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SOME THEOREMS IN FOURIER ANALYSIS ON SYMMETRIC SETS

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Let R be the real line and $A = A(R)$ the space of continuous functions on R which are the Fourier transforms of functions in $L^1(R)$. $A(R)$ is a Banach Algebra when it is given the $L^1(R)$ norm. For a closed $F \subseteqq R$ one defines $A(F)$ as the restrictions of $f \in A$ to F with the norm of $g \in A(F)$ the infimum of the norms of elements of A whose restrictions are g. Let $F_r \subseteqq R$ be of the form

$$F_r = \{\textstyle\sum_1^\infty \varepsilon_j r(j) : \varepsilon_j \text{ either 0 or 1}\}.$$

This paper shows that if

$$\sum (r(j+1)/r(j))^2 < \infty \quad \text{and} \quad \sum (s(j+1)/s(j))^2 < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. We also show that, in some sense square summability is the best possible criterion. In the course of the proof we show that F_r is a set of synthesis and uniqueness if $\sum (r(j+1)/r(j))^2 < \infty$. This is almost a converse to a theorem of Salem.

We shall also consider sets $E_m \subseteqq \prod_1^\infty Z_{m(j)}$ of the form

$$E_m = \{x : j^{\text{th}} \text{ coordinate is 0 or 1}\} .$$

The E_m will have analogous properties to the F_r that will depend on the $m(j)$.

The original work on isomorphisms of the algebras was done in [2] where Beurling and Helson show that any automorphism of A must arise from a map φ by $f \circ \varphi$ where $\varphi(x) = ax + b$. For restriction algebra the situation is more complex. In [5] it is shown that an isomorphism between $A(F_1)$ and $A(F_2)$ of norm one must be given by $f \to f \circ \varphi$ where $\varphi : F_2 \to F_1$ is continuous and $e^{i\varphi}$ is a restriction to F_2 of a character of the discrete reals. Further if F_2 is thick in some appropriate sense the character is continuous. However, McGehee [11] gives examples of F_1 and F_2 for which the restriction algebras $A(F_1)$ and $A(F_2)$ are isomorphic under an isomorphism induced by a discontinuous character. Meyer [12] has shown that if

$$\sum r(j + 1)/r(j) < \infty \quad \text{and} \quad \sum s(j + 1)/s(j) < \infty$$

then $A(F_r)$ is isomorphic to $A(F_s)$. For appropriate $r(j)$ this is an example of an isomorphism induced by a φ with $e^{i\varphi}$ not even a discontinuous character. He also showed that under these hypothesis F_r was a set of synthesis and uniqueness.

DEFINITIONS AND NOTATIONS. For background material and notation not defined here we refer the reader to [7] and [15].

In this paper G will always be a locally compact abelian group with dual group Γ. If g and γ are elements of G and Γ respectively, the value of the character γ at the point g will be denoted by (γ, g).

When we have a sequence of compact abelian groups G_j, we shall denote their *direct product* (complete direct sum [15]) by ΠG_j. If Γ_j is the dual of G_j, then the direct sum [15] $\Sigma\Gamma_j$ is the dual of ΠG_j. The j^{th} *coordinate* of elements g of ΠG_j or γ of $\Sigma\Gamma_j$ will be denoted by g_j and γ_j. One has:

$$(\gamma, g) = \Pi(\gamma_j, g_j)$$

where all but a finite number of elements in the product are 1.

We shall be dealing with the following basic groups:

(i) The *multiplicative circle group* will be denoted by T. T shall be identified with the unit interval by $x \in [0, 1) \to \exp(x)$ where $\exp(x) = e^{2\pi i x}$. The additive group of integers Z is the dual group of T. If $x \in [0, 1)$ represents an element of T and $n \in Z$ then $(n, x) = \exp(nx)$.

(ii) R will denote the *additive group* of *reals*. R is isomorphic to its dual under the pairing given by

$$(y, x) = \exp(xy),$$

$x, y \in R$.

(iii) Z_n for $n \geqq 2$ will denote the *additive group* of *integers mod n*. Z_n is also isomorphic to its dual under the pairing given by

$$(r, s) = \exp(rs/n),$$

$r, s \in Z_n$.

Any nonzero regular translation invariant measure on a locally compact abelian group G is called a Haar measure. If μ_G and μ_Γ are Haar measures on G and its dual group Γ respectively, the Fourier transform $\hat{f}$ of f in $L^1(\Gamma, \mu_\Gamma)$ is defined by

$$\hat{f}(g) = \int_\Gamma f(\gamma)(\gamma, g)d\mu_\Gamma$$

for $g \in G$. The inversion theorem gives

$$\int_G \hat{f}(g)(\gamma, -g)d\mu_G = Cf(\gamma) .$$

We shall normalize μ_G and μ_Γ so that $C = 1$. If G is compact we can place $\mu_G(G) = 1$ and if Γ is discrete $\mu_\Gamma(\gamma) = 1$ for $\gamma \in \Gamma$. $L^1(G)$ will denote $L^1(G, \mu_G)$ for a normalized Haar measure.

For $f, h \in L^1(\Gamma)$ define the convolution $f_* h$ by

$$f * h(\gamma) = \int_{\lambda \in \Gamma} f(\gamma - \lambda) h(\lambda) d\mu_\Gamma .$$

In [15] it is shown that $L^1(\Gamma)$ is a commutative Banach algebra under convolution and for $g \in G$

$$\widehat{f * h}(g) = \hat{f}(g)\hat{h}(g) .$$

We denote by $M(G)$ the space of all regular, complex valued Borel measures on G of finite total variation. In [15] the Fourier transform $\hat{\mu}$ of $\mu \in M(G)$ and the convolution $\mu * \nu$ of measures in $M(G)$ are defined. It is shown that $M(G)$ is a Commutative Banach Algebra under convolution and

$$\widehat{\mu * \nu}(\gamma) = \hat{\mu}(\gamma) \cdot \hat{\nu}(\gamma)$$

for $\gamma \in \Gamma$.

Let $A = A(G)$ be defined by

$$A(G) = \{\hat{f} \colon f \in L^1(\Gamma)\} .$$

$A(G)$ is a Banach algebra under pointwise multiplication and with norm $\| \cdot \|_A$ defined by $\|\hat{f}\|_A = \|f\|_{L^1(\Gamma)}$ and is isomorphic to $L^1(\Gamma)$ under $*$. For a closed set $E \subseteqq G$ define the restriction algebra

$$A(E) = \{\hat{f}/E \colon f \in L^1(\Gamma)\}$$

with norm $\| \cdot \|_{A(E)}$ defined by

$$\| h \|_{A(E)} = \inf \{\|\hat{f}\|_A \colon \hat{f}/E = h\} .$$

$A(E)$ is again a Banach algebra under pointwise multiplication. Set

$$I(E) = \{\hat{f} \colon \hat{f}/E = 0 \quad \text{and} \quad f \in L^1(\Gamma)\}$$

$A(E)$ can be identified with the quotient algebra $A(G)/I(E)$.

