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SUMMABILITY OF FOURIER SERIES BY TRIANGULAR MATRIX TRANSFORMATIONS

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Hille and Tamarkin have proved a result for the Nörlund summability of the Fourier series of $f(t)$ at $t = x$, under the hypothesis (i) $\varphi(t) = \{f(x+t) + f(x-t) - 2f(x)\}/2 = o(1), t \to 0$, which includes as a special case the corresponding result for the Cesàro summability. However, under the lighter condition (ii) $\int_0^t \varphi(u)du = o(t), t \to 0$, Astrachan has proved a theorem for the Nörlund summability which does not cover the corresponding Cesàro case. The object of the present paper is to prove theorems for the Nörlund summability and another triangular matrix method of summability which are subtler than Astrachan's theorem in the sense that they include as a special case the corresponding result for the Cesàro summability.

1. Definitions and notations. Let $\sum_{n=0}^{\infty} v_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. We shall consider sequence-to-sequence transformation of the type

$$u_n = \sum_{k=0}^{\infty} d_{nk} s_k \tag{1.1}$$

in which the elements of the matrix $D = ((d_{nk}))$ are real or complex constants and $d_{nk} = 0$ for $k > n$. The sequence $\{u_n\}$ is said to be the sequence of D-means of $\{s_n\}$. If $\lim_{n\to\infty} u_n$ exists and is equal to u then we say that the series $\sum_{n=0}^{\infty} v_n$ or the sequence $\{s_n\}$ is summable D to the sum u.

Let $\{p_n\}$ be a sequence of constants, real or complex and let us write $P_n = p_0 + p_1 + \cdots + p_n \neq 0, P_{-1} = p_{-1} = 0$. Then the matrix D defines a Nörlund matrix (N, p_n) [7], if

$$d_{nk} = p_{n-k}/P_n\,, \qquad (n \geqq k \geqq 0)\,. \tag{1.2}$$

The conditions for the regularity of the (N, p_n) mean are

$$\lim_{n\to\infty} p_n/P_n = 0 \quad \text{and} \quad \sum_{k=0}^{n} |p_k| = O(|P_n|)\,, \qquad n \to \infty\,. \tag{1.3}$$

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha > -1) \tag{1.4}$$

the (N, p_n) mean reduces to the familiar (C, α) mean.

The product of the matrix $(C, 1)$ with the matrix (N, p_n) defines the matrix $(C, 1)\cdot(N, p_n)$. Thus D defines the matrix $(C, 1)\cdot(N, p_n)$ if

$$d_{nk} = \frac{1}{n+1}\sum_{\nu=k}^{n} p_{\nu-k}/P_\nu\ , \qquad (0 \leqq k \leqq n)\ . \tag{1.5}$$

Similarly, one defines the $(N, p_n)\cdot(C, 1)$ matrix as a product of the (N, p_n) matrix with the $(C, 1)$ matrix. In Astrachan's notations [1] the $(N, p_n)\cdot(C, 1)$ summability is denoted by $(N, p_n)\cdot C_1$.

Let $f(t)$ be a periodic function, with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without any loss of generality that the constant term in the Fourier series of $f(t)$ is zero, so that $\int_{-\pi}^{\pi} f(t)dt = 0$ and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)\ . \tag{1.6}$$

We write throughout:

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}\ ;$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du,\ \alpha > 0;\ \Phi_0(t) = \varphi(t)\ ;$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\Phi_\alpha(t)/t^\alpha;\ \alpha \geqq 0\ ;$$

$$R_n = np_n/P_n;\ S_n = \sum_{\nu=0}^{n} P_\nu(\nu+1)^{-1}/P_n\ ;$$

$$\Delta\mu_n, \text{ or more precisely } \Delta_n\mu_n = \mu_n - \mu_{n+1}\ ;$$

$$\tau = [1/t];\ P_{[\lambda]} = P(\lambda);\ p_{[\lambda]} = p(\lambda)\ ;$$

where $[\lambda]$ denotes the greatest integer not greater than λ.

K, denotes a positive constant not necessarily the same at each occurrence.

