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DISTRIBUTION OF ZEROS OF SOLUTIONS OF A FOURTH ORDER DIFFERENTIAL EQUATION

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The primary concern of this paper is to study the distribution of zeros of solutions, that have at least four zeros, two or more of which are distinct zeros, of the canonical fourth order equation

$$(E_4) \qquad L_4[y] = (r_3L_3[y])' + q_3r_2L_2[y] + q_4y = 0\ ,$$

where $r_i(x) > 0,\ r_i(x),\ q_j(x) \in C[a, \infty),\ \ i = 1, 2, 3,\ \ j = 1, 2, 3, 4,$ **which was introduced by Barrett.**

The canonical second order equation

$$(E_2) \qquad L_2[y] = (r_1y')' + q_1y = 0\ ,$$

where $r_1(x) > 0$, $r_1(x), q_1(x) \in C\,[a, \infty)$, has been studied extensively. The canonical third order linear differential equation

$$(E_3) \qquad L_3[y] = (r_2L_2[y])' + q_2(r_1y') = 0\ ,$$

where $r_i(x) > 0$, $r_i(x), q_i(x) \in C\,[a, \infty)$, $i = 1, 2,$ which was introduced by Barrett, is a generalization of the second order equation (E_2).

Dolan studied the distribution of zeros of extremal solutions of (E_3) for the first conjugate point $\eta_1(t)$. In paragraph 2 the same study is made for the equation (E_4) and many of Dolan's ideas and techniques are used there. The results in paragraph 2 substantially complete the investigation begun in a paper by Aliev. Aliev defined and investigated the numbers $r_{ijk}(t)$ and $r_{1111}(t)$, which are extensions of the two-point nonoscillation numbers $r_{ij}(t)$ of Azbelev and Caljuk. Several of his results were reported in sources, which did not include the proofs, and these proofs were unavailable to the author, e.g.,

$$r_{31}(t) \geqq r_{211}(t) \geqq \min\,[r_{22}(t), r_{31}(t)]\ .$$

Aliev also proved that

$$r_{1111}(t) = \min\,[r_{121}(t), r_{112}(t)]\ ,$$

and purported to prove that $r_{1111}(t) = \min\,[r_{211}(t), r_{112}(t)]$ but his proof is incorrect and this remains an open question. In paragraph 3 these results of Aliev are reproved and a much more complete picture is presented in the ordering of the numbers $\eta_1(t)$, $r_{ij}(t)$, $r_{ijk}(t)$, and $r_{1111}(t)$. The main results of this paper appear in paragraph 3.

1. Introduction. We will be concerned with the fourth order quasi differential equation

$$(E_4) \qquad L_4[y] = (D_3y)' + q_3D_2y + q_4y = 0 \,,$$

where

$$D_1y = r_1y',\ D_2y = r_2L_2[y],\ D_3y = r_3L_3[y] \,.$$

For a discussion of the basic properties of (E_4) see Chapter III of [6].

An adjoint differential equation to (E_4) is

$$(E_4^+) \qquad L_4^+[y] = (D_3^+y)' + q_1D_2^+y + q_4y = 0 \,,$$

where D_i^+y is obtained from D_iy, $i = 1, 2, 3$, by interchanging r_1 with r_3, and q_1 with q_3. Note then that equation (E_4^+) is obtained from (E_4) by interchanging r_1 with r_3, and q_1 with q_3. If (E_4) and (E_4^+) are equivalent, then L_4 (and the corresponding equation (E_4)) is said to be self adjoint; e.g., if $r_1(x) \equiv r_3(x)$ and $q_1(x) \equiv q_3(x)$ in equation (E_4), then we have a canonical form for the self adjoint equation of order four [6], i.e.,

$$(E_4^*) \qquad \begin{aligned} L_4^*[y] = {} & \{r_1[(r_2[(r_1y')' + q_1y])' + q_2r_1y']\}' \\ & + q_1r_2[(r_1y')' + q_1y] + q_4y = 0 \,. \end{aligned}$$

A fundamental set of solutions $u_j(x, t)$ of (E_4) for $t \in [a, \infty)$ is defined by

$$D_iu_j(t, t) = \delta_{ij},\ i = 0, 1, 2, 3,\ j = 0, 1, 2, 3 \,.$$

Similarly, a fundamental set of solutions $u_j^+(x, t)$, $j = 0, 1, 2, 3$, is defined for the adjoint equation (E_4^+). We call $u_3(x, t)$ the first principal solution of (E_4). Leighton and Nehari [10] made use of the identity

$$u_3(s, t) = -u_3^+(t, s) \,.$$

which is a special case of the following theorem which follows from the Lagrange Identity [6] for (E_4).

THEOREM 1.1. *For $\alpha, \beta = 0, 1, 2, 3$ and $s, t \in [a, \infty)$*

$$(1.1) \qquad D_\alpha u_\beta(s, t) = (-1)^{\alpha+\beta} D_{3-\beta}^+ u_{3-\alpha}^+(t, s) \,.$$

The derivation of equations (1.1) is similar to that given by Dolan [7] for a similar set of formulas for the equation (E_3).