The dual space of $A(G)$ is denoted by PM (or $PM(G)$). Its elements are called *pseudomeasures*. Each $S \in PM$ can be identified with a function $\hat{S} \in L^\infty(\Gamma)$ as follows. The action of $S \in PM$ as a linear functional on $\hat{f} \in A(G)$ is given by

$$(S, \hat{f}) = \int_\Gamma f(\gamma)\overline{\hat{S}(\gamma)} d\mu_\Gamma .$$

We shall denote by $\| S \|_{PM}$ the $L^\infty(\Gamma)$ norm of $\hat{S}$. Thus PM under $\| \cdot \|_{PM}$ is identical with $L^\infty(\Gamma)$ under the sup norm.

Since $A(E)$ is the quotient of $A(G)$ by $I(E)$, the dual of $A(E)$ consists of those $S \in PM$ which annihilate every function in $I(E)$.

We shall denote this dual of $A(E)$ by $N(E)$. If $N(E)$ is the set of all $S \in PM$ with supp $S \subseteqq E$ [7, p. 161], then E is said to be a set of synthesis. The set of all $\mu \in M(G)$ with support in E we denote by $M(E)$. $M(E)$ can be considered a subspace of $N(E)$ with $(\mu, \hat{f}) = \int \hat{f}\, d\bar{\mu}$. The two definitions for $\hat{\mu}$ coincide.

If G_1 and G_2 are locally compact abelian groups and E_1 and E_2 are closed subsets of G_1 and G_2 respectively we say that $\Phi: A(E_1) \rightarrow A(E_2)$ is an isomorphism into if and only if it is an injective algebraic homomorphism and is continuous. If the range of Φ is dense in $A(E_2)$ there exists a continuous $\varphi: E_2 \rightarrow E_1$ with $\Phi f = f \circ \varphi$ [9]. We always denote the adjoint of Φ taking $N(E_2)$ into $N(E_1)$ by Φ^*.

Symmetric sets in R are defined as follows. For any sequence $r = \{r(j): j = 1, \cdots\}$ of positive reals with the property

$$\sum_k^\infty r(j) < r(k-1)$$

we define the subset F_r of R by

$$F_r = \left\{\sum_1^\infty \varepsilon_j r(j): \varepsilon_j \text{ either } 0 \text{ or } 1\right\}.$$

The representation of the elements of F_r as an infinite sum is unique. For each positive integer k, the subset F_r^k or F_r is defined by

$$F_r^k = \left\{\sum_1^k \varepsilon_j r(j): \varepsilon_j \text{ either } 0 \text{ or } 1\right\}.$$

We define the subspace $N_1(F_r)$ of $N(F_r)$ by

$$N_1(F_r) = \bigcup_{k=1}^\infty M(F_r^k).$$

For any given sequence $m = \{m(j): j = 1, 2, \cdots\}$ of positive integers we define the subset E_m of $\Pi_j Z_{m(j)}$ by

$$E_m = \{x: x \in \Pi Z_{m(j)}; x_j \text{ either } 0 \text{ or } 1\}.$$

For each positive integer k the subset E_m^k of E_m is defined by

$$E_m^k = \{x: x \in E_m; x_j = 0 \text{ if } j > k\}.$$

Define the subspace $N_1(E_m)$ of $N(E_m)$ by

$$N_1(E_m) = \bigcup_{k=1}^\infty M(E_m^k).$$

For r and m as above there is a standard homeomorphism $\varphi: E_m \rightarrow F_r$ which takes $x \rightarrow \Sigma x_j r(j)$. Let the inverse of φ be called ψ.

We shall frequently write E for E_m, E^k for E_m^k, F for F_r, and F^k for F_r^k when the respective sequences are clear.

Throughout this work ε_j will always denote a quantity that may take on the values 0 or 1.

1. The symbols r and m shall always denote $\{r(j): j = 1, 2, \cdots\}$ and $\{m(j): j = 1, 2, \cdots\}$ respectively. F_r and E_m will then represent the previously defined sets with $\varphi: E_m \rightarrow F_r$ and $\psi: F_r \rightarrow E_m$ the standard homeomorphisms. The maps φ and ψ induce maps between $N_1(E_m)$ and $N_1(F_r)$ which we shall again denote by φ and ψ. The maps have the form

$$\varphi(\mu)(\{\varphi(x)\}) = \mu(\{x\})$$

for $\mu \in N_1(E)$, and

$$\psi(\mu)(\{\psi(x)\}) = \mu(\{x\})$$

for $\mu \in N_1(F)$.

If $x = \langle \varepsilon_1, \cdots \varepsilon_k, 0, \cdots \rangle$ is an element of E_m^k and $\mu \in M(E^k)$ set

$$a(\varepsilon_1, \cdots, \varepsilon_k) = \mu(\{x\}) .$$

If $y = \sum_1^k \varepsilon_j r(j)$ is an element of F^k and $\nu \in M(F^k)$ set

$$b(\varepsilon_1, \cdots, \varepsilon_k) = \nu(\{y\}) .$$

We see that

$$\| \mu \|_{PM} = \sup_{\xi_1, \cdots, \xi_k} | \sum a(\varepsilon_1, \cdots, \varepsilon_k) \xi_1^{\varepsilon_1} \cdots \xi_k^{\varepsilon_k} |$$

where ξ_j is an arbitrary $m(j)$ root of unity and the sum is taken over all combinations with ε_j being 0 or 1. Similarly

$$\| \nu \|_{PM} = \sup_x \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left(x \sum_1^k \varepsilon_j r(j) \right) \right|$$

where $x \in R$.

For any $\mu \in N_1(E)$ we define

$$\| \mu \|_{\text{MAX}} = \sup_{\theta_1, \cdots \theta_k} \left| \sum a(\varepsilon_1 \cdots, \varepsilon_k) \exp \left(\sum \varepsilon_j \theta_j \right) \right|$$

where $\theta_j \in R$. Define $\| \nu \|_{\text{MAX}}$ for $\nu \in N_1(F)$ by

$$\| \nu \|_{\text{MAX}} = \sup_{\theta_1, \cdots \theta_k} \left| \sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp \left(\sum \varepsilon_j \theta_j \right) \right| .$$

It is clear that $\|\mu\|_{PM} \leqq \|\mu\|_{\mathrm{MAX}}$ and $\|\nu\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}$. For any standard homeomorphism φ we have

$$\|\varphi\mu\|_{PM}/\|\mu\|_{PM} \leqq \|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} .$$

Similarly

$$\|\psi\nu\|_{PM}/\|\nu\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}/\|\nu\|_{PM} .$$

One should note that if r is a sequence of reals independent mod 1 over the rationals, Kronecher's Theorem [4, p. 99] implies that $\|\nu\|_{\mathrm{MAX}} = \|\nu\|_{PM}$ for $\nu \in N_1(E_r)$.

In order to achieve isomorphisms between certain quotient algebras we shall first study the ratios $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM}$ and $\|\nu\|_{\mathrm{MAX}}/\|\nu\|_{PM}$.

LEMMA 1.1. *If $\sum(1/m(j))^2 < \infty$ then there is a C depending only on m so that $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} \leqq C$ for all nonzero $\mu \in N_1(E_m)$.*

Proof. For each k, since $M(E^k)$ is finite dimentional, there is a smallest constant $A(k)$ so that $\|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} \leqq A(k)$ for all nonzero $\mu \in M(E^k)$. We shall show that there are constants C_k with $\Pi C_k < \infty$ so that $A(k)/A(k-1) \leqq C_k$.