2. Introduction.

Concerning the Cesàro summability of Fourier series Bosanquet [2] has proved the following.

THEOREM A. *If $\varphi_\alpha(t) = o(1)$ as $t \to 0$, then the Fourier series of $f(t)$, at $t = x$, is summable $(C, \alpha + \delta)$ for every $\delta > 0$ and $\alpha \geqq 0$.*

Theorem A is known to be the best possible in the sense that it breaks down if $\delta = 0$.

For the Nörlund summability of Fourier series we have the following result due to Hille and Tamarkin [5].

THEOREM B. *A regular (N, p_n) method is Fourier effective, if the sequence $\{p_n\}$ satisfies the hypotheses:*

$$R_n = O(1) \ , \tag{2.1}$$

$$\sum_{k=1}^{n} k \, |\Delta p_{k-1}| = O(|P_n|) \ , \tag{2.2}$$

$$\sum_{k=1}^{n} |P_k|/k = O(|P_n|) \ , \tag{2.3}$$

as $n \to \infty$.

Theorem B implies *inter alia* that if $\varphi(t) = o(1)$ as $t \to 0$, and $\{p_n\}$ satisfies the hypotheses (2.1)–(2.3), then the Fourier series of $f(t)$ is summable by a regular (N, p_n) method.

Replacing the hypothesis: $\varphi(t) = o(1)$ as $t \to 0$ of Theorem B by the lighter hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$, Astrachan [1] proved the following.

THEOREM C. *A regular (N, p_n) method is K_α effective* $(0 < \alpha \leqq 1)$, *if the sequence $\{p_n\}$ satisfies the hypotheses* (2.1), (2.2) *and*

$$\sum_{k=1}^{n} k(n - k) \, |\Delta^2 p_{k-2}| = O(|P_n|) \ , \tag{2.4}$$

$$\sum_{k=1}^{n} |P_k|/k^2 = O(|P_n|/n) \ , \tag{2.5}$$

as $n \to \infty$.

Hille and Tamarkin have also pointed out in [5] that the sequence $\{p_n\}$ defined by (1.4) satisfies the hypotheses of Theorem B for $1 > \alpha > 0$ and therefore, (C, α) summability for such a α is Fourier effective. Thus Bosanquet's Theorem A when $\alpha = 0$ is an immediate consequence of Theorem B. It is therefore natural to expect that the hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$, may lead to (N, p_n) summability of the Fourier series of $f(t)$ and that such a result may include Theorem A when $\alpha = 1$, as a special case. However, Astrachan's Theorem C in this direction only implies the summability (C, δ) for $\delta \geqq 2$, whereas one needs the summability (C, δ), $\delta > 1$, in order to cover Bosanquet's Theorem A when $\alpha = 1$. Thus there is a gap of approximately 1 between the orders of (C) summability implied by Theorem C and the corresponding case of Theorem A. This emerges from the following reasoning.

The result of Lemma 8.1 in Astrachan [1], which is required for the proof of his Theorem C states that

$$\sum_{k=0}^{n} (n - k) \, |\Delta^2 p_{k-2}| = O(|P_n|/n) \ , \tag{2.6}$$

as $n \to \infty$. Since the left hand side of (2.6) is greater than Kn we observe that $Kn^2 \leqq |P_n|$. It may be pointed out that for Astrachan's proof of Lemma 8.1 one has to assume $p_0 = 0$.

The object of our Theorem 1 is to show that it is indeed, possible to obtain a result for the (N, p_n) summability of Fourier series which has also the scope of covering Bosanquet's Theorem A for $\alpha = 1$.

Astrachan [1, Th. II] has also obtained the following result for the $(N, p_n)\cdot(C, 1)$ summability of the Fourier series.

THEOREM D. *The* $(N, p_n)\cdot(C, 1)$ *method is* K_α *effective* $(0 < \alpha \leqq 1)$ *provided the sequence* $\{p_n\}$ *satisfies the hypotheses* (2.1)–(2.3) *and the regularity condition* (1.3).