Instead of the usual Wronskians, $\det(y_i^{(j)})$, involving pure derivatives we introduce the more convenient generalized Wronskians,

$$W[y_1, \cdots, y_n] = \det(D_i y_j), \qquad \begin{array}{l} i = 0, 1, \cdots, n-1 \\ j = 1, 2, \cdots, n \\ n = 1, 2, 3, 4 \end{array}$$

$$W^+[y_1, \cdots, y_n] = \det(D_i^+ y_j),$$

which are given in terms of quasi derivatives ($D_0 y = D_0^+ y = y$). When we speak of a zero of a solution $y(x)$ of (E_4) of order k at we mean $D_i y(t) = 0$, $i = 0, \cdots, k-1$, $k = 1, 2, 3, 4$. If (E_4) is disconjugate on $[t, \infty)$ we write $\eta_1(t) = \infty$. Otherwise $\eta_1(t)$ will denote the first conjugate point of t. For properties of $\eta_1(t)$ see [9], [12]. A nontrivial solution of (E_4) having four zeros on $[t, \eta_1(t)]$ is called an extremal solution of (E_4) for $\eta_1(t)$. A nontrivial solution y of (E_4) is said to have an $i_0 - i_1 - \cdots - i_\nu (\nu = 1, 2, 3; i_k = 1, 2, 3)$ distribution of zeros on $[t, b] \subset [a, \infty)$ provided y has a zero at t_k of order at least i_k where $t \leqq t_0 < t_1 < \cdots < t_\nu \leqq b$. We now can introduce the following concepts, introduced by Dolan [7] for (E_3).

DEFINITION 1.1. For $t \in [a, \infty)$, the number $z_{i_0 i_1 \cdots i_\nu}(t)$ is the infimum of the set of numbers $b > t$ such that there is a nontrivial solution y of (E_4) having an $i_0 - i_1 - \cdots - i_\nu$ distribution of zeros on $[t, b]$ and a zero of order at least i_0 at t_0. By $z_{i_0 i_1 \cdots i_\nu}(t) = \infty$ we mean there is no such distribution of zeros on $[t, \infty)$.

DEFINITION 1.2. For $t \in [a, \infty)$, the number $r_{i_0 i_1 \cdots i_\nu}(t)$ is the infimum of the set of numbers $b > t$ such that there is a nontrivial solution y of (E_4) having an $i_0 - i_1 - \cdots - i_\nu$ distribution of zeros on $[t, b]$, by $r_{i_0 i_1 \cdots i_\nu}(t) = \infty$ we mean there is no such distribution of zeros on $[t, \infty)$. The numbers $z^+_{i_0 i_1 \cdots i_\nu}(t)$ and $r^+_{i_0 i_1 \cdots i_\nu}(t)$ are defined similarly for the adjoint equation (E_4^+).

If $z_{ij}(t) < \infty \{r_{ij}(t) < \infty\}$, then the word "infimum" in Definition 1.1 {1.2} can be replaced by "minimum". However, if $\nu > 1$ in Definitions 1.1 and 1.2 then you cannot in general do this (see paragraphs 2 and 3). There is a close relation between zeros of solutions and uniqueness problems, for example if α, β, γ are numbers such that $t \leqq \alpha < \beta < \gamma < r_{121}(t) \leqq \infty$, then there is a unique solution of (E_4) satisfying

$$y(\alpha) = A, \; y(\beta) = B, \; D_1 y(\beta) = C, \; y(\gamma) = D$$

where A, B, C, D are constants. Corresponding statements hold for the other numbers in Definitions 1.1 and 1.2 (see [2] or [4]). For known properties of these numbers see [1]—[6].

It is convenient to use further notation introduced by Dolan [7].

DEFINITION 1.3. If $z_{ij}(t) < \infty\{r_{ij}(t) < \infty\}$ and $i + j = 4$, then

$$Z_1(t) = \max\{z_{31}(t), z_{22}(t), z_{13}(t)\}$$
$$\{R_1(t) = \max\{r_{31}(t), r_{22}(t), r_{13}(t)\}\} .$$

By $Z_1(t) = \infty\{R_1(t) = \infty\}$ we mean at least one of the $z_{ij}(t)$ $\{r_{ij}(t)\}$, $i + j = 4$, is infinite.

The next lemma appears and is used quite often ([4], [10], [12]) and is stated here in terms of the equation (E_4).

LEMMA 1.2. *Let $u(x)$ be a nontrivial solution of (E_4) with a zero at $t_1 \in [a, \infty)$ of order $n \geqq 1$ and a zero at $t_2 \in (t_1, \infty)$ of order $m \geqq 1$ and $u(x) > 0$ on (t_1, t_2). Let $v(x)$ be a solution of (E_4) such that $v(x) > 0$ on (t_1, t_2) and $v(x)$ does not have a zero of order $\geqq n$ at t_1 and does not have a zero of order $\geqq m$ at t_2. Then there are constants c_1, c_2 such that $c_1c_2 < 0$ and $z(x) = c_1u(x) + c_2v(x)$ is a solution of (E_4) with a zero of order two in (t_1, t_2).*

2. The distribution of zeros of extremal solutions.

I. *Distribution of zeros when $\eta_1(t) < Z_1(t)$.* In this part of paragraph 2 we study the distribution of zeros of extremal solutions of (E_4) for $\eta_1(t)$ when $\eta_1(t) < Z_1(t)$. Since $\eta_1(t) < z_{ij}(t)$ if and only if $\eta_1(t) < r_{ij}(t)$, $\eta_1(t) < Z_1(t)$ if and only if $\eta_1(t) < R_1(t)$. Thus the theorems in this chapter are true when the assumption $\eta_1(t) < Z_1(t)$ is replaced by $\eta_1(t) < R_1(t)$.

Hartman [8] proves for an n^{th} order linear homogeneous differential equation that no nontrivial solution has n zeros, counting multiplicities, on an open interval (α, β) if and only if no nontrivial solution has n distinct zeros on the open interval (α, β). Hartman raised the question as to whether you could or could not replace "open interval (α, β)" by "closed interval $[\alpha, \beta]$" in the preceding statement. The next lemma shows that you can not replace "open interval (α, β)" by "closed interval $[\alpha, \beta]$". Dolan [7] has established a similar theorem for a third order differential equation. A similar result would hold for an n^{th} order linear homogeneous differential equation.