The quotient $\|\mu\|_{PM}/\|\mu\|_{\mathrm{MAX}}$ is equal to

$$\frac{\sup_{\xi_j}\left|\sum_{\varepsilon_j}[(a(\varepsilon_1, \cdots, \varepsilon_{k-1}, 0) + a(\varepsilon_1, \cdots \varepsilon_{k-1}, 1)\xi_k)(\xi_1^{\varepsilon_1} \cdots \xi_{k-1}^{\varepsilon_{k-1}})]\right|}{\sup_{Z_j}\left|\sum_{\varepsilon_j}[(a(\varepsilon_1, \cdots, \varepsilon_{k-1}, 0) + a(\varepsilon_1, \cdots, \varepsilon_{k-1} 1)Z_k)(Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}})]\right|} \tag{1.2}$$

where ξ_j are $m(j)$ roots of unity and Z_j are complex numbers of modulus 1. By a division and multiplication $\|\mu\|_{PM}/\|\mu\|_{\mathrm{MAX}}$ becomes

$$\begin{aligned} &\frac{\sup_{\xi_j}\left|\sum[(a(\cdots, 0) + a(\cdots, 1)\xi_k)\xi_1^{\varepsilon_1} \cdots \xi_{k-1}^{\varepsilon_{k-1}}]\right|}{\sup_{\xi_k, Z}\left|\sum[(a(\cdots, 0) + a(\cdots, 1)\xi_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|} \\ &\quad\times \frac{\sup_{\xi_k, Z_j}\left|\sum[(a(\cdots, 0) + a(\cdots, 1)\xi_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|}{\sup_{Z_j}\left|\sum[(a(\cdots, 0) + a(\cdots, 1)Z_k)Z_1^{\varepsilon_1} \cdots Z_{k-1}^{\varepsilon_{k-1}}]\right|} . \end{aligned} \tag{1.3}$$

The factor used in division and multiplication in (1.3) is nonzero. If it were zero $\|\mu\|_{PM}$ would be zero and hence μ would be zero. The fraction on the left of (1.3) is greater than or equal to $1/A(k-1)$. Choose $z_j = y_j$ so that the maximum of the denominator in (1.2) is achieved. The fraction on the right in (1.3) is greater than or equal to

$$\left|1 + \frac{\sum [a(\cdots, 1)(\xi_k - y_k)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}]}{\sum [(a(\cdots, 0) + a(\cdots, 1)y_k)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}]}\right| . \tag{1.4}$$

If $\sum a(\cdots, 1)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}$ is zero (1.4) is equal to one. Otherwise set $e^{ix} = \xi_k/y_k$ and (1.4) is equal to

$$\left|1 + \frac{e^{ix} - 1}{\left[\dfrac{\sum [a(\cdots, 0)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}]}{y_k \sum [a(\cdots, 1)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}]}\right] + 1}\right| . \tag{1.5}$$

However, in order that the choice $z_j = y_j$ give $\|\mu\|_{\mathrm{MAX}}$, the quotient

$$\frac{\sum a(\cdots, 0)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}}{y_k \sum a(\cdots, 1)y_1^{\varepsilon_1} \cdots y_{k-1}^{\varepsilon_{k-1}}}$$

must be a real positive real number. Call that number s and (1.5) becomes

$$\left|1 + \frac{(\cos x - 1) + i \sin x}{s + 1}\right|$$

which is greater than or equal to

$$1 - x^2/2 .$$

For an appropriate ξ_k, $|x|$ is less than or equal to $2\pi/m(k)$.

From the above calculation we get

$$\|\mu\|_{PM}/\|\mu\|_{\mathrm{MAX}} \geqq \frac{(1 - 2\pi^2/(m(k))^2)}{A(k-1)}$$

and therefore

$$A(k) \leqq A(k-1)\cdot\left(1 + \frac{C^1}{(m(k))^2}\right)$$

for some absolute constant C^1 and for all $m(k)$ sufficiently large. Since $\sum (1/m(j))^2 < \infty$ the theorem is proven.

For the symmetric sets F_r we shall need the following lemma similar to Lemma 1.1.

LEMMA 1.6. *Suppose that* $\sum (r(j+1)/r(j))^2 < 1/24$. *Choose a real number* x_0 *and define the interval* I *to be*

$$\left\{x : |x - x_0| < 2\left(\sum_1^k 1/r(j)\right)\right\} .$$

There is then a constant C_1 *independent of* k *and* x_0, *so that*

$$\|\nu\|_{\mathrm{MAX}}/\sup|\hat{\nu}(x)| < C_1, \text{ for all nonzero } \nu \in M(F_r^k) .$$

Proof. Fix k and choose a nonzero $\nu \in M(F^k_r)$. There exists real numbers $\theta_1, \cdots, \theta_k$ less than or equal to one, for which

$$\|\nu\|_{\mathrm{MAX}} = |\sum b(\varepsilon_1, \cdots, \varepsilon_k) \exp(\sum \varepsilon_j \theta_j)| .$$

Define the functions $\hat{\nu}_k, \cdots, \hat{\nu}_2, \hat{\nu}_1 = \hat{\nu}$ on R by

$$\hat{\nu}_j(\mathrm{x}) = \sum \left[b(\varepsilon_1, \cdots, \varepsilon_k) \exp\left(\sum_1^{j-1} \varepsilon_j \theta_j\right) \exp x\left(\sum_j^k \varepsilon_j r(j)\right)\right] .$$

Let us estimate $\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/\|\nu\|_{\mathrm{MAX}}$ where

$$I_1 = \left\{x \colon |x - x_0| \leqq \sum_{k-1}^k (2/r(j))\right\} .$$

There is an x'_0 within $(1/r(k))$ of x_0 for which $x'_0 \cdot r(k) = \theta_k \pmod 1$. Pick x_1 within $1/r(k-1)$ of x'_0 so that $x_1 \cdot r(k-1) = \theta_{k-1} \pmod 1$. Then

$$\sup_{x \in I_1} |\hat{\nu}_{k-1}(x)|/\|\nu\|_{\mathrm{MAX}} \geqq |\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} .$$

As a function of $x, \hat{\nu}_k(x)$ is the Fourier Stieltjes transform of a measure ν_k having support in $[0, r(k)]$. Now,

$$\begin{aligned} |\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} &= |\hat{\nu}_k(x_1)|/|\hat{\nu}_k(x'_0)| \\ &= \left|1 + \frac{\hat{\nu}'_k(x'_0)}{\hat{\nu}_k(x'_0)}(x_1 - x'_0) + \frac{\hat{\nu}''_k(x'_0)}{\hat{\nu}_k(x'_0)} \frac{(x_1 - x'_0)^2}{2} + \cdots\right| \end{aligned}$$