Due to possible oversight, Astrachan has not shown that the regularity conditions follow from his statement of Theorem D. Further, his proof of Theorem D contains a deficiency, which has been pointed out and supplied by the present author in [4].

Silverman has shown in [8, Th. 1] that a necessary and sufficient condition for a (N, p_n) matrix to be permutable with the $(C, 1)$ matrix is that it be a Cesàro matrix. This implies that

$$(C, 1)\cdot(N, p_n) \neq (N, p_n)\cdot(C, 1)$$

except when $\{p_n\}$ is defined by (1.4). In view of this Astrachan's technique of obtaining his Theorem D from Theorem B fails in the case of the $(C, 1)\cdot(N, p_n)$ summability and one has to give a direct proof to conclude the $(C, 1)\cdot(N, p_n)$ summability of Fourier series of $f(t)$ under the hypothesis: $\varphi_1(t) = o(1)$ as $t \to 0$. More precisely, we observe that since the $(C, 1)$ mean is a very special case of the (N, p_n) mean viz. the case in which $p_n = 1$, the convenience of expressing the $(C, 1)$ mean of the Fourier series of $f(t)$, essentially as a difference of the Fejér's and Dirichlet's kernels of $\varphi_1(t)$ [1, p. 546], disappears totally in the case of the (N, p_n) mean.

Thus for the $(C, 1)\cdot(N, p_n)$ summability of Fourier series, we obtain Theorem 2 which also covers Theorem A when $\alpha = 1$.

3. We prove the following results.

THEOREM 1. *If* $\varphi_1(t) = o(1)$ *as* $t \to 0$ *and* $\{p_n\}$ *is nonnegative, monotonic nondecreasing sequence such that* $p_n \to \infty$ *as* $n \to \infty$, $\{p_{n+1} - p_n\}$ *is nonincreasing,* $R_n = O(1)$ *and* (2.5) *holds, then the Fourier series of* $f(t)$, *at* $t = x$, *is summable* (N, p_n).

THEOREM 2. *If* $\varphi_1(t) = o(1)$ *as* $t \to 0$ *and* $\{p_n\}$ *is a nonnegative, monotonic nonincreasing sequence such that* $S_n = O(1)$, *then the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $(C, 1)\cdot(N, p_n)$.

REMARKS. It is easy to see that if $\{p_n\}$ is nonnegative and nondecreasing then $(n+1)p_n \geqq P_n$ and therefore $S_n = O(1)$. Further, in this case

$$\sum_{k=1}^{n} k\,|\varDelta p_{k-1}| = -\sum_{k=1}^{n-1}\sum_{\mu=1}^{k}(p_\mu - p_{\mu-1}) + n\sum_{\mu=1}^{n}(p_\mu - p_{\mu-1}) = O(P_n)\ ,$$

if $R_n = O(1)$. Thus the sequence $\{p_n\}$ used in Theorem 1 also satisfies the hypotheses of Theorem B.

As demonstrated by the present author in [3] if $\{p_n\}$ is a nonnegative sequence then the hypotheses: $R_n = O(1)$ and $S_n = O(1)$ imply that

$$P_k \sum_{n=k+1}^{\infty} \frac{1}{(n+1)P_{n-1}} = O(1)\ , \qquad (k = 1, 2, 3, \cdots)\ ,$$

from which it is immediate that $P_n \to \infty$ as $n \to \infty$. It may be observed that with a slight modification in author's analysis in [3] it is possible to even drop the condition $R_n = O(1)$ to get the same conclusion.

4. We require the following lemmas for the proof of our results.

LEMMA 1. *If $\{q_n\}$ is nonnegative and nonincreasing, then for $0 \leqq a \leqq b \leqq \infty$ and $0 \leqq t \leqq \pi$,*

$$\left|\sum_{k=a}^{b} q_k \exp ikt\right| \leqq KQ_\tau\ ,$$

where $\tau = [1/t]$ and $Q_m = q_0 + q_1 + \cdots + q_m$.

This lemma may be proved by following the technique of proof of Lemma 5.11 in McFadden [6].