LEMMA 2.1. *If $\eta_1(t) < Z_1(t) \leqq \infty$ for $t \in [a, \infty)$, then no extremal solution of (E_4) for $\eta_1(t)$ has four distinct zeros on $[t, \eta_1(t)]$.*

Proof. Assume $u(x)$ is an extremal solution of (E_4) for $\eta_1(t)$ with four distinct zeros on $[t, \eta_1(t)]$. Since $\eta_1(t)$ is a strictly increasing function $u(x)$ has exactly four simple zeros on $[t, \eta_1(t)]$ and $u(t) =$

$u(\eta_1(t)) = 0$. Since $\eta_1(t) < Z_1(t) \leqq \infty$, there are three possibilities.

Case 1. $\eta_1(t) < z_{13}(t) \leqq \infty$.

For $\varepsilon > 0$, sufficiently small, let $\{u_\varepsilon(x)\}$ be a set of nontrivial solutions of (E_4) satisfying

$$\begin{aligned} u_\varepsilon(t) = 0 &= u_\varepsilon(\eta_1(t) - \varepsilon) \\ D_1 u_\varepsilon(\eta_1(t) - \varepsilon) &= D_1 u(\eta_1(t)) \neq 0 \\ D_2 u_\varepsilon(\eta_1(t) - \varepsilon) &= D_2 u(\eta_1(t)) . \end{aligned}$$

Since $u_3(t, \eta_1(t)) \neq 0$, it is easy to see that as $\varepsilon \to 0$

$$u_\varepsilon(x) \to u(x) \text{ uniformly for } x \in [t, \eta_1(t)] .$$

Since the zeros of $u(x)$ in $(t, \eta_1(t))$ are simple, there is an $\varepsilon_0 > 0$ such that $u_{\varepsilon_0}(x)$ is a solution of (E_4) with four zeros on $[t, \eta_1(t) - \varepsilon_0]$. This contradicts the definition of $\eta_1(t)$.

Case 2. $\eta_1(t) < z_{22}(t) \leqq \infty$.

For $\varepsilon > 0$, sufficiently small, let $\{u_\varepsilon(x)\}$ be a set of nontrivial solutions of (E_4) satisfying

$$\begin{aligned} u_\varepsilon(t) &= 0 = u_\varepsilon(\eta_1(t) - \varepsilon) \\ D_1 u_\varepsilon(t) &= D_1 u(t) \neq 0 \\ D_1 u_\varepsilon(\eta_1(t) - \varepsilon) &= D_1 u(\eta_1(t)) . \end{aligned}$$

Using the fact that $W[u_2(\eta_1(t), t), u_3(\eta_1(t), t)] \neq 0$, we proceed as in Case 1 to obtain a contradiction.

Case 3. $\eta_1(t) < z_{31}(t) \leqq \infty$.

A similar argument disposes of this case.

Aliev [5] proved for the classical fourth order equation that if $r_{13}(t) < \min[r_{31}(t), r_{22}(t)]$, $r_{31}(t) < \min[r_{13}(t), r_{22}(t)]$, or $r_{22}(t) < \min[r_{13}(t), r_{31}(t)]$, then no nontrivial solution of $l_4[y] = 0$ has four distinct zeros on $[t, \eta_1(t)]$. Lemma 2.1 generalizes his Theorems 1, 3, and 5. Another way in which Lemma 2.1 can be generalized, which is also similar to a result of M. Dolan [7] for (E_3), is the following theorem.

THEOREM 2.2. *If, for* $t \in [a, \infty)$, $\eta_n(t) < \infty$ *and if one of the inequalities* $u_3(\eta_n(t), t) \neq 0$, $W[u_2(\eta_n(t), t), u_3(\eta_n(t), t)] \neq 0$, $u_3(t, \eta_n(t)) \neq 0$ *holds, then no extremal solution of* (E_4) *for* $\eta_n(t)$ *has all simple zeros on* $[t, \eta_n(t)]$.

The proof of Theorem 2.2 is similar to the proof of Lemma 2.1.

Obviously a theorem like this can be stated for a more general linear n^{th} order equation.

LEMMA 2.3. *If, for* $t \in [a, \infty)$, $\eta_1(t) < Z_1(t) \leqq \infty$, *then no extremal solution of* (E_4) *for* $\eta_1(t)$ *has a* $1-2-1$ *distribution of zeros on* $[t, \eta_1(t)]$.

Proof. Assume $u(x)$ is a nontrivial solution of (E_4) with a $1-2-1$ distribution of zeros on $[t, \eta_1(t)]$, then $u(x)$ has a simple zero at t, a simple zero at $\eta_1(t)$, a double zero at some point $\tau \in (t, \eta_1(t))$, and we can assume $u(x) > 0$ for $x \in (t, \tau) \cup (\tau, \eta_1(t))$. It follows that there is a nontrivial linear combination of $u(x)$ and $u_3(x, t)$ with four distinct zeros on $[t, \eta_1(t)]$ which contradicts Lemma 2.1.

LEMMA 2.4. *If, for* $t \in [a, \infty)$, $\eta_1(t) < Z_1(t) \leqq \infty$, *then no extremal solution of* (E_4) *for* $\eta_1(t)$ *has a* $2-1-2$ *distribution of zeros on* $[t, \eta_1(t)]$.

Proof. Let $u(x)$ be a nontrivial solution of (E_4) with a $2-1-2$ distribution of zeros on $[t, \eta_1(t)]$, then $u(x)$ has a double zero at t, a double zero at $\eta_1(t)$, a simple zero at some point $\tau \in (t, \eta_1(t))$, and no other zeros on $[t, \eta_1(t)]$. It is easy to see that $u_3(x, t)$ does not have a multiple zero at $\eta_1(t)$, and so we can apply Lemma 1.2 to $u(x)$ and $u_3(x, t)$ to get a contradiction.