$|\hat{\nu}_k|^2$ has a maximum at x'_0. Therefore, if $\hat{\nu}_k = f + ig$, with f and g real, $f \cdot f' + g \cdot g' = 0$ at x'_0. But, at x'_0,

$$\begin{aligned} \hat{\nu}'_k/\hat{\nu}_k &= f' + ig'/f + ig \\ &= (ff' + gg' + i(fg' - f'g))/f^2 + g^2 , \end{aligned}$$

which is purely imaginary. Therefore,

$$|\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} \geqq 1 - \left|\frac{\hat{\nu}''_k(x'_0)}{\hat{\nu}_k(x'_0)} \frac{(x_1 - x'_0)^2}{2} + \cdots\right| .$$

If a measure μ has support in $[0, \delta]$ a theorem of Bernstein [1, p. 138] shows that for all x

$$[\hat{\mu}'(x)| \leqq \delta \|\mu\|_{PM}$$

and hence its *nth* derivative $\hat{\mu}^{(n)}$ has

$$|\hat{\mu}^{(n)}(x)| \leqq \delta^n \|\mu\|_{PM} .$$

Since ν_k has support in $[0, r(k)]$ we obtain

$$|\hat{\nu}_{k-1}(x_1)|/\|\nu\|_{\mathrm{MAX}} \geqq 1 - (r(k)^2/r(k-1)^2) .$$

In effect, we have just shown that there is an $x_1 \in I_1$ for which

$$\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-1}(x_1)| \leqq 1 + 2(r(k)/r(k-1))^2 .$$

Assume that for some $j < k - 1$ there is an

$$x_j \in I_j = \left\{x: |x - x_0| \leqq \sum_{k-j}^{k} (2/r(l))\right\}$$

for which

$$\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-j}(x_j)| \leqq \prod_{l=k-j}^{\infty} (1 + 24(r(l+1)/r(l))^2) .$$

We shall show there is then an $x_{j+1} \in I_{j+1}$ for which

$$\begin{aligned} &\|\nu\|_{\mathrm{MAX}}/|\hat{\nu}_{k-(j+1)}(x_{j+1})| \\ &\quad \leqq \prod_{l=k-j-1}^{\infty} (1 + 24(r(l+1)/r(l))^2) . \end{aligned} \tag{1.7}$$

Consider $S = \{x: |x - x_j| \leqq 1/r(k-(j+1))\}$. If $|\hat{\nu}_{k-j}|$ does not have a relative maximum in S greater than or equal to $|\hat{\nu}_{k-j}(x_j)|$, then $|\hat{\nu}_{k-j}|$ would be greater than or equal to $|\hat{\nu}_{k-j}(x_j)|$ on some interval in S of length equal to $1/r(k-(j+1))$. However there would be an x_{j+1} in the interval for which $x_{j+1} \cdot r(k-(j+1)) = \theta_{k-(j+1)} \pmod 1$ and hence $\hat{\nu}_{k-(j+1)}(x_{j+1}) = \hat{\nu}_{k-j}(x_{j+1})$, which implies the induction step. Let us assume therefore that there is an x'_j where

$$|x'_j - x_0| \leqq (1/r(k-(j+1)) + \sum_{k-j}^{k} 2/r(l)),$$

$|\hat{\nu}_{k-j}(x'_j)| \geqq |\hat{\nu}_{k-j}(x_j)|$ and at which $|\hat{\nu}_{k-j}|$ has a relative maximum. As before, choosing x_{j+1} within $1/r(k-j+1)$ of x'_j and satisfying $x_{j+1} \cdot r(k-(j+1)) = \theta_{k-(j+1)}$ gives

$$\begin{aligned} |\hat{\nu}_{k-(j+1)}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| &= |\hat{\nu}_{k-j}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| \\ &\geqq \sum 1 - \left| \frac{\hat{\nu}''_{k-j}(x'_j)}{\hat{\nu}_{k-j}(x'_j)} \cdot \frac{(x_{j+1} - x'_j)^2}{2} + \cdots \right| . \end{aligned} \tag{1.8}$$

$\hat{\nu}_{k-j}$ as a function of x is the Fourier Stieltjes of a measure ν_{k-j} having support in $[0, 2r(k-j)]$. Since $\|\nu_{k-j}\|_{PM} \leqq \|\nu\|_{\mathrm{MAX}}$, the previously stated theorem of Bernstein gives

$$|\hat{\nu}^{(n)}_{k-j}(x')| \leqq (2r(k-j))^n \|\nu\|_{\mathrm{MAX}} .$$

However

$$\begin{aligned} \|\nu\|_{\mathrm{MAX}} &\leqq \left[\prod_{l=k-j}^{\infty} (1 + 24(r(l+1)/r(l))^2)\right] \times |\hat{\nu}_{k-j}(x'_j)| \\ &\leqq e^{24\Sigma(r(l+1)/r(l))^2} \cdot |\hat{\nu}_{k-j}(x'_j)| \\ &\leqq 3\,|\hat{\nu}_{k-j}(x'_j)| \end{aligned}$$

Since $\Sigma(r(l+1)/r(l))^2 \leqq (1/24)$. Therefore in (1.8),

$$|\hat{\nu}_{(k-j+1)}(x_{j+1})/\hat{\nu}_{k-j}(x'_j)| \geqq 1 - 12(r(k-j)/r(k-(i+1))^2$$

and hence (1.7) is true, finishing the induction.

Lemma 1.6 in its present form is an adaptation and extension of a lemma of Meyer [12]. Previously we had much more stringent conditions on the r, to arrive at a similar conclusion to Lemma 1.6.

To utilize the Lemmas 1.1 and 1.6 to obtain isomorphisms of restriction algebras we shall introduce some functional analysis.

Let V represent a Banach Space and V^* its dual. For $r > 0$ let $B_r = \{t: t \in V^*, ||t|| \leqq r\}$. A set $O \subseteqq V^*$ is said to be open in the *bounded topology* on V^* if and only if $O \cap B_r$ is open in the relative weak* topology of B_r for all r > 0. For a distribution of the bounded topology the reader should consult [6, p. 427].

LEMMA 1.10. *Let V, W be Banach spaces with duals V^* and W^*. Let $K \subset V^*$ be a weak* dense subspace of V^*. Suppose that $T: K \rightarrow W$ is linear and continuous when K has the topology induced by the bounded topology on V^* and W^* has the weak topology. Then there exists a bounded linear transformation S: $W \rightarrow V$ for which $T = S^*/K$.*

Proof. For each $w \in W$, define the linear functional T_w on K by

$$T_w(t) = Tt(w) .$$

Each T_w is continuous in the topology induced by the bounded topology of V^* which is a locally convex topology by Corollary 5, page 428 of [6]. Hence by the Hahn-Banach theorem there exists an extension $\tilde{T}_w$ of T_w to all of V^*, continuous in the bounded topology of V^*.

By Theorem 6, page 428 of [6], $\tilde{T}_w$ is continuous in the weak* topology on V^*. Hence there exists an element $v \in V$ such that $T_w(t) = t(v)$ for all $t \in K$. Since K is assumed weak* dense in V^*, the element v is determined by w. Define S: $W \rightarrow V$ by $S(w) = v$. S is linear. Since K is weak* dense S is closed. Therefore by the Closed Graph Theorem S is bouned. If $t \in K, w \in W$

$$S^*t(w) = t(S(w)) = Tt(w) ,$$

which completes the proof.