LEMMA 2. *If $\{p_n\}$ is a nonnegative and monotonic nondecreasing sequence such that $\{p_{n+1} - p_n\}$ is nonincreasing and $R_n = O(1)$, then as $n \to \infty$*

$$\sum_{k=0}^{n} p_k(n-k)\exp(ikt) = O(nP_\tau) + O(t^{-2}p_n)$$

uniformly in $0 < t \leqq \pi$.

Proof. We write by Abel's transformation

$$\begin{aligned}
&\sum_{k=0}^{n} p_k(n-k)\exp(ikt)\\
&\quad = \sum_{k=0}^{n-1} \varDelta_k\{p_k(n-k)\}\sum_{\nu=0}^{k}\exp(i\nu t)\\
&\quad = (1-\exp it)^{-1}\left[\sum_{k=0}^{n-1}\varDelta_k\{p_k(n-k)\} - \sum_{k=0}^{n-1}\varDelta_k\{p_k(n-k)\}\exp i(k+1)t\right]
\end{aligned}$$

$$= (1 - \exp it)^{-1}\Big[np_0 - \sum_{k=0}^{n-1}(n-k)\Delta p_k \exp i(k+1) - \sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big]$$

$$= (1 - \exp it)^{-1}\Big[np_0 - \sum_{k=0}^{n-1}\sum_{\nu=0}^{k}\Delta p_\nu \exp i(\nu+1)t - \sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big].$$

Thus

$$\Big|\sum_{k=0}^{n} p_k(n-k)\exp ikt\Big|$$

$$\leqq |1 - \exp it|^{-1}\Big[np_0 + \sum_{k=0}^{n-1}\Big|\sum_{\nu=0}^{k}\Delta p_\nu \exp i(\nu+1)t\Big| + \Big|\sum_{k=0}^{n-1} p_{k+1}\exp i(k+1)t\Big|\Big]$$

$$\leqq Kt^{-1}\Big[np_0 + K\sum_{k=0}^{n-1}\sum_{\nu=0}^{\tau}(p_{\nu+1} - p_\nu) + p_n \max_{1\leqq\nu\leqq n}\Big|\sum_{k=1}^{\nu}\exp ikt\Big|\Big]$$

(by Lemma 1 and Abel's Lemma, since $\{p_{\nu+1} - p_\nu\}$ is nonnegative, nonincreasing and $\{p_n\}$ is nondecreasing)

$$\leqq Kt^{-1}[np_0 + Knp_{\tau+1} + Kp_n t^{-1}]$$

$$\leqq Knt^{-1}p_{\tau+1} + Kt^{-2}p_n$$

$$\leqq KnP_\tau + Kt^{-2}p_n\ ,$$

since $\{p_n\}$ is nondecreasing and $R_n = O(1)$ which also implies $P_{n+1}/P_n = O(1)$.

This completes the proof of Lemma 2.

LEMMA 3. *If $\{p_n\}$ is nonnegative and nonincreasing, then as $n\to\infty$*

$$\sum_{\nu=0}^{n}\frac{1}{P_\nu}\sum_{k=0}^{\nu}(\nu-k)p_k\exp i(\nu-k)t = O(t^{-2}) + O\Big(t^{-1}P_t\sum_{\nu=\tau}^{n}\frac{1}{P_\nu}\Big) + O\Big(\frac{nt^{-1}P_\tau}{P_{n+1}}\Big),$$

uniformly in $0 < t \leqq \pi$.

Proof. Applying Abel's transformation we get

$$\sum_{\nu=k}^{n}\frac{\nu-k}{P_\nu}\exp i(\nu-k)t$$

$$= \sum_{\nu=k}^{n}\Delta_\nu\Big(\frac{\nu-k}{P_\nu}\Big)\sum_{\mu=k}^{\nu}\exp i(\mu-k)t + \frac{n-k+1}{P_{n+1}}\sum_{\mu=k}^{n}\exp i(\mu-k)t$$

$$= (1-\exp it)^{-1}\Big[\sum_{\nu=k}^{n}\Delta_\nu\Big(\frac{\nu-k}{P_\nu}\Big)\{1-\exp i(\nu-k+1)t\} + \frac{n-k+1}{P_{n+1}}\{1-\exp i(n-k+1)t\}\Big]$$

$$= (1-\exp it)^{-1}\Big[-\sum_{\nu=k}^{n}\frac{p_{\nu+1}}{P_\nu P_{\nu+1}}(\nu-k)\exp i(\nu-k+1)t + \sum_{\nu=k}^{n}\frac{1}{P_{\nu+1}}\exp i(\nu-k+1)t - \frac{n-k+1}{P_{n+1}}\exp i(n-k+1)t\Big].$$

