp$ and $A_i[1] \leqq A_0[1]$ for $i = 1$ or 2. Let $u = (a_0 + a_1)(a_0 + a_2)(a_1 + a_2)$, and $a_i' = a_i + u$ for $i = 0, 1, 2, 3$. Then $a_i' + a_j' = a_i + a_j$ for $i, j = 0, 1, 2$, $(a_0' + a_1')(a_0' + a_2')(a_1' + a_2') = u$, and $\{a_0', a_1', a_2'\} \perp$ in $[u, 1]$. Thus, letting $t_i' = t_i + u$ for $i = 0, 1, 2, 3$, and $A_0' = a_1' + a_2' + a_3'$ and cyclically, since

$$t_i'A_i' = (t_i + u)A_i = u + t_iA_i = u \ ,$$
$$d[t_i'] = d[t_i] + d[u] - d[t_iu] = d[t_i] + d[u] \geqq d[c'] \ ,$$

we can apply Case 1 to $\{a_i', b_i'\}_{i=0}^3$ to conclude that

$$(10) \qquad a_0' \leqq a_1' + p_2'(p_0' + p_1') \ ,$$

where

$$p_i' = (a_j' + a_k')(b_j' + b_k') = (a_j + a_k)(b_j + b_k + u) = u + p_i \ .$$

We infer from (10) that

$$\begin{aligned} a_0 &\leqq u + a_1 + (u + p_2)(u + p_0 + p_1) \\ &= u + a_1 + p_2B_3(u + p_0 + p_1) \\ &= u + a_1 + p_2(p_0 + p_1 + uB_3) \ . \end{aligned}$$

Inasmuch as $A_i[1] \not\leqq A_0[1]$ for $i = 1$ or 2, and $A_i = B_i$ for $i = 0, 1, 2$, we see that $B_3[1]$ must be distinct from $B_k[1]$ for some $k = 0, 1, 2$, whence, by Lemma 1.3,

$$uB_3 \leqq B_0B_1B_2B_3 \leqq B_kB_3 = b_i + b_j \ ,$$

and $uB_3 \leqq p_k$ for this k. Thus

$$u + a_1 + p_2(p_0 + p_1 + uB_3) = u + a_1 + p_2(p_0 + p_1) \ ,$$
$$a_0 \leqq a_1 + a_0(a_1 + a_2) + p_2(p_0 + p_1) \ ,$$

and

$$a_0 = a_0(a_1 + a_2) + a_0(a_1 + p_2(p_0 + p_1)) .$$

Since a_0 is a cycle, we conclude that $a_0 \leqq a_1 + a_2$ or $a_0 \leqq a_1 + p_2(p_0 + p_1)$. However, inasmuch as $a_0 \leqq a_1 + a_2$ implies that $A_1, A_2 \leqq A_0$, contradicting the assumption that $A_i[1] \not\leqq A_0[1]$ for $i = 1$ or 2, we have the desired conclusion.

DEFINITION. A lattice L is said to be *Arguesian* if, for any six elements $a_0, a_1, a_2, b_0, b_1, b_2$ of L,

(11) $\quad (a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leqq a_0(a_1 + p_2(p_0 + p_1)) + b_0(b_1 + p_2(p_0 + p_1))$

where for i, j, k distinct in $\{0, 1, 2\}$, $p_i = (a_j + a_k)(b_j + b_k)$.

Notation. We will denote the left and right hand sides of the inequality (11) by l and r respectively. For i, j and k distinct in $\{0, 1, 2\}$, we will write $g_i = (a_j + b_j)(a_k + b_k)$. If there is, in the same situation another ordered sextuplet $a'_0, a'_1, a'_2, b'_0, b'_1, b'_2$ of elements from the lattice L, we will denote the polynomials formed from them as above by l', r', p'_i, and g'_i.