It is clear that $N_1(E_m)$ and $N_1(F_r)$ are weak* dense in $N(E_m)$ and $N(F_r)$, respectively. By studying the continuity of the standard maps between $N_1(E_m)$ and $N_1(F_r)$, we shall be able to use Lemma 1.10 to obtain isomorphisms between $A(E_m)$ and $A(F_r)$ for certain classes of sequences m and r.

Choose $\mu \in N_1(E)$. For each k we define an approximating measure μ_k in $M(E^k)$ by

$$\mu_k(\{x\}) = \sum_{y \in D} \mu(\{y\})$$

where $x \in E^k$ and $D = \{y : y \in E \text{ and } y_j = x_j \text{ for } j \leqq k\}$. Let

$$\Gamma^k = \{\gamma : \gamma \in \Sigma Z(m(j)) \quad \text{and} \quad \gamma_j = 0 \quad \text{if} \quad j > k\}.$$

If $\gamma \in \Gamma^k \hat{\mu}_k(\gamma) = \hat{\mu}(\gamma)$. It is easy to see that

$$\| \mu_k \|_{PM} = \sup_{\gamma \in \Gamma} {}_k \, | \hat{\mu}_k(\gamma) | \, .$$

To each $\lambda \in M(E^k)$ we associate the measure λ' in $M(E^k)$ defined by

$$\lambda'(\{x\}) = \begin{cases} 0 & \text{if} \quad x_k = 0 \\ \lambda(\{x\}) & \text{if} \quad x_k = 1 \end{cases} .$$

It is not hard to see that

$$\| \lambda' \|_{PM} \leqq 2 \| \lambda \|_{PM} \, .$$

Choose $\nu \in N_1(F)$. For each k define an approximating measure ν_k in $M(F^k)$ by

$$\nu_k(\{x\}) = \sum_{y \in D} \nu(\{y\})$$

where $x = \sum_1^k x_j r(j)$ and $D = \{y : y = \Sigma \varepsilon_j r(j) \text{ and } \varepsilon_j = x_j \text{ for } j \leqq k\}$.

To each $\beta \in M(F^k)$ we associate the measure β' in $M(F^k)$ defined by

$$\beta'(\{x\}) = \begin{cases} 0 & \text{if} \quad x = \sum_1^k \varepsilon_j r(j) \quad \text{and} \quad \varepsilon_k = 0 \\ 1 & \text{if} \quad x = \sum_1^k e_j r(j) \quad \text{and} \quad \varepsilon_k = 1 \end{cases} .$$

We are now ready to prove the following theorem.

THEOREM 1.11. *If $\Sigma(1/m(j))^2 < \infty$ and $\Sigma(r(j+1)/r(j))^2 < \infty$ then $A(E_m)$ is isomorphic to $A(F_r)$.*

We shall break the proof into two lemmas.

LEMMA *A. Let F_r be any symmetric set. Let $\Sigma(1/m(j))^2 < \infty$ $\varphi \colon E_m \to F_r$ the standard homeomorphism. Then there is an iso-*

morphism into $\Phi\colon A(F_r) \to A(E_m)$ *given by*

$$\Phi(f) = f \circ \varphi, \qquad f \in A(F_r) .$$

Proof. We shall study the continuity properties of

$$\varphi\colon N_1(E) \to N_1(F) .$$

For $f \in A(F)$ define

$$U_{\varepsilon,f} = \{\nu\colon \nu \in N_1(F) \quad \text{and} \quad |(\nu, f)| < \varepsilon\} .$$

To establish that φ is continuous from the bounded weak* topology of $N_1(E)$ to the weak* topology of $N_1(F)$ it is sufficient to prove that the zero element of $N_1(E)$ is an interior point of $\varphi^{-1}(U_{\varepsilon,f})$ (i.e., that φ is continuous at 0). This follows at once if we prove that given a and ε, there exists δ, k such that if for $\mu \in N_1(E)$

$$\begin{aligned} &\|\mu\|_{PM} \leqq a \quad \text{and} \quad |\hat{\mu}(\gamma)| < \delta \quad \text{for} \quad \gamma \in \Gamma^k \\ &\varphi(\mu) \quad \text{is an element of} \quad U_{\varepsilon,f}. \end{aligned} \tag{1.12}$$

In view of Lemma 1.1 (1.12) follows if we can show that given a, ε, and M then there exists δ, k such that for $\mu \in N_1(E)$,

$$\begin{gathered} \|\mu\|_{PM} \leqq a \quad \text{and} \quad \hat{\mu}(\gamma) < \delta \quad \text{for} \quad \gamma \in \Gamma^k \\ \text{then} \\ |\widehat{\varphi(\mu)}(x)| < \varepsilon \quad \text{for} \quad |x| \leqq M . \end{gathered} \tag{1-13}$$

We first estimate $|\widehat{\varphi(\mu)} - \widehat{\varphi(\mu_k)}|$ for $\mu \in M(E^s)$.

$$\begin{aligned} |\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| &\leqq \sum_k^{s-1} |\widehat{\varphi(\mu_{j+1})}(x) - \widehat{\varphi(\mu_j)}(x)| \\ &\leqq \sum_k^{s-1} |\exp(-xr(j+1)) - 1| \cdot \|\varphi(\mu'_{j+1})\|_{PM} . \end{aligned}$$

By Lemma 1.1, for any s

$$|\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| \leqq 4\pi C|x| \, \|\mu\|_{PM} \cdot \sum_{k+1}^{\infty} r(j) .$$

For μ with $\|\mu\|_{PM} \leqq a$, pick $\delta < \varepsilon/2C$ where C is the constant of Lemma 1.1 and choose k so that $4\pi CMa \sum_{k+1}^{\infty} r(j) < \varepsilon/2$. If $|\hat{\mu}(\gamma)| < \delta$ for $\gamma \in \Gamma^k$, then $\|\mu_k\|_{PM} < \delta$ and by Lemma 1.1 $\|\varphi(\mu_k)\|_{PM} < \varepsilon/2$. If $|x| \leqq M$, then $|\widehat{\varphi(\mu)}(x) - \widehat{\varphi(\mu_k)}(x)| < \varepsilon/2$ so

$$|\widehat{\varphi(\mu)}(x)| < \varepsilon, \quad \text{for} \quad |x| \leqq M .$$

The conditions of Lemma 1.10 are satisfied so $\varphi = \Phi^*$ for some linear $\Phi: A(F) \to A(E)$. For $\mu \in N_1(E)$ and $f \in A(F)$

$$(\Phi f, \mu) = (f, \varphi(\mu)) .$$

Therefore if $x \in \bigcup_1^\infty E^s$

$$\Phi f(x) = f(\varphi(x)) .$$

Since φ, f and Φf are continuous, Φ is the linear map wanted.