Changing the order of summation of the inner sums, thus we have

$$
\begin{aligned}
\sum &\equiv \left|\sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k)p_k \exp i(\nu - k)t\right| \\
&= \left|\sum_{k=0}^{n} p_k \sum_{\nu=k}^{n} \frac{\nu - k}{P_\nu} \exp i(\nu - k)t\right| \\
&\leqq Kt^{-1}\left[\left|\sum_{k=0}^{n} p_k \sum_{\nu=k}^{n} \frac{p_{\nu+1}}{P_\nu P_{\nu+1}}(\nu - k) \exp i(\nu - k + 1)t\right|\right. \\
&\quad + \left|\sum_{k=0}^{n} p_k \sum_{\nu=k}^{n} \frac{1}{P_{\nu+1}} \exp i(\nu - k + 1)t\right| \\
&\quad \left.+ \frac{1}{P_{n+1}}\left|\sum_{k=0}^{n} p_k(n - k + 1) \exp i(n - k + 1)t\right|\right] \\
&= \sum_1 + \sum_2 + \sum_3 ,
\end{aligned}
$$

say.

Again by a change of order of summation we have

$$
\begin{aligned}
\sum_1 &\leqq Kt^{-1} \sum_{\nu=0}^{n} \frac{p_{\nu+1}}{P_\nu P_{\nu+1}} \left|\sum_{k=0}^{\nu} p_k (\nu - k) \exp i(\nu - k + 1)t\right| \\
&\leqq Kt^{-1} \sum_{\nu=0}^{\tau-1} \frac{R_{\nu+1}}{P_\nu} \sum_{k=0}^{\nu} p_k + Kt^{-1} \sum_{\nu=\tau}^{n} \frac{R_{\nu+1}}{P_\nu} \max_{0\leqq\rho\leqq\nu} \left|\sum_{k=0}^{\rho} p_k \exp i(\nu - k + 1)t\right|
\end{aligned}
$$

(by Abel's Lemma. If $\tau = 0$ the first part is taken as 0.)

$$
\leqq Kt^{-2} + Kt^{-1}P_\tau \sum_{\nu=\tau}^{n} \frac{1}{P_\nu} ,
$$

by virtue of Lemma 1 and the fact that $(n + 1)p_n \leqq P_n$.

Similarly,

$$
\begin{aligned}
\sum_2 &\leqq Kt^{-1} \sum_{\nu=0}^{n} \frac{1}{P_{\nu+1}} \left|\sum_{k=0}^{\nu} p_k \exp i(\nu - k + 1)t\right| \\
&\leqq Kt^{-1} \sum_{\nu=0}^{\tau-1} \frac{1}{P_{\nu+1}} \sum_{k=0}^{\nu} p_k + Kt^{-1} \sum_{\nu=\tau}^{n} \frac{1}{P_{\nu+1}} P_\tau \\
&\leqq Kt^{-1} + Kt^{-1}P_\tau \sum_{\nu=\tau}^{n} \frac{1}{P_\nu} ,
\end{aligned}
$$

by Lemma 1.

Finally, by Lemma 1 and Abel's Lemma we have

$$
\sum_3 \leqq Kt^{-1} \frac{n}{P_{n+1}} P_\tau .
$$

This completes the proof of Lemma 3.