It follows as a corollary to Lemma 2.4 that if $\eta_1(t) < \infty$, then there is an extremal solution of (E_4) for $\eta_1(t)$ which has a sum of at least four zeros at t and $\eta_1(t)$ and is nonzero in $(t, \eta_1(t))$. This is a special case of the general n^{th} order results of Sherman [12] and Hinton [9].

It is evident that if $z_{31}(t) = z_{22}(t) < z_{13}(t)\{z_{13}(t) = z_{22}(t) < z_{31}(t)\}$ and if $\eta_1(t) < z_{32}(t)\{\eta_1(t) < z_{23}(t)\}$, then there is an extremal solution of (E_4) for $\eta_1(t)$ with a $2-1-1\{1-1-2\}$ distribution of zeros on $[t, \eta_1(t)]$. Hence, the condition $\eta_1(t) < Z_1(t) \leqq \infty$ is not enough to ensure that no extremal solution of (E_4) for $\eta_1(t)$ has a $2-1-1$ or a $1-1-2$ distribution of zeros on $[t, \eta_1(t)]$. Lemmas 2.5 and 2.6 give partial answers to this quandary.

LEMMA 2.5. *If, for*

$$t \in [a, \infty),\ \eta_1(t) < \max[z_{22}(t), z_{13}(t)]\{\eta_1(t) < \max[z_{22}(t), z_{31}(t)]\}\ ,$$

then no extremal solution of (E_4) *for* $\eta_1(t)$ *has a* $1-1-2\ \{2-1-1\}$ *distribution of zeros on* $[t, \eta_1(t)]$.

Proof. Assume $\eta_1(t) < \max[z_{22}(t), z_{13}(t)]$ and $u(x)$ is a nontrivial solution of (E_4) with a $1-1-2$ distribution of zeros on $[t, \eta_1(t)]$, then $u(x)$ has a simple zero at t, a double zero at $\eta_1(t)$, a simple zero at some point $\tau \in (t, \eta_1(t))$, and no other zeros on $[t, \eta_1(t)]$. If $z_{13}(t) > \eta_1(t)$ we get a contradiction by applying Lemma 1.2 to $u_3(x, t)$ and $u(x)$. Otherwise there is a nontrivial linear combination of $u(x)$ and $u_3(x, \eta_1(t))$ with a double zero at t and at $\eta_1(t)$ which contradicts $z_{22}(t) > \eta_1(t)$. The proof of the second half of this theorem is similar.

EXAMPLE 2.1. $y^{iv} - y' = 0$. Aliev [4] noted that $\eta_1(t) = r_{13}(t) \approx t + 5.9$, $r_{31}(t) = r_{22}(t) = \infty$. It follows from Lemma 2.5 that no nontrivial solution of $y^{iv} - y' = 0$ has a $2-1-1$ distribution of zeros on $[t, \eta_1(t)]$.

EXAMPLE 2.2. $y^{iv} + y' = 0$. Since, for t real, $\eta_1(t) = r_{31}(t) \approx t + 5.9$ and $r_{13}(t) = r_{22}(t) = \infty$, we have by Lemma 2.5 that no nontrivial solution has a $1-1-2$ distribution of zeros on $[t, \eta_1(t)]$.

LEMMA 2.6. *If, for* $t \in [a, \infty)$, *equation* (E_4) *is self adjoint and if* $\eta_1(t) < Z_1(t) \leqq \infty$, *then no extremal solution of* (E_4) *for* $\eta_1(t)$ *has a* $1-1-2$ or $2-1-1$ *distribution of zeros on* $[t, \eta_1(t)]$.

Proof. Assume $u(x)$ is a nontrivial solution of (E_4) with a $2-1-1$ distribution of zeros on $[t, \eta_1(t)]$, then $u(x)$ has a double zero at t, a simple zero at $\eta_1(t)$, a simple zero at some $\tau \in (t, \eta_1(t))$, and no other zeros on $[t, \eta_1(t)]$. Since (E_4) is self adjoint $z_{13}(t) = z_{31}(t)$ and $\eta_1(t) < Z_1(t)$ is possible in two ways.

If $\eta_1(t) = z_{22}(t) < z_{31}(t) = z_{13}(t)$, then there is a nontrivial solution $v(x)$ of (E_4) with a double zero at t and a double zero at $\eta_1(t)$. It follows that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a triple zero at t and a zero at $\eta_1(t)$. This contradicts the inequality $\eta_1(t) < z_{31}(t)$.

If $\eta_1(t) = z_{13}(t) = z_{31}(t) < z_{22}(t)$, then for $\varepsilon > 0$, sufficiently small, let $\{u_\varepsilon(x)\}$ be a set of nontrivial solutions of (E_4) satisfying

$$u_\varepsilon(t) = D_1 u_\varepsilon(t) = 0$$
$$u_\varepsilon(\eta_1(t) - \varepsilon) = 0$$
$$D_1 u_\varepsilon(\eta_1(t) - \varepsilon) = D_1 u(\eta_1(t)) \neq 0 \ .$$

Since $\eta_1(t) < z_{22}(t)$, $W[u_2(\eta_1(t), t), u_3(\eta_1(t), t)] \neq 0$ and it follows that as $\varepsilon \to 0$

$$u_\varepsilon(x) \to u(x) \text{ uniformly for } x \in [t, \eta_1(t)] \ .$$

Since τ is a simple zero of $u(x)$, there is an $\varepsilon_0 > 0$ such that $u_{\varepsilon_0}(x)$ is a nontrivial solution of (E_4) with a $2-1-1$ distribution of zeros on

$[t, \eta_1(t) - \varepsilon_0]$. This contradicts the definition of $\eta_1(t)$. The other half of this theorem is proved similarly.