THEOREM 1.6. *A primary lattice L of geometric dimension at least 4 is Arguesian.*

Proof. Inasmuch as the formula (11) holds trivially for a sextuplet $a_0, a_1, a_2, b_0, b_1, b_2$ when one of the elements is an atom and the rest are zero, we can proceed by induction on $\sum_0^2 (d[a_i] + d[b_i])$. Letting $a'_0 = a_0(b_0 + g_0)$, and $b'_0 = b_0(a_0 + g_0)$, if $a'_0 < a_0$ or $b'_0 < b_0$, we can apply the inductive hypothesis to conclude that $l = l' \leqq r' \leqq r$. Thus, we might as well assume that $a'_0 = a_0$ and $b'_0 = b_0$, or, equivalently, that $a_0 + g_0 = b_0 + g_0$. Similarly, we can assume that $a_i + g_i = b_i + g_i$, for $i = 1, 2$. Next suppose that a_0 is not a cycle. Then $a_0 = a'_0 + a''_0$ where $a'_0 < a_0$ and $a''_0 < a_0$, and, by the inductive hypothesis applied to $(a'_0, a_1, a_2, b_0, b_1, b_2)$ and $(a''_0, a_1, a_2, b_0, b_1, b_2)$, we infer that $l' \leqq r' \leqq r$, and $l'' \leqq r'' \leqq r$. Inasmuch as $l' + l'' = l$, we conclude that $l \leqq r$. Thus we can assume that the elements $a_0, a_1, a_2, b_0, b_1, b_2$ are cycles. Finally, let c be any cycle contained in l and let $a'_i = a_i(b_i + c)$ and $b'_i = b_i(a_i + c)$ for $i = 0, 1, 2$. It is easily shown that $a'_i + b'_i = a'_i + c = b'_i + c$, so that, since L is of geometric dimension at least 4, we can apply Theorem 1.5 to conclude that $a'_0 \leqq a'_1 + p'_2(p'_0 + p'_1)$, and $b'_0 \leqq b'_1 + p'_2(p'_0 + p'_1)$. Thus $r \geqq r' = a'_0 + b'_0 \geqq c$, and we have that r contains every cycle that is contained in l. Inasmuch as l is the sum of the cycles it contains, it follows that $r \geqq l$, as was to be shown.

2. In this section we exhibit an Arguesian primary lattice of

geometric dimension 2 which cannot be represented as the lattice of submodules of a finitely generated module over a completely primary uniserial ring, thus showing that the assumption that L be Arguesian or of geometric dimension at least 4 is necessary for the representation theorem.

Notation. The submodule lattice of a module M will be denoted by $L(M)$. If M consists of n-tuples from the ring R, we will denote by $[r_1, r_2, \cdots, r_n]$, the member of $L(M)$ spanned by the element $(r_1, r_2, \cdots, r_n)$.

We will consider throughout this section a fixed field K and a one-to-one map φ of K onto itself such that φ is *not* an automorphism but has the property: $\varphi(0) = 0, \varphi(1) = 1$. We also fix the vector space V of 5-tuples from K and denote by u_i, the element of V with 1 in the ith place and 0 elsewhere. Then, letting $Q = [u_1] + [u_2]$, and $P = Q + [u_3]$ in $L(V)$, we define a one-to-one map of the elements of $L(V)$ covered by Q onto the elements of $L(V)$ covering P by

$$F([r, 1, 0, 0, 0]) = P + [0, 0, 0, \varphi(r), 1] ,$$
$$F([u_1]) = P + [u_4] .$$

Finally, we define the following subset of $L(V)$:

$$L_\varphi = [0, P] \cup [Q, V] \cup \bigcup_{X<<Q} [X, F(X)] .$$

We will refer to the intervals $[0, P]$, $[Q, V]$ and $[X, F(X)]$ for $X \ll Q$ as the intervals used to define L_φ.

Observe that $[0, P]$ and $[Q, V]$ are subspace lattices of 3-dimensional vector spaces over K and hence can be viewed as subspace lattices of projective geometries S_1 and S_2 respectively. In this way F is a one-to-one map of the points on the line Q in S_1 to the lines containing the point P of S_2. After showing that L_φ is an Arguesian primary lattice of geometric dimension 2, we will prove that if L_φ were representable as the submodule lattice of a finite dimensional module over a completely primary uniserial ring, then the function F would have properties that imply that the map φ used to define it is an automorphism, thus contradicting our assumption on φ.