5. *Proof of Theorem* 1. For the Fourier series of $f(t)$, at $t = x$ we have

$$s_k(x) - f(x) = \frac{1}{\pi}\int_0^\pi \varphi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt\ .$$

Therefore, if t_n denotes the (N, p_n) mean of $\{s_k(x)\}$ then

$$t_n - f(x) = \frac{1}{\pi P_n}\int_0^\pi \varphi(t)\left\{\sum_{k=0}^{n} p_{n-k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right\}dt\ .$$

Integrating by parts, we get

$$\begin{aligned} t_n - f(x) &= \frac{\Phi_1(\pi)}{\pi P_n}\sum_{k=0}^{n} p_{n-k}(-1)^k \\ &\quad - \frac{1}{\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n} p_{n-k}k\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &\quad - \frac{1}{2\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t\right\}dt \\ &\quad + \frac{1}{2\pi P_n}\int_0^\pi \frac{\Phi_1(t)}{\tan(t/2)}\left\{\sum_{k=0}^{n} p_{n-k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right\}dt \\ &= L_1 + L_2 + L_3 + L_4\ , \end{aligned}$$

say.

Thus, in order to prove the theorem it is sufficient to show that as $n \to \infty$,

(5.1) $$L_j = o(1)\ ; \qquad (j = 1, 2, 3 \text{ and } 4)\ .$$

Since $\Phi_1(t)\cot t/2 = o(1)$ as $t \to 0$, it follows from Theorem B that $L_4 = o(1)$ as $n \to \infty$, when one appeals to the remarks contained in § 3 of the present paper.

We write

$$\frac{1}{P_n}\left|\sum_{k=0}^{n} p_{n-k}(-1)^k\right| \leqq K\frac{p_n}{P_n} = o(1)\ ,$$

as $n \to \infty$, since $\{p_n\}$ is nonnegative and nondecreasing and $R_n = O(1)$. Thus, we have $L_1 = o(1)$ as $n \to \infty$.

Also, $L_3 = o(1)$ as $n \to \infty$, by virtue of Riemann-Lebesgue Theorem and the regularity of the (N, p_n) mean which is implied by the hypotheses: $\{p_n\}$ is nonnegative and $R_n = O(1)$.

Finally, to show that $L_2 = o(1)$ as $n \to \infty$, we observe that

$$\frac{\Phi_1(t)}{\sin t/2} = o(1)$$

as $t \to 0$ and that the kernel occuring in L_2 is the real part of the complex valued function

$$\frac{1}{\pi P_n}\left\{\exp - i\left(n+\frac{1}{2}\right)t\right\}\sum_{k=0}^{n} p_k(n-k)\exp ikt = M_n(t)\ ,$$

say.

Therefore, in order to prove that $L_2 = o(1)$ as $n \to \infty$, it is enough to show that as $n \to \infty$

$$I \equiv \int_0^\pi g(t)M_n(t)dt = o(1) , \tag{5.2}$$

where $g(t) = o(1)$ as $t \to 0$.

We write, for a fixed δ such that $0 < \delta \leqq \pi$,

$$I = \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\delta} + \int_\delta^\pi\right) g(t)M_n(t)dt = I_1 + I_2 + I_3 , \tag{5.3}$$

say.

Since

$$M_n(t) = O\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k(n-k)\right) = O(n) ,$$

we have, as $n \to \infty$

$$I_1 = O\left(n\int_0^{n^{-1}} |g(t)|\, dt\right) = o(1) . \tag{5.4}$$

For the interval $o < \delta \leqq t \leqq \pi$, we have from Lemma 2

$$M_n(t) = O\left(\frac{n}{P_n}\right) + O\left(\frac{p_n}{P_n}\right) = O\left(\frac{1}{p_n}\right) + o(1) = o(1) ,$$

as $n \to \infty$, by the hypotheses: $R_n = O(1)$ and that $p_n \to \infty$ as $n \to \infty$. Therefore, as $n \to \infty$,

$$I_3 = o(1) . \tag{5.5}$$

Since $g(t) = o(1)$ as $t \to 0$, to demonstrate the truth of $I_2 = o(1)$ as $n \to \infty$ we prove that

$$I_2^* \equiv \int_{n^{-1}}^{\delta} |M_n(t)|\, dt \leqq K .$$

By Lemma 2, we have

$$\begin{aligned} I_2^* &\leqq K\frac{n}{P_n}\int_{n^{-1}}^{\delta} P(1/t)dt + K\frac{p_n}{P_n}\int_{n^{-1}}^{\delta} t^{-2}dt \\ &= K\frac{n}{P_n}\int_{\delta^{-1}}^{n} \frac{P(s)}{s^2}ds + KR_n , \\ &\leqq K , \end{aligned}$$

by virtue of the hypotheses: $R_n = O(1)$ and (2.5). Thus, as $n \to \infty$.