COROLLARY 2.7. *If, for* $t \in [a, \infty)$, $\eta_1(t) < \min [r_{13}(t), r_{22}(t)]$, $\eta_1(t) < \min [r_{31}(t), r_{22}(t)]$ *or* $\eta_1(t) < \min [r_{13}(t), r_{31}(t)]$, *then no extremal solution of* (E_4) *for* $\eta_1(t)$ *has three distinct zeros on* $[t, \eta_1(t)]$ *of which one is at least a double zero.*

Proof. This corollary follows from Lemmas 2.3 and 2.4, and a closer look at the proof of Lemma 2.6.

Corollary 2.7 gives Theorems 2, 4, and 5 of Aliev [5] for the more general equation (E_4). Lemmas 2.3—2.6 are generalizations of these theorems of Aliev [5] for the equation (E_4).

The next theorem characterizes extremal solutions of (E_4) for $\eta_1(t)$ when (E_4) is self adjoint and $\eta_1(t) < Z_1(t) \leqq \infty$. In particular, it shows that the extremal solutions guaranteed by Sherman [12] are the only ones in certain cases. It follows easily from Lemmas 2.1, 2.3—2.6.

THEOREM 2.8. *If equation* (E_4) *is self adjoint and* $\eta_1(t) < Z_1(t) \leqq \infty$, *then no extremal solution of* (E_4) *for* $\eta_1(t)$ *has a zero on* $(t, \eta_1(t))$.

In the special case of Theorem 2.8, when equation (E_4) is self adjoint with $\eta_1(t) < z_{22}(t)$, it is interesting to note that, even though no nontrivial solution of (E_4) with four zeros on $[t, \eta_1(t)]$ has a zero in $(t, \eta_1(t))$, given any $\varepsilon > 0$ there is a nontrivial solution to (E_4) with a $2-1-1$, $1-2-1$, and $1-1-2$ distribution of zeros on $[t, \eta_1(t) + \varepsilon]$ the first zero being at t. This is the essence of part (i) of Corollary 3.9 in paragraph 3. Theorems 3.4, 3.6, and 3.7 are generalizations of part (i) of Corollary 3.9. In the other case of Theorem 2.8, when equation (E_4) is self adjoint with $\eta_1(t) < z_{31}(t)$, it is interesting to note that, even through no nontrivial solution of (E_4) with four zeros on $[t, \eta_1(t)]$ has a zero in $(t, \eta_1(t))$, given any $\varepsilon > 0$ there is a solution with a $2-1-1$ and $1-1-2$ distribution of zeros on $[t, \eta_1(t) + \varepsilon]$. This is the essence of part (ii) of Corollary 3.9 in paragraph 3, where we establish a generalization (Theorem 3.8) of this result.

II. *Distribution of zeros when* $\eta_1(t) = Z_1(t)$. In this part of paragraph 2 we study the distribution of zeros of extremal solutions of (E_4) for $\eta_1(t)$ when $\eta_1(t) = Z_1(t)$, i.e., when

$$\eta_1(t) = z_{13}(t) = z_{22}(t) = z_{31}(t) < \infty .$$

The following lemma is very useful.

LEMMA 2.9. *If $v_1(x)$, $v_2(x)$, and $v_3(x)$ are three linearly independent solutions of (E_4) with zeros at t and $\eta_1(t)$, then there is an extremal solution of (E_4) for $\eta_1(t)$ with four distinct zeros on $[t, \eta_1(t)]$.*

Proof. Since v_1, v_2, and v_3 are three linearly independent solutions of (E_4), $w^+(x) = W[v_1, v_2, v_3]$ is a nontrivial solution of the adjoint equation (E_4^+) [6] and it follows by the formulas (3.19) in [6] that $w^+(x)$ has a triple zero at t and at $\eta_1(t)$. Let α, β be distinct numbers in $(t, \eta_1(t))$, then $w(x) = W^+[w^+(x), u_3^+(x, \alpha), u_3^+(x, \beta)]$ is a nontrivial solution of (E_4) which has zeros at t, α, β, and $\eta_1(t)$.

We can now easily prove the following theorem.

THEOREM 2.10. *If, for $t \in [a, \infty)$, $\eta_1(t) = Z_1(t) < \infty$ and*

$$\eta_1(t) < \min\{z_{23}(t), z_{32}(t)\} ,$$

then for each of the distributions $1-1-1-1, 1-1-2, 1-2-1$, and $2-1-1$ of zeros on $[t, \eta_1(t)]$ there is an extremal solution of (E_4) for $\eta_1(t)$. In fact for any $\alpha, \beta, t \leqq \alpha \leqq \beta \leqq \eta_1(t)$ there is a nontrivial solution of (E_4) satisfying $y(t) = y(\alpha) = y(\beta) = y(\eta_1(t)) = 0$. (By $t = \alpha < \beta < \eta_1(t)$ is meant the boundary conditions $y(t) = D_1 y(t) = y(\beta) = y(\eta_1(t)) = 0$, etc.).

Proof. Let $u(x)$ be a nontrivial solution of (E_4) which has exactly a double zero at t and at $\eta_1(t)$. Since $\eta_1(t) < \min\{z_{23}(t), z_{32}(t)\}$ it follows that $u_3(x, t)$, $u_3(x, \eta_1(t))$, and $u(x)$ are three linearly independent solutions of (E_4) with zeros at t and $\eta_1(t)$. It follows from the proof of Lemma 2.9 that for any $\alpha, \beta, t \leqq \alpha < \beta < \eta_1(t)$, there is a nontrivial solution of (E_4) satisfying $y(t) = y(\alpha) = y(\beta) = y(\eta_1(t)) = 0$.

If $t < \alpha = \beta < \eta_1(t)$ then let $\{a_i\}$ and $\{b_i\}$ be sequences of numbers in (t, α) and $(\alpha, \eta_1(t))$ respectively such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = \alpha$ and let $v_i(x)$ be a solution of (E_4) satisfying

$$v_i(t) = v_i(a_i) = v_i(b_i) = v_i(\eta_1(t)) = 0 .$$

It follows by Ascoli's theorem that there is a nontrivial solution of (E_4) satisfying $y(t) = y(\alpha) = D_1 y(\alpha) = y(\eta_1(t)) = 0$. The other parts of this theorem are proved similarly.