THEOREM 2.1. *The set L_φ is a sublattice of $L(V)$ which is Arguesian and primary and is such that its identity element V can be written as the sum of cycles*:

$$V = ([u_1] + (u_4]) \dotplus ([u_2] + [u_5]) \dotplus [u_3] .$$

Proof. If the elements x and y of L_φ are in a common interval used to define it, then clearly their sum and product are in L_φ. On the other hand, if $x \in [A, B]$, and $y \in [C, D]$, then $x+y \in [A + C, B + D]$, and $xy \in [AC, BD]$. Inasmuch as the set of intervals used to define L_φ is closed under the operations

$$[A, B] \oplus [C, D] = [A + C, B + D] ,$$
$$[A,B] \odot [C, D] = [AC, BD] ,$$

L_φ is a sublattice of $L(V)$. Showing tnat L_φ is semi-primary clearly reduces, by symmetry, to showing that every element A of L_φ is the sum of cycles of L_φ. Inasmuch as an element in a complemented lattice is the sum of atoms, a proof by induction on $d[A]$ clearly reduces to showing that if A is an atom of one of the intervals used to define L_φ and in no other such interval, then A is a cycle. This is clearly true of atoms of $[0, P]$ and $[X, F(X)]$ for $X \ll Q$, so we can assume that A is an atom of $[Q, V]$. But then either $A = P \in [0, P]$ or $A + P \gg P$, whence $A + P = F(X)$ for some $X \ll Q$, and $A \in [X, F(X)]$. In either event we have a contradiction, and we conclude that L_φ is semi-primary.

To see that L_φ is primary, note first that this reduces to showing that an interval of length 2 cannot have exactly two atoms, and note further that an interval in the lattice of subspaces of a vector space is itself the subspace lattice of a vector space and can therefore be shown to have at least three atoms. Thus, proving that L_φ is primary reduces to proving that every interval $[A, B]$ of length 2 with distinct atoms X_1 and X_2 is contained in one of the intervals used to define L_φ. We might as well assume then that the elements X_1 and X_2 are in distinct intervals $[C_1, D_1]$ and $[C_2, D_2]$ used to define L_φ for otherwise we would be through. By symmetry this reduces to three cases: (i) $C_1 = 0, C_2 = Q$, (ii) $C_2 \ll C_1, D_2 \ll D_1$, (iii) $C_1, C_2 \ll Q$. However, we can immediately dismiss the first case because, since X_1 and X_2 are of the same dimension, we have that $X_1 = P \in [Q, V]$ or $X_2 = Q \in [0, P]$. Since $X_2 \ll X_1 + X_2$ we infer that $X_2 = (X_1 + X_2)D_2$ or $X_1 + X_2 \leqq X_2$, whence $X_2 \geqq (C_1 + C_2)D_2 = C_2 + C_1D_2$ or $X_1 \leqq D_2$. In case (ii) $C_1 \leqq D_2$, so that $X_2 \geqq C_1$ or $X_1 \leqq D_2$ and we are through while, in case (iii) $C_1 + C_2 = Q$ and $D_1D_2 = P$ so that $X_2 \in [Q, V]$ or $X_1 \in [0, P]$ which is case (ii). Thus L_φ is primary.

That L_φ is Arguesian follows from the fact that it is a sublattice of $L(V)$ which, by [2; Th. 2.14], is Arguesian and the observation that the condition defining Arguesian lattices can be written as an identity.