$$I_2 = o(1) . \tag{5.6}$$

Combining (5.3)-(5.6), we get (5.2) and therefore $L_2 = o(1)$ as $n \to \infty$. This completes the proof of Theorem 1.

6. *Proof of Theorem* 2. If t_n^1 denotes the $(C, 1)\cdot(N, p_n)$ mean of the sequence $\{s_k(x)\}$ then

$$\begin{aligned} t_n^1 - f(x) &= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{\nu=k}^{n} \frac{p_{\nu-k}}{P_\nu} s_k(x) - f(x) \\ &= \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} s_k(x) - f(x) \\ &= \frac{1}{(n+1)\pi} \int_0^\pi \varphi(t) \left\{ \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} \frac{\sin (k+1/2)t}{\sin (t/2)} \right\} dt \, . \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} t_n^1 - f(x) = {} & \frac{\Phi_1(\pi)}{\pi(n+1)} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k}(-1)^k \\ & - \frac{1}{\pi(n+1)} \int_0^\pi \frac{\Phi_1(t)}{\sin (t/2)} \left\{ \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} k \cos\left(k + \frac{1}{2}\right)t \right\} dt \\ & - \frac{1}{2\pi(n+1)} \int_0^\pi \frac{\Phi_1(t)}{\sin (t/2)} \left\{ \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} \cos\left(k + \frac{1}{2}\right)t \right\} dt \\ & + \frac{1}{2\pi(n+1)} \int_0^\pi \frac{\Phi_1(t)}{\tan (t/2)} \left\{ \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} \frac{\sin [k + (1/2)]t}{\sin (t/2)} \right\} dt \\ = {} & C_1 + C_2 + C_3 + C_4 \, , \end{aligned}$$

say.

Thus, in order to prove the theorem it is sufficient to show that as $n \to \infty$

$$C_j = o(1)\, ; \qquad (j = 1, 2, 3 \text{ and } 4)\, . \tag{6.1}$$

Since $\{p_n\}$ is nonnegative and nonincreasing, we have by Abel's Lemma

$$\frac{1}{P_\nu} \left| \sum_{k=0}^{\nu} p_{\nu-k}(-1)^k \right| \leqq K \frac{p_0}{P_\nu} = o(1)\, ,$$

as $\nu \to \infty$, by virtue of the fact that $P_n \to \infty$ as $n \to \infty$. By virtue of the regularity of the $(C, 1)$ mean we now get $C_1 = o(1)$ as $n \to \infty$.

Further, since $[\Phi_1(t)/\sin (t/2)] \cos t/2 = o(1)$ as $t \to 0$ and the $(C, 1)$ mean is regular, Theorem B implies that $C_4 = o(1)$ as $n \to \infty$, when one observes that the sequence $\{p_n\}$ used in our Theorem 2 satisfies all the hypotheses of Theorem B.

That $C_3 = o(1)$ as $n \to \infty$, follows from the Riemann-Lebesgue Theorem and the fact that the $(C, 1)$ and the (N, p_n) mean are both regular.