LEMMA B. *Let F_r be a symmetric set with $\Sigma(r(j+1)/r(j))^2 < \infty$. Let $\psi: F_r \to E_m$ be the standard homeomorphism of F_r with some E_m. Then there is an isomorphism into $\Psi: A(E_m) \to A(F_r)$ given by*

$$\Psi(f) = f \circ \psi, \qquad f \in A(E_m) .$$

Proof. There is an l so that $\sum_{l+1}^\infty (r(j+1)/r(j))^2 < 1/24$. F is a union of 2^l sets which are translations of the set $F' = \{x: x = \sum_{l+1}^\infty \varepsilon_j r(j)\}$. It is therefore sufficient to prove the theorem for F'. For convenience, assume F_r has the property $\sum_1^\infty (r(j+1)/r(j))^2 < 1/24$. We shall show as in Lemma A that $\psi: N_1(F_r) \to N_1(E_m)$ has the required continuity properties to be the adjoint of a continuous linear map $\Psi: A(E_m) \to A(F_r)$ satisfying $\Psi(f) = f \circ \psi$.

Using Lemmas 1.6 and 1.10 as in Lemma A, it is enough to show that if a, ε, M are given, then there exists $\delta, x_1, \cdots, x_t$ so that the following holds.

If $\nu \in N_1(F)$, $\|\nu\|_{PM} \leqq a$ and $\hat{\nu}(x_j) < \delta$ for $j = 1, \cdots, t$, then $|\widehat{\psi(\nu)}(\gamma)| < \varepsilon$ for $\gamma \in \Gamma^M$.

Choosing $\nu \in N_1(F)$ with $\|\nu\|_{PM} \leqq a$ and estimating $|\hat{\nu} - \hat{\nu}_k|$ gives

$$\begin{aligned} |\hat{\nu}(x) - \hat{\nu}_k(x)| &\leqq \sum_k^S |\hat{\nu}_{j+1}(x) - \hat{\nu}_j(x)| \\ &\leqq \sum_k^\infty |\exp(-xr(j+1)) - 1| \, \|\nu'_{j+1}\|_{PM} . \end{aligned}$$

Lemma 1.1 and 1.6 show that the PM norm on $N_1(F_r)$ and $N_1(E_{m'})$ are equivalent when $\sum (1/m'(j))^2 < \infty$. Hence

$$\begin{aligned} |\hat{\nu}(x) - \hat{\nu}_k(x)| &\leqq 4\pi x C_1 C \, \|\nu\|_{PM} \sum_{k+1}^\infty r(j) \\ &\leqq 8\pi |x| C_1 C a \cdot r(k+1) . \end{aligned}$$

An easy consequence of the condition $\Sigma(r(j+1)/r(j))^2 < 1/24$ is that

$$\lim_{k \to \infty} 8\pi C_1 . C . a . \left(\sum_1^k 2/r(j)\right) \cdot r(k+1) = 0 .$$

Pick $k \geqq M$ large enough so that

$$8\pi C_1 Ca\Big(\sum_1^k 2/r(j)\Big)r(k+1) < \varepsilon/4C_1 .$$

Then

(1.14) $$|\hat{\nu}(x) - \hat{\nu}_k(x)| < \varepsilon/4C_1$$

for $|x| < \sum_1^k (2/r(j))$. By Lemma 1.6 there is an x_0 with

$$|x_0| < \sum_1^k (2/r(j)$$

so that for $\nu_k \in M(F^k)$

$$\|\nu_k\|_{\mathrm{MAX}}/\|\hat{\nu}_k(x_0)\| < C_1 .$$

By a theorem of Bernstein [1, p. 138]

$$|\hat{\nu}_k(x_0) - \hat{\nu}_k(x_*)| \leqq C_1 |\hat{\nu}_k(x_0)| \Big(\sum_1^\infty r(j)\Big) |x_* - x_0| .$$

Therefore, if $|x_* - x_0| < 1/2(\sum r(j))\cdot C_1$

(1.15) $$\|\nu_k\|_{\mathrm{MAX}}/|\hat{\nu}_k(x_*)| \leqq 2C_1 .$$

Choose for $i = 1, \cdots, t$; x_i with $|x_i| \leqq \sum_1^k (2/r(j))$ so that for every x with $|x| \leqq \sum_1^k (2/r(j))$ there is an x_j with $|x - x_j| < 1/2(\Sigma r(j))\cdot C_1$. If $|\hat{\nu}(x_j)| < \varepsilon/4C_1$ for $x_j, j = 1, \cdots, t$, then $|\hat{\nu}_k(x_j)| < \varepsilon/2C_1$ by (1.14), and by (1.15) $\|\nu_k\|_{\mathrm{MAX}} < \varepsilon$. Consequently, $\|\psi(\nu_k)\|_{PM} < \varepsilon$. Since $k > M$ we see that $|\widehat{\psi(\nu)}(\gamma)| < \varepsilon$ for $\gamma \in \Gamma^M$.

As in Lemma A, the continuity conditions of Lemma 1.10 are satisfied and

$$\bar{\Psi}(f) = f \circ \psi .$$

Theorem 1.11 is an immediate consequence of Lemmas A and B. Meyer [12] has proven that if $\Sigma(r(j+1)/r(j)) < \infty$ and

$$\Sigma(s(j+1)/s(j)) < \infty$$

then $A(F_r) \cong A(F_s)$. Lemmas 1.6 was an analogue and improvement on his main lemma which allowed us to obtain the theorem with square summability.

If $r_0(j) = \{e^{-j}\cdot 2^{-j^2}\}$ then every $A(F_r)$ and $A(E_m)$ with

$$\Sigma(r(j+1)/r(j))^2 < \infty \quad \text{and} \quad \Sigma(1/m(j))^2 < \infty$$

is isomorphic to $A(F_{r_0})$. The isomorphisms are given by

$$f \longrightarrow f \circ \varphi$$

where f is in an appropriate restriction algebra and φ one of the standard homeomorphisms. We shall call an isomorphism between any two restriction algebras induced in this manner a *standard isomorphism.* If $A(F_r)$ or $A(E_m)$ is isomorphic to $A(F_{r_0})$ by standard isomorphisms, F_r or E_m will then be said to belong to the *class* M_y. One should note that for $\mu \in N_1(F_{r_0})$, $\|\mu\|_{PM} = \|\mu\|_{\mathrm{MAX}}$.

Define sets of multiplicity and uniqueness as in [7, p. 52]. In [7, p. 100] it is shown that if $\alpha \in [0, 1/2)$ one can construct sets F_r of multiplicity with $r(j+1)/r(j) = 0(j^{-\alpha})$. The next theorem shows, in particular, that if $r(j+1)/r(j) = 0(j^{-\alpha})$ with $\alpha \in (1/2, \infty)$ then F_r is a set of uniqueness.