The equation [6]

$$y^{iv} + 10y'' + 9y = 0, t \in [a, \infty) ,$$

is self adjoint with $\eta_1(t) = z_{33}(t) = t + \pi$. Let $\alpha, \beta \in (t, \eta_1(t))$, then since $y_1(x) = \cos^2(x + \pi/2 - \alpha)\sin(x - t)$, $y_2(x) = \sin^2(x - t)\cos(x + \pi/2 - \alpha)$, and $y_3(x) = \sin(x - t)\sin(x - \alpha)\sin(x - \beta)$ are solutions of

$$y^{iv} + 10y'' + 9y = 0 ,$$

there is for each of the distributions $1-2-1$, $2-1-1$, $1-1-2$, and $1-1-1-1$ of zeros on $[t, \eta_1(t)]$ an extremal solution of $y^{iv} + 10y'' + 9y = 0$ for $\eta_1(t)$. This example suggests the next theorem.

THEOREM 2.11. *If, for $t \in [a, \infty)$, equation (E_4) is self adjoint with $\eta_1(t) = z_{33}(t)$, then the assertions in Theorem 2.10 hold.*

Proof. Since $\eta_1(t) = z_{33}(t)$, $u_3(x, t)$ has a triple zero at t and $\eta_1(t)$. Let α, β be distinct numbers in $(t, \eta_1(t))$, then since (E_4) is self adjoint $w(x) \equiv W[u_3(x, t), u_3(x, \alpha), u_3(x, \beta)]$ is a nontrivial solution to (E_4) satisfying $y(t) = y(\alpha) = y(\beta) = y(\eta_1(t)) = 0$. The remainder of the proof is the same as in Theorem 2.10.

3. Ordering theorems for $r_{ij}(t)$, $z_{ij}(t)$, $r_{ijk}(t)$, $z_{ijk}(t)$, and $r_{1111}(t)$.

Theorem 3.1 was proved by Hartman [8], and, more recently, by Opial [11]. We state this theorem here without proof.

THEOREM 3.1. *For $t \in [a, \infty)$, $\eta_1(t) = r_{1111}(t)$.*

The reader should compare Theorem 3.1 to Lemma 2.1 and Theorem 2.10. In particular, if $\eta_1(t) < Z_1(t)$, then in the definition of $r_{1111}(t)$ we cannot replace the word "infimum" by the word "minimum".

By use of Theorem 3.1 it is fairly easy to prove the next theorem. R. G. Aliev [5] proved the first case of Theorem 3.2 in a somewhat different manner. He also claims that $r_{1111}(t) = \min[r_{211}(t), r_{112}(t)]$ but his proof is incomplete. However, no counterexample has been produced.

THEOREM 3.2. *For $t \in [a, \infty)$,*

$$\eta_1(t) = \min[r_{121}(t), r_{112}(t)] = \min[r_{121}(t), r_{211}(t)] .$$

Proof. Since $\eta_1(t) = r_{1111}(t)$, it suffices to show that

$$r_{1111}(t) \geqq \min[r_{121}(t), r_{112}(t)] \quad \text{and} \quad r_{1111}(t) \geqq \min[r_{121}(t), r_{211}(t)] .$$

Let $\rho(t) = \min[r_{121}(t), r_{112}(t)]$ and assume $r_{1111}(t) < \rho(t)$, then there are points α, β, γ, and δ such that $t \leqq \alpha < \beta < \gamma < \delta < \rho(t)$ and a nontrivial solution $u(x)$ of (E_4) satisfying $u(\alpha) = u(\beta) = u(\gamma) = u(\delta) = 0$ and $u'(\alpha)u'(\beta)u'(\gamma) = 0$. Since $\delta < \rho(t) \leqq r_{112}(t)$, there is a unique solution $v(x)$ of (E_4) satisfying $v(\alpha) = 0$, $v(\beta) = 1$, $v(\delta) = v'(\delta) = 0$. Since $\delta < \rho(t) \leqq r_{112}(t)$, $v(x) < 0$ for $x \in (\alpha, \delta)$. By Lemma 1.2, there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in

(β, γ), a zero at α and at δ. This contradicts the condition that $\delta < r_{121}(t)$. Similarly $r_{1111}(t) \geqq \min [r_{121}(t), r_{211}(t)]$.

Several interesting examples illustrate Theorem 3.2. In Example 2.1 Aliev [4] noted that $r_{211}(t) = \infty$, it follows from Theorem 3.2 that $\eta_1(t) = z_{121}(t)$. In Example 2.2, $r_{112}(t) = \infty$ and hence $\eta_1(t) = z_{121}(t)$. As a third example if in equation (E_4), $q_i(x) \leqq 0$, $i = 1, 2, 3, 4$, then $r_{121}(a) = r_{31}(a) = r_{13}(a) = \infty$ and so by Theorem 3.2 $\eta_1(a) = r_{112}(a) = r_{22}(a) = r_{211}(a)$.

Aliev [4] quotes the first inequality in the next theorem and states that he proved it in a paper [1] unavailable to the author. Theorem 3.3 follows easily from Lemma 1.2.