Finally, we observe that $[\Phi_1(t)/\sin(t/2)] = o(1)$ as $t \to 0$ and therefore, in order to prove that $C_2 = o(1)$ as $n \to \infty$, it is sufficient to show that as $n \to \infty$

(6.2) $$E \equiv \int_0^\pi g(t)J_n(t)dt = o(1) ,$$

where $g(t) = o(1)$ as $t \to 0$ and

$$J_n(t) = \frac{\exp(it/2)}{\pi(n+1)} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k)p_k \exp i(\nu - k)t .$$

Let us write for a fixed δ such that $0 < \delta \leqq \pi$,

(6.3) $$E = \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\delta} + \int_\delta^\pi\right) g(t)J_n(t)dt = E_1 + E_2 + E_3 ,$$

say. Since

$$|J_n(t)| < \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} (\nu - k)p_k \leqq Kn ,$$

we have as $n \to \infty$

(6.4) $$E_1 = O\left(n\int_0^{n^{-1}} (g(t))dt\right) = o(1) .$$

For the interval $0 < \delta \leqq t \leqq \pi$, we have by Lemma 3

$$J_n(t) = o(1) + O\left(\frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu}\right) + O\left(\frac{1}{P_{n+1}}\right) = o(1)$$

as $n \to \infty$, since $P_n \to \infty$ as $n \to \infty$ and $(C, 1)$ mean is regular. Thus, as $n \to \infty$,

(6.5) $$E_3 = o(1) .$$

Since $g(t) = o(1)$ as $t \to 0$, to prove that $E_2 = o(1)$ as $n \to \infty$, it is enough to demonstrate that

$$E_2^* = \int_{n^{-1}}^{\delta} |J_n(t)|\, dt \leqq K .$$

By Lemma 3 we get

$$\begin{aligned} E_2^* &\leqq \frac{K}{n+1}\int_{n^{-1}}^{\delta} \frac{dt}{t^2} + \frac{K}{n+1}\int_{n^{-1}}^{\delta} \frac{P(1/t)}{t}\left\{\sum_{\nu=[1/t]}^{n} \frac{1}{P_\nu}\right\}dt \\ &\quad + \frac{K}{P_n}\int_{n^{-1}}^{\delta} \frac{P(1/t)}{t}dt \\ &\leqq K + \frac{K}{n+1}\int_{\delta^{-1}}^{n} \frac{P(s)}{s}\left\{\sum_{\nu=[s]}^{n} \frac{1}{P_\nu}\right\}ds + K\frac{1}{P_n}\int_{\delta^{-1}}^{n} \frac{P(s)}{s}ds \\ &\leqq K + \frac{K}{n+1}\int_{\delta^{-1}}^{n} \frac{P(s)}{s}\left\{\sum_{\nu=[s]}^{n} \frac{1}{P_\nu}\right\}ds , \end{aligned}$$

since $S_n = O(1)$. That $E_2^* \leqq K$, now follows from the fact that

$$\frac{1}{n+1}\sum_{k=1}^{n}\frac{P_k}{k}\sum_{\nu=k}^{n}\frac{1}{P_\nu} = \frac{1}{n+1}\sum_{\nu=1}^{n}\frac{1}{P_\nu}\sum_{k=1}^{\nu}\frac{P_k}{k}$$
$$= \frac{1}{n+1}\sum_{\nu=1}^{n} S_\nu \leqq K ,$$

since $S_n = O(1)$. Therefore, as $n \to \infty$

$$E_2 = o(1) . \tag{6.6}$$

Combining (6.3)-(6.6), we get (6.2) and therefore, $C_2 = o(1)$ as $n \to \infty$.

This completes the proof of Theorem 2.

References

1. Max Astrachan, *Studies in the summability of Fourier series by Nörlund means*, Duke Math. J. **2** (1936), 543-568.
2. L. S. Bosanquet, *On the summability of Fourier series*, Proc. London Math. Soc. **31** (1930), 144-164.
3. H. P. Dikshit, *Absolute summability of Fourier series by Nörlund means*, Math. Zeit. **102** (1967), 166-170.
4. ———, *A note on a theorem of Astrachan on the* $(N, p_n)\cdot(C, 1)$ *summability of Fourier series*, Math. Student **33** (1964), 77-79.
5. E. Hille and J. D. Tamarkin, *On the summability of Fourier series I*, Trans. Amer. Math. Soc. **34** (1932), 757-783.
6. L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168-207.
7. N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Årsskrift (2) **16** (1919).
8. S. L. Silverman, *Products of Nörlund transformation*, Bull. Amer. Math. Soc. **43** (1937), 95-101.

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