THEOREM 1.16. *Suppose that* $\Sigma(r(j+1)/r(j))^2 < \infty$. *Then* F_r *is a set of synthesis and there is a constant* B *so that for all* $S \in N(F_r)$

$$\|S\|_{PM} \leqq B\,\overline{\lim}\,|\hat{S}(x)|\,.$$

Hence F_r *is a set of uniqueness.*

Proof. Choose l so that $\sum_{l+1}^{\infty}(r(j+1)r(j))^2 < 1/24$. Then F is a union of 2^l disjoint sets of the form $a(\varepsilon) + F(l)$ where $\varepsilon = \langle \varepsilon_1, \cdots, \varepsilon_l \rangle$ and $F(l) = \{x: x = \sum_{l-1}^{\infty} \varepsilon_j r(j)\}$. We can find 2^l functions φ_ε in $A(R)$ where $\varphi_\varepsilon = 1$ on $a(\varepsilon) + F(l)$ and 0 on the other sets. Let $S \in PM$ with support in F_r. $S = \Sigma_\varepsilon \varphi_\varepsilon S$ and hence if $\varphi_\varepsilon S \in N(a(\varepsilon) + F(l))$ for each ε, $S \in N(F_r)$. Moreover, for some ε the inequality

$$\|\varphi_\varepsilon S\|_{PM} \geqq 2^{-l}\|S\|_{PM}$$

must hold. If $\|S\|_{PM} > B \lim |\hat{S}(x)|$ we see that

$$\|\varphi_\varepsilon S\|_{PM} \geqq \frac{2^{-l}B}{\|\varphi_\varepsilon\|_A}\,\overline{\lim}\,|\widehat{\varphi_\varepsilon S}(x)|\,.$$

We may therefore assume that $\Sigma(r(j+1)/r(j))^2 < 1/24$.

Lemma 1.6 and [12, Proposition 2.2.3] imply that there is a natural isomorphism T from $A(F_r^k \times [-2r(k+1), 2r(k+1)])$ in $A(R \times R)$ to $A(F_r^k + [-2r(k+1), 2r(k+1)]$ with norm

$$T \leqq (1 - \alpha\, 4r(k+1)\cdot(\Sigma_1^k 1/r(j)))^{-1}$$

and $\|T^{-1}\| = 1$, where $\alpha \leqq 1$ and is independent of k. For large enough k the norm is smaller than some constant B_1. For each $x \in R$ consider the funtion $f_x \in A(F_r^k + [-2r(k+1), 2r(k+1)])$

$$f_x(y) = \exp(xy) - \exp(x\cdot\Sigma_1^k \varepsilon_j r(j)) \quad \text{for} \quad |y - \Sigma_1^k \varepsilon_j r(j)| \leqq 2r(k+1)\,.$$

Its image in $A(F_r^k \times [-2r(k+1), 2r(k+1)]$ is

$$\tilde{f}_x(t, y) = \exp(xt) \cdot (\exp(xy) - 1) .$$

Then

$$\| f_x \|_{A(F_r^k + [\cdot])} \leqq B_1 \| \tilde{f}_x \|_{A(F_r^k \times [\cdot])} \leqq B_2 | x | r(k + 1) .$$

Define $v_k \in M(F_r^k)$ by

$$v_k(\{\Sigma \varepsilon_j r(j)\}) = \widehat{(S|_{\Sigma_1^k \varepsilon_j r(j) + [\cdot]})(0)} .$$

where S is a given element of PM with support in F_r. Then for sufficiently large k

$$| \hat{S}(x) - \hat{v}_k(x) | = | (S, f_x) | \leqq B_2 \cdot | x | \cdot \| S \|_{PM} \cdot r(k + 1) .$$

By Lemma 1.6 we have that

$$\hat{v}_k(x) \longrightarrow \hat{S}(x) \forall x \in R; \lim \| v_k \|_{PM} \leqq C \| S \|_{PM}$$

and hence $S \in N(F_r)$ and F_r is a set of synthesis.

For convenience assume that $\| S \|_{PM} = 1$ and $| \hat{S}(0) | > 1/2$. Suppose that $| \hat{S}(x) | < \varepsilon$ for $x > x_0$. Pick a constant k_0 so that

$$(x_0 + 4 \cdot \Sigma_1^k r(j)) B_2 \| S \|_{PM} \cdot r(k + 1) < \varepsilon$$

for $k > k_0$. Then if $k > k_0$

$$| \hat{v}_k(x) | < 2\varepsilon$$

for all x satisfying $| x - x_* | \leqq \Sigma_1^k (2/r(j))$ where x_* is the center of the interval $[x_0, x_0 + 4\Sigma_1^k (1/r(j))]$. Since $| \hat{v}_k(0) | > 1/2$ Lemma 1.6 shows that

$$\varepsilon > 1/4C_1 .$$

Theorem 1.16 is essentially methods of McGehee and Meyer utilizing Lemma 1.6.

We next examine the sets E_m. By [15, p. 166] they are sets of synthesis. If $m(j) = 2$ for all but a finite number of j, E_m has positive measure and there is an $S \in N(E_m)$ with $\inf_T \sup_{\gamma \in \sim T} | \hat{S}(\gamma) | = 0$. The following is a converse.

THEOREM 1.17. *Let $m(j)$ be a sequence of integers with infinitely many $m(j) \geqq 3$. Then there is a constant C so that for all $S \in N(E_m)$*

$$\| S \|_{PM} \leqq C \inf_T \sup_{\gamma \in \sim T} | \hat{S}(\gamma) |$$

where T is any finite set in $\Sigma Z_{m(j)}$.

Proof. Let $S \in N(E)$ and assume for simplicity that $\| S \|_{PM} = 1$ and $\hat{S}(0) > 3/4$. Let $\{\mu_k\}$ be the measure defined by

$$\mu_k\{x\} = \left(\widehat{S|_{x+\prod_{j=k+1}^{\infty} Z_{m(j)}}}\right)(0)$$

where $x = \langle \varepsilon_1, \cdots, \varepsilon_k, 0, 0, \cdots \rangle$. Let $\gamma^s \varepsilon \sum \Gamma_{m(j)}$ be that element with

$$\gamma_j^s = \begin{cases} 0 & \text{if} \quad j \neq s \\ 1 & \text{if} \quad j = s \end{cases}.$$

Then for $1 \leqq s \leqq k$

$$\begin{aligned} \hat{\mu}_k(\gamma^s) = & \sum_{\varepsilon(s)=0} a(\varepsilon(1), \cdots, \varepsilon(k)) \\ & + \sum_{\varepsilon(s)=1} a(\varepsilon(1), \cdots, \varepsilon(k)) \exp(1/m(s)) . \end{aligned}$$

If we call $\sum_{\varepsilon(s)=0} a(\varepsilon(1), \cdots, \varepsilon(k)) = \alpha$

$$\sum_{\varepsilon(s)=1} a(\varepsilon(1), \cdots, \varepsilon(k)) = \beta \quad \text{then} \quad \hat{\mu}_k(0) = \alpha + \beta .$$

It is easy to see that $\alpha \leqq 1$ and $\beta \leqq 2$. Therefore

$$\begin{aligned} | \hat{\mu}_k(\gamma^s) - \hat{\mu}_k(0) | & \leqq 2 | \exp(1/m(s) - 1) | \\ & \leqq 4\pi/m(s) . \end{aligned}$$

Therefore, if $m(s) > 8\pi$

$$| \hat{\mu}_k(\gamma^s) | > 1/4 .$$

Let $\tilde{\gamma}^s \in \Sigma\Gamma_{m(j)}$ be the element with

$$\tilde{\gamma}_j^s = \begin{cases} 0 & \text{if} \quad j \neq s \\ m(s) - 1 & \text{if} \quad j = s \end{cases}.$$

Then

$$\hat{\mu}_k(\tilde{\gamma}^s) = \alpha + \beta \exp(-1/m(s))$$

and hence

$$| \hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s) | = 2\beta \sin(2\pi/m(s)) .$$

If $3 \leqq m(s) < 8\pi$ and $| \hat{\mu}_k(\gamma^s) | < (1/100)$ then $\beta > (1/3)$ and

$$| \hat{\mu}_k(\gamma^s) - \hat{\mu}_k(\tilde{\gamma}^s) | > 1/50$$

and hence $| \hat{\mu}_k(\tilde{\gamma}^s) | > 1/50$. Therefore we may conclude that for all k either $| \hat{\mu}_k(\gamma^s) |$ or $| \hat{\mu}_k(\tilde{\gamma}^s) |$ is greater than 1/100 provided $m(s) \geqq 3$.