Finally, note that, since $[u_1] + [u_4]$ is not contained in P, it cannot be the sum of atoms of L_φ. But since $[u_1] + [u_4]$ is of dimension 2, this means that it must be a 2-cycle in L_φ. Similarly $[u_2] + [u_5]$ is a 2-cycle in L_φ. Inasmuch as the atoms of L_φ contained in the cycles $[u_1] + [u_4], [u_2] + [u_5]$ and $[u_3]$ are $[u_1], [u_2]$ and $[u_3]$ respectively we conclude from 4.14 of PAL that these cycles are independent, whence, by a dimension argument.

$$V = ([u_1] + [u_4]) \dotplus ([u_2] + [u_5]) \dotplus [u_3] .$$

LEMMA 2.2. *Given an element X of L_φ that is covered by Q, there is a 2-cycle X' of L_φ covering X. Further, for any 2-cycle X'' covering X, $F(X) = X'' + P$.*

Proof. Inasmuch an $[X, F(X)]$ is complemented, $F(X)$ is the sum of elements covering X. If none of these elements is a 2-cycle of L_φ, then each must be the sum of atoms, and $F(X)$ is the sum of atoms. This contradicts the fact that $F(X)$ covers P, and every atom of L_φ is contained in P. Thus there is a 2-cycle X' covering X. Now, if X'' is any 2-cycle covering X, since $[0, P]$ is complemented we infer that $X'' \nleqq P$. Thus, either $X'' \leqq F(X)$, and $X'' + P = F(X)$, or $X'' \nleqq F(X)$, and $X'' + F(X) = V$. However, in the latter case V is the sum of elements covering X, whence $[X, V]$ is complemented, and X is the product of elements covered by V. Inasmuch as the only dual atoms of L_φ are in $[Q, V]$, this implies that $X \geqq Q$, a contradiction.

LEMMA 2.3. *If L_φ is representable as the submodule lattice of a finitely generated module over a completely primary uniserial ring, then there is an isomorphism ψ of $[0, P]$ onto $[Q, V]$ such that*

(a) $\psi[u_1] = [u_4] + Q, \psi[u_2] = [u_5] + Q, \psi[u_3] = [u_3] + Q$,

(b) $F(X) = \psi(X) + P$ *for every* $X \ll Q$.

Proof. Suppose λ' is an isomorphism of L_φ onto the submodule lattice of a module M' over a completely primary uniserial ring R. For definiteness we will take M' to be a left R-module, although it will be apparent from the proof that there is no loss of generality in this assumption. Furthermore, since L_φ if of rank 2, we can view M' as a module over R/J^2 or, equivalently, assume that $J^2 = (0)$. Then, since

$$M' = \lambda'([u_1] + [u_4]) \oplus \lambda'([u_2] + [u_5]) \oplus \lambda'([u_3]) ,$$

and R is completely primary and uniserial, there is an isomorphism λ of L_φ onto $L(M)$, where $M = R \times R \times R/J$, such that

$$\lambda([u_1] + [u_4]) = [m_1], \lambda([u_2] + [u_5]) = [m_2], \lambda([u_3]) = [m_3] ,$$

where, by m_i we mean the element of M with 1 in the ith place and 0 elsewhere. If we fix a generator p of J, it is clear that $[pm_1]$ and $[pm_2]$ are the unique elements of $L(M)$ covered by $[m_1]$ and $[m_2]$ respectively. Thus, since $[u_1]$ and $[u_2]$ are the unique elements of L_φ covered by $[u_1] + [u_4]$ and $[u_2] + [u_5]$, it follows that

$$\lambda[u_1] = [pm_1], \lambda[u_1] = [pm_2]$$

and if, $Q' = \lambda(Q)$ and $P' = \lambda(P)$, then

$$Q' = [pm_1] + [pm_2], P' = Q' + [m_3] .$$

Moreover,

$$\lambda([u_4] + Q) = \lambda([u_1] + [u_4] + Q) = [m_1] + Q, \lambda([u_5] + Q) = [m_2] + Q' .$$

Inasmuch as $Q' = [pm_1] + [pm_2]$, the elements of $L(M)$ covered by Q' are of the form $[pr_1, pr_2, 0]$ for r_1 and r_2 in R such that r_1 or r_2 is not in J. Clearly, $[r_1, r_2, 0]$ is a 2-cycle covering such an element, so that, if we define $F' = \lambda F \lambda^{-1}$, according to 2.2,

$$F'[pr_1, pr_2, 0] = [r_1, r_2, 0] + P' , \tag{1}$$

for $r_1, r_2 \in R$ with r_1 or $r_2 \notin J$.