THEOREM 3.3. *If $t \in [a, \infty)$, then*

$$r_{211}(t) \geqq \min [r_{31}(t), r_{22}(t)] ,$$
$$r_{112}(t) \geqq \min [r_{13}(t), r_{22}(t)] .$$

In the next three theorems we consider the cases where either $r_{13}(t)$ or $r_{31}(t)$ is less than $r_{22}(t)$. Note that in Example 2.1 $r_{13}(t) < r_{31}(t) = r_{22}(t) = \infty$ and in Example 2.2 $r_{31}(t) < r_{13}(t) = r_{22}(t) = \infty$. Also, in the more familiar self adjoint cases, $r_{31}(t) = r_{13}(t)$. In particular, for the differential equation $y^{iv} + y = 0$ we have $r_{13}(t) = r_{31}(t) < r_{22}(t) = \infty$, and for the differential equation $y^{iv} - y = 0$ we have

$$r_{22}(t) < r_{13}(t) = r_{31}(t) = \infty .$$

THEOREM 3.4. *If, for $t \in [a, \infty)$, $r_{13}(t) < r_{22}(t)$, then*

$$r_{13}(t) = r_{112}(t) = z_{112}(t) .$$

Proof. By Theorem 3.3, $r_{112}(t) \geqq \min [r_{13}(t), r_{22}(t)] = r_{13}(t)$. Hence to complete the proof of this theorem it suffices to show that given $\varepsilon > 0$, but small enough so that $r_{13}(t) + \varepsilon < r_{22}(t)$, there is a nontrivial solution of (E_4) with a $1 - 1 - 2$ distribution of zeros on $[t, r_{13}(t) + \varepsilon]$ and with a zero at t. Since $r_{13}(t) + \varepsilon < r_{22}(t)$ there is a point $\alpha \in (t, r_{13}(t))$ such that $r_{13}(\alpha) = z_{13}(\alpha) \in (r_{13}(t), r_{13}(t) + \varepsilon)$ [4]. Let $\beta = z_{13}(\alpha)$, then there is a nontrivial solution $u(x)$ of (E_4) with a triple zero at β and a zero at α where $t < \alpha < \beta < r_{13}(t) + \varepsilon$. If $u(t) = 0$, then $r_{13}(t) = r_{112}(t) = z_{112}(t)$. If $u(t) \neq 0$, then let $v(x)$ be a nontrivial solution of (E_4) with a zero at t and a double zero at β. If $v(x) \neq 0$ for $x \in (t, \beta)$, then by Lemma 1.2 there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in (α, β) and a double zero at β. This contradicts $\beta < r_{22}(t)$. Therefore $v(x)$ has a zero in (α, β) and we have $r_{13}(t) = r_{112}(t) = z_{112}(t)$.

Lemma 3.5 was proved by R. G. Aliev in a paper [3] unavailable to the author and is reproved here for the sake of completeness.

LEMMA 3.5. *For* $t \in [a, \infty)$, $r_{31}(t) \geqq r_{211}(t)$.

Proof. Let $\beta = r_{31}(t)$ and assume $\beta < r_{211}(t)$. There is a point $\alpha \in [t, \beta)$ such that $u_3(\beta, \alpha) = 0$ and $u_3(x, \alpha) > 0$ for $x \in (\alpha, \beta)$. If β is a double zero of $u_3(x, \alpha)$, then $u_3(x, \alpha) > 0$ for $x \in (\beta, r_{211}(t))$. Let $\gamma \in (\beta, r_{211}(t))$ and $\mu \in (\alpha, \beta)$ and take $v(x)$ to be the solution of (E_4) with $v(\alpha) = v'(\alpha) = 0 = v(\gamma)$ and $v(\mu) = u_3(\mu, \alpha)$. Since $\gamma < r_{211}(t)$, $v(x) > 0$ for $x \in (\alpha, \gamma)$. It follows that the difference $u_3(x, \alpha) - v(x)$ is a nontrivial solution of (E_4) with a double zero at α, a zero at μ, and a zero in (β, γ). This contradicts the inequality $\gamma < r_{211}(t)$. Therefore β is either a simple zero or a triple zero of $u_3(x, \alpha)$. In either case $u_3(x, \alpha) < 0$ for $x \in (\beta, r_{211}(t))$. Let $\mu \in (\alpha, \beta)$ and $\nu \in (\beta, r_{211}(t))$, then, since $t < \mu < \nu < r_{211}(t)$, there is a unique solution $z(x)$ of (E_4) satisfying $z(\alpha) = z'(\alpha) = 0$, $z(\mu) = (1/2)u_3(\mu, \alpha)$, and $z(\nu) = (1/3)u_3(\nu, \alpha)$. Since $z(x)$ and $u_3(x, \alpha)$ are linearly independent, $D_2z(\alpha) \neq 0$, and there are two possibilities. If $D_2z(\alpha) > 0$, then $u_3(x, \alpha) - z(x)$ has a double zero at α, a zero in (α, μ) and a zero in (μ, ν). If $D_2z(\alpha) < 0$, then $z(x)$ has a double zero at α, a zero in (α, μ) and a zero in (μ, ν). In both cases we contradict the inequality $\nu < r_{211}(t)$.

THEOREM 3.6. *If, for* $t \in [a, \infty)$, $r_{31}(t) < r_{22}(t)$, *then*

$$r_{31}(t) = r_{211}(t) = z_{211}(t) .$$

Proof. It follows from Theorem 3.3 and Lemma 3.5 that $r_{31}(t) = r_{211}(t)$. To show that $r_{31}(t) = z_{211}(t)$ it suffices to show that given $\varepsilon > 0$, but small enough so that $r_{31}(t) + \varepsilon < r_{22}(t)$, that there is a nontrivial solution of (E_4) with a $2 - 1 - 1$ distribution of zeros on $[t, r_{13}(t) + \varepsilon]$ with at least a double zero at t. Since $r_{31}(t) + \varepsilon < r_{22}(t)$, $r_{31}(t) = z_{31}(t)$ and hence $u_3(r_{31}(t), t) = 0$ [4]. The number ε can be taken so that $u_3(x, t) \neq 0$ for $x \in (r_{31}(t), r_{31}(t) + \varepsilon)$. Let $\alpha \in (r_{31}(t), r_{31}(t) + \varepsilon)$ and let $v(x)$ be a nontrivial solution of (E_4) with $v(t) = v'(t) = 0$, $v(\alpha) = 0$. If $v(x) \neq 0$ for $x \in (t, \alpha)$, then if we apply Lemma 1.2 to $v(x)$ and $u_3(x, t)$ we contradict the inequality $r_{31}(t) < r_{22}(t)$. Hence $v(x)$ has a zero in $(t, r_{31}(t))$ and so $r_{31}(t) = z_{211}(t)$.