On Γ^k, $\hat{\mu}_k$ and $\hat{S}$ are identical. Suppose there is a t so that

(1.19) $$|\hat{S}(\gamma)| < 1/200$$

for $\gamma \notin \Gamma^t$. Pick a $k > t$ so that there is an s with $k > s > t$ for which $m(s) \geqq 3$. Then either $|\hat{\mu}_k(\tilde{\gamma}^s)|$ or $|\hat{\mu}_k(\tilde{\gamma}^s)|$ is greater than 1/100. Hence $|\hat{S}(\gamma^s)|$ or $|\hat{S}(\tilde{\gamma}^s)|$ is greater than 1/100 contradicting (1.19).

2. In this section we shall exhibit sets E_m, F_r that do not have $A(E_m)$ or $A(F_r)$ isomorphic to $A(F_{r_0})$ by standard isomorphisms. They are then not in the class M_y.

The first theorem is a converse to Lemma A.

THEOREM 2.1. *If $\Sigma(1/m(j))^2 = \infty$, then E_m is not an element of the class M_y.*

Proof. It is sufficient to show that

$$\sup_{\mu \in N(E)} \|\mu\|_{\mathrm{MAX}}/\|\mu\|_{PM} = \infty$$

since for $\nu \in N_1(F_{r_0})$ $\|\nu\|_{PM} = \|\nu\|_{\mathrm{MAX}}$. For each integer s, let $x^s \in \Pi Z_{m(j)}$ be that element with $x^s_j = \delta^s_j$. Let α_s be the two point measure

$$\alpha_s\{x^s\} = \exp(1/3m(s)) .$$

For each k, define an element μ_k of $M(E^k)$ by

$$\mu_k = \alpha_1 * \cdots * \alpha_k .$$

we see that

$$\|\mu_k\|_{\mathrm{MAX}} = 2^k$$

while

$$\|\mu_k\|_{PM} = \sup_{\xi_s}\left|\prod_{s=1}^{k}(1 + \exp(1/(3m(s)))\cdot\xi_s\right|,$$

where the ξ_s are $m(s)$ roots of unity. Since

$$|1 + \exp(1/3m(s))| \geqq |1 + \exp(1/3m(s))\xi_s|$$

for ξ_s any $m(s)$ root of unity, and since $\cos(\theta) < 1 - \theta^2/4$ for $\theta < 1$

$$\begin{aligned}\|\mu_k\|_{PM} &= 2^k \prod_{s=1}^{k} \cos(\pi/3m(s)) \\ &\leqq 2^k \prod_{s=1}^{k}(1 - (1/3m(s))^2) .\end{aligned}$$

Therefore

$$\|\mu_k\|_{\mathrm{MAX}}/\|\mu_k\|_{PM} \geqq 1/\prod_{s=1}^{k}(1 - (1/3m(s))^2)$$

and since $\Sigma(1/m(s))^2 = \infty$, $\|\mu_k\|_{\mathrm{MAX}}/\|\mu_k\|_{PM} \to \infty$ as $k \to \infty$.

We have actually shown more than claimed in Theorem 2.1. The proof shows that if $\{r(j)\}$ is any independent sequence and $\Sigma(1/m(j))^2 = \infty$, then $A(E_m)$ is not isomorphic to $A(F_r)$ by a standard isomorphism.

The next theorem will imply that no condition on the convergence of $(r(j+1)/r(j))$ weaker than

$$\Sigma(r(j+1)/r(j))^2 < \infty ,$$

is sufficient for a set F_r to be a member of the class M_y.

THEOREM 2.2. *Suppose that n_j is an increasing sequence of integers. Let $b \geqq 2$ be an integer and put $r(j) = b^{-n_j}$. If*

$$\Sigma(r(j+1)/r(j))^2 = \infty$$

then F_r is not an element of the class M_y.

Proof. Let us assume for convenience that $\Sigma_1^\infty(r(2j)/r(2j-1))^2 = \infty$ and $b = 10$. We can also assume our set F to be on the circle. For any integer j define the two point measure γ_j by

$$\gamma_j\{0\} = 1$$
$$\gamma_j\{r(j)\} = \exp\left(-\frac{1}{2}\right).$$

For each k, define an element ν_k of $M(F^k)$ by

$$\nu_k = \gamma_1 * \cdots * \gamma_k .$$

Then for any integer s

$$|\hat{\nu}_{2k}(s)| = 2^{2k}\left|\prod_1^{2k} \cos\left(\pi\left(s\cdot 10^{-n_j} - \frac{1}{2}\right)\right)\right| .$$

In this product, consider terms $\delta_j(s)$ of the form

$$\left|\cos\left(\pi\left(s\cdot 10^{-n_{2j-1}} - \frac{1}{2}\right)\right)\cdot\cos\left(\pi\left(s\cdot 10^{-n_{2j}} - \frac{1}{2}\right)\right)\right| .$$

If

$$\left|s\cdot 10^{-n_{2j-1}} - \frac{1}{2}\right| < 1/10 \bmod 1 ,$$

then

$$\left|s\cdot 10^{-n_{2j}} - \frac{1}{2}\right| \geqq \frac{1}{10}\cdot(10^{n_{2j-1}}/10^{n_{2j}}) \bmod 1.$$

Then

$$\begin{aligned}|\hat{\nu}_{2k}(s)| &= 2^{2k}\prod_{j=1}^{k}|\delta_j(s)| \\ &\leqq 2^{2k}\prod_{j=1}^{k}(1 - D\cdot(10^{n_{2j-1}}/10^{n_{2j}})^2)\ ,\end{aligned}$$

where D is an absolute constant. Therefore

$$\|\nu_{2k}\|_{PM} \leqq 2^{2k}\prod_{j=1}^{k}(1 - D(r(2j)/r(2j-1))^2)\ .$$

However, $\|\nu_{2k}\|_{\text{MAX}} = 2^{2k}$, so

$$\|\nu_{2k}\|_{\text{MAX}}/\|\nu_{2k}\|_{PM} \geqq \left|/\prod_{j=1}^{k}(1 - D(r(2j)/r(2j-1))^2)\right|\ .$$

Therefore $\|\nu_{2k}\|_{\text{MAX}}/\|\nu_{2k}\|_{PM} \longrightarrow \infty$ as $k \longrightarrow \infty$. Hence F_r is not a member of the class M_y. The proof with $b \neq 10$ is completely analogous to the proof with $b = 10$.

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