Since $JP' = 0$ and $JM \subseteqq Q'$, P' and M/Q' are modules over the division ring R/J with bases $\{pm_1, pm_2, m_3\}$ and $\{m_1 + Q', m_2 + Q' \; m_3 + Q'\}$ respectively. Thus the correspondence

$$pm_1 \longrightarrow m_1 + Q', pm_2 \longrightarrow m_2 + Q', m_3 \longrightarrow m_3 + Q'$$

defines a lattice isomorphism $\psi' \colon [0, P'] \cong [Q', M]$ which has the properties

$$\psi'[pm_1] = [m_1] + Q', \psi'[pm_2] = [m_2] + Q', \psi'[m_3] = [m_3] + Q'$$
$$\psi'[pr_1, pr_2, 0] = [r_1, r_2, 0] + Q' ,$$

so that, by (1), for every $X \ll Q'$,

$$F'(X') = \psi'(X') + P' . \tag{2}$$

Therefore, defining $\psi = \lambda^{-1}\psi'\lambda$, we have

$$\psi[u_1] = [u_4] + Q, \psi[u_2] = [u_5] + Q, \psi[u_3] = [u_3] + Q .$$

Taking X' to be $\lambda(X)$ in (2) for $X \ll Q$, we also have $F'\lambda(X) = \psi'\lambda(X) + P'$ so that, applying λ^{-1} to both sides, we conclude that $F(X) = \psi(X) + P$, as was to be shown.

THEOREM 2.4. *The lattice L_φ cannot be represented as a submodule lattice of a finitely generated module over a completely primary uniserial ring.*

Proof. Denoting by V_3, the vector space of triples from K, and by w_i the element of V_3 with 1 in the ith place and zero elsewhere, it is clear that the correspondences $\{w_i \to u_i\}_{i=1}^3$ and

$$\{w_1 \longrightarrow u_4 + Q,\ w_2 \longrightarrow u_5 + Q,\ w_3 \longrightarrow u_3 + Q\}$$

give lattice isomorphisms σ and τ of $L(V_3)$ onto $[0, P]$ and $[Q, V]$ respectively, so that the map $\theta = \tau^{-1}\psi\sigma$ is an automorphism of $L(V_3)$ with $\theta[w_i] = [w_i]$ for $i = 1, 2, 3$, and

$$F\sigma[r, 1, 0] = \tau[\varphi(r), 1, 0] + \tau[w_3]\ .$$

However, in view of 2.3,

$$F\sigma[r, 1, 0] = \psi\sigma[r, 1, 0] + \psi\sigma[w_3]\ ,$$

whence

$$\psi\sigma[r, 1, 0] + \psi\sigma[w_3] = \tau[\varphi(r), 1, 0] + \tau[w_3]\ ,$$

and, applying τ^{-1} to both sides of this equation, we conclude that

$$\theta[r, 1, 0] + \theta[w_3] = [\varphi(r), 1, 0] + [w_3]\ ,$$

and $\theta[r, 1, 0] = [\varphi(r), 1, 0]$. According to the fundamental theorem of projective geometry, θ is induced by a semilinear transformation (T, φ') where T is an automorphism of the additive group of V_3 and φ' is an automorphism of K. Since $\theta[w_i] = [w_i]$ for $i = 1, 2, 3$, there are nonzero elements s_1, s_2, s_3 in K such that $T(w_i) = s_i w_i$ for $i = 1, 2, 3$. On the other hand,

$$\begin{aligned}[\varphi(r), 1, 0] &= \theta[r, 1, 0] = [T(r, 1, 0)] = [\varphi'(r)T(w_1) + T(w_2)] \\ &= [(\varphi'(r)s_1, s_2, 0]\ ,\end{aligned}$$

so that $\varphi'(r)s_1 = \varphi(r)s_2$. However, since $\varphi'(1) = 1 = \varphi(1)$, we see that $s_1 = s_2$, whence $\varphi'(r) = \varphi(r)$. Thus φ must be an automorphism, contradicting the original assumption on φ, and the theorem is established.

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