THEOREM 3.7. *If, for* $t \in [a, \infty)$, $r_{13}(t) = r_{31}(t) < r_{22}(t)$, *then*

$$r_{31}(t) = r_{13}(t) = r_{121}(t) = z_{121}(t) .$$

Proof. Since $\eta_1(t) = \min[r_{13}(t), r_{31}(t), r_{22}(t)] = r_{31}(t)$, it suffices to show that given $\varepsilon > 0$, but small enough so that $\eta_1(t) + \varepsilon < r_{22}(t)$,

$u_3(x, t) < 0$ for $x \in (\eta_1(t), \eta_1(t) + \varepsilon)$, and $u_3(x, \eta_1(t) > 0$ for $x \in (\eta_1(t), \eta_1(t) + \varepsilon)$, that there is a nontrivial solution of (E_4) with a $1-2-1$ distribution of zeros on $[t, \eta_1(t) + \varepsilon]$ with a zero at t. Let $v(x)$ be a nontrivial solution of (E_4) with a double zero at t and a zero in $(\eta_1(t), \eta_1(t) + \varepsilon)$. If $v(\eta_1(t)) = 0$, then it is easy to see that there would be a nontrivial linear combination of $u_3(x, t)$ and $v(x)$ with a double zero at t and at $\eta_1(t)$. This contradicts $\eta_1(t) < r_{22}(t)$ and so $v(\eta_1(t)) \neq 0$. If $v(x) \neq 0$ for $x \in (t, \eta_1(t))$, then if we apply Lemma 1.2 to $u_3(x, t)$ and $v(x)$ we contradict the inequality $\eta_1(t) < r_{22}(t)$. Hence $v(x)$ has a zero (and only one, call it α) in $(t, \eta_1(t))$. Let β be the first zero of $v(x)$ in $(\eta_1(t), \eta_1(t) + \varepsilon)$. It follows by Lemma 1.2 that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a zero at t, a double zero in (t, α) and a zero in $(\eta_1(t), \beta)$.

THEOREM 3.8. *If* $r_{22}(t) < \min[r_{31}(t), r_{13}(t)]$, *then*

$$r_{22}(t) = r_{211}(t) = z_{211}(t) = r_{112}(t) .$$

Proof. Let $\rho(t) \equiv \min[r_{31}(t), r_{13}(t)]$ and let $u(x)$ be a nontrivial solution of (E_4) with exactly a double zero at t and a double zero at $\eta_1(t)$. By Lemma 2.4 $u(x)$ does not have a zero in $(t, \eta_1(t))$. It is easy to see that there is a nontrivial linear combination of $u_3(x, t)$ and $u(x)$ in $(t, \eta_1(t))$, and a zero at p where $p < \eta_1(t) + \varepsilon < \rho(t)$. Hence $r_{22}(t) = z_{211}(t)$.

To show $\eta_1(t) = r_{112}(t)$ it suffices to show that for $\varepsilon < 0$, but small enough so that $\eta_1(t) + \varepsilon < \rho(t)$, there is a nontrivial solution of (E_4) with a $1-1-2$ distribution of zeros on $[t, \eta_1(t) + \varepsilon]$. Let $\delta \in (t, \eta_1(t))$ such that $\eta_1(\delta) \in (\eta_1(t), \eta_1(t) + \varepsilon)$. Since $\eta_1(t) + \varepsilon < \rho(t)$, $\eta_1(\delta) = r_{22}(\delta)$. Let $w(x)$ be a nontrivial solution of (E_4) with a double zero at δ and a double zero at $r_{22}(\delta)$. If $u(x)$ has a zero in (t, δ), then $\eta_1(t) = r_{112}(t)$. If $u(x)$ does not have a zero in (t, δ), then there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a zero in (t, δ), a zero in $(\delta, r_{22}(\delta))$ and a double zero at $r_{22}(\delta) < \eta_1(t) + \varepsilon$.

For the equation $y^{iv} + y'' = 0$, $\eta_1(t) = r_{22}(t) = t + 2\pi$ and $r_{31}(t) = r_{13}(t) = \infty$. It follows from Theorem 3.8 that $z_{211}(t) = r_{112}(t) = t + 2\pi = z_{112}(t)$.

COROLLARY 3.9. *Let* (E_4) *be self adjoint.*

(i) *If* $\eta_1(t) < r_{22}(t)$, *then* $\eta_1(t) = z_{211}(t) = z_{121}(t) = z_{112}(t)$.

(ii) *If* $\eta_1(t) < r_{31}(t)$, *then* $\eta_1(t) = z_{211}(t) = r_{112}(t)$.

Proof. Corollary 3.9 follows directly from Theorems 3.4, 3.6, 3.7, and 3.8.

One notices the absence of $r_{121}(t)$ in part (ii) of Corollary 3.9. For the equations of the form $(ry'')'' + py = 0$ where $r(x) > 0$, $p(x) \leqq 0$, $x \in [a, \infty)$, $r(x), p(x) \in C[a, \infty)$ for which $\eta_1(t) = r_{22}(t) < \infty$ [6] the hypothesis of part (ii) of Corollary 3.9 is fulfilled but $\eta_1(t) < r_{31}(t) = r_{13}(t) = r_{121}(t) = \infty$ [10].

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