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MEASURE ALGEBRAS ON IDEMPOTENT SEMIGROUPS

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Taylor has shown that for every commutative convolution measure algebra M there is a compact topological semigroup S, called the structure semigroup of M, and an embedding $\mu \to \mu_S$ of M into $M(S)$ such that every complex homomorphism of M has the form $h_f(\mu) = \int_S f d\mu_s$ for some semicharacter f on S.

This paper deals with commutative convolution measure algebras whose structure semigroups are idempotent. The measure algebra on the interval $[0, 1]$, where the interval is given the semigroup operation of maximum multiplication, is an algebra of this type. These algebras are studied in this general setting in the hope of shedding new light on the known theory of measure algebras on locally compact idempotent semigroups and in the hope of extending attempts to classify a convolution measure algebra in terms of the algebraic nature of its structure semigroup.

An example is given of a measure algebra on a compact idempotent semigroup whose structure semigroup is not idempotent.

Our goal in this paper is to apply the structure theory for commutative convolution measure algebras developed by Taylor [5] to a special class of algebras which includes those studied by Hewitt and Zuckerman [3], Ross [4], and Baartz [1]. We will assume that each convolution measure algebra mentioned in this paper is commutative and, in addition, that each semigroup mentioned is commutative. We begin by giving the essential features of Taylor's structure theory.

A convolution measure algebra is roughly an ordered Banach space of measures with a multiplication which makes it a Banach algebra and which relates appropriately to the norm and the order. For a precise definition, see [5]. Examples include $L^1(G)$, the algebra of all absolutely continuous measures on a locally compact group G; $M(G)$, the measure algebra on G (all bounded regular Borel measures on G); and $M(S)$, the measure algebra on a locally compact semigroup S. Convolution is the multiplication operation in each of these examples. Both $L^1(G)$ and $M(G)$ are semisimple algebras as is $M(S)$ under certain not-too-restrictive conditions. We will therefore focus our attention only upon semisimple algebras.

Let M denote a semisimple convolution measure algebra. Taylor has shown in [5] that there is a compact topological semigroup S, called the structure semigroup of M, and an embedding $\mu \to \mu_S$ of M

into $M(S)$ with the following properties.

(A) $\mu \rightarrow \mu_S$ is an algebraic isomorphism and an order preserving isometry.

(B) The image M_S of the map $\mu \rightarrow \mu_S$ is weak* dense in $M(S)$; i.e., M_S separates points in $C(S)$.

(C) $C(S)$ is the closed linear span of $\hat{S}$; i.e., $\hat{S}$ separates points of S. ($\hat{S}$ is the collection of all continuous semicharacters on S),

(D) Each complex homomorphism of M has the form $h_f(\mu) = \int_S f d\mu_s$ for some f in $\hat{S}$.

The reader will recall that a semicharacter is a nonzero, bounded, complex valued function f defined on the semigroup S which satisfies $f(x \cdot y) = f(x)f(y)$ for all x and y in S. As a result of (D), the set $\hat{S}$ of semicharacters with the weak* topology induced by M can be considered the maximal ideal space of M. We will regard a semicharacter f in S as both a continuous function on S and a complex homomorphism of M via the indentification given by (D), Thus, we write $f(\mu)$ in place of $h_f(\mu)$.

We are now in a position to define the type of algebra which will be our object of study.

DEFINITION 1. A semisimple convolution measure algebra M will be called a P-algebra provided $f(\mu) \geqq 0$ for every positive measure μ in M and every complex homomorphism f of M.

Examples of P-algebras are the measure algebra $M(T)$, under convolution, of the compact semigroup $T = [a, b]$ with multiplication $x \cdot y = \max\{x, y\}$ [3], and more generally, the measure algebra $M(T)$ of a finite product T of locally compact, totally ordered spaces with co-ordinatewise maximum multiplication [1]. In both examples, each complex homomorphism of $M(T)$ has the form

$$h_A(\mu) = \int_T \chi_A d\mu \qquad \mu \in M(T) ,$$

for some subsemigroup A of T whose complement $T \backslash A$ is a (prime) ideal of T (Definition 1.5, [1]). Consequently, $M(T)$ is a P-algebra. In § 4 we will give an alternate proof the $M(T)$ is a P-algebra, based on the results of § 3.

We pause to define several terms with which the reader may not be familiar. The reader is referred to Taylor [5] for terms not defined here.

Let M be a convolution measure algebra.

DEFINITION 2. A closed subspace (subalgebra, ideal) N of M is called an L-subspace (subalgebra, ideal) if whenever $\mu \in N$, then $\nu \in N$ for all $\nu < \mu$(ν absolutely continuous with respect to μ).

DEFINITION 3. An L-ideal N of M is called a prime L-ideal if $N^{\perp} = \{\mu \in \mu \mid \mu \perp \nu$ (μ and ν are mutually singular) for all $\nu \in N\}$ is a subalgebra of M.

2. Some characterizations of P-algebras. Our first theorem gives six equivalent conditions for a semisimple convolution measure algebra M to be a P-algebra. The identity e mentioned in statements (5) and (6) of the theorem is the identity in M if M has an identity and is the identity adjoined to M in the usual manner if M does not have an identity. Similarly, the inversion mentioned in statement (6) takes place in the algebra M if M has an identity, and in the algebra "M with identity adjoined" if M does not have an identity.

THEOREM 1. *Let M be a semisimple convolution measure algebra. Then the following statements are equivalent.*

(1) *M is a P-algebra.*

(2) *$\hat{S}$ is an idempotent semigroup.*

(3) *S is an idempotent semigroup.*

(4) *For each $f \in \hat{S}$, $M = N_f + N_f^{\perp}$ where N_f is a prime L-ideal such that if $\mu = \mu_1 + \mu_2(\mu_1 \in N_f, \mu_2 \in N_f^{\perp})$, then $f(\mu) = (\mu_2)_S(S)$.*

(5) *The spectral radius of $\mu - e$ is less than or equal to one for every positive measure μ of norm one.*

(6) *$\mu + e$ is invertible for every positive measure μ of norm one.*

Proof. The order of proof will be (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1). (1) $\Rightarrow$ (2). Since the integral of each semicharacter in $\hat{S}$ with respect to any positive measure in M is nonnegative, each semicharacter in S is a nonnegative function by (B). Let f be in $\hat{S}$. Then f is nonnegative and hence for fixed z, f^z is in $\hat{S}$ if $\operatorname{Re} z > 0$. Now $g_s(z) = f^z(s)$ is analytic in $\operatorname{Re} z > 0$ for fixed $s \in S$. But then $g_s(z)$ is a nonnegative analytic function and is therefore constant. If we evaluate $g_s(z)$ at $z = 1$, we obtain $f^z(s) = f(s)$ for all $\operatorname{Re} z > 0$. If we let $z = 2$ in the above equality, we obtain $f^2(s) = f(s)$ for each $s \in S$ and hence $f^2 = f$. Therefore $\hat{S}$ is an idempotent semigroup.

(2) $\Leftrightarrow$ (3). Let $f \in \hat{S}$ and $s \in S$. We conclude that $f(s) = f^2(s) = f(s)f(s) = f(s \cdot s)$ since $f^2 = f$. Thus $s \cdot s = s$ by (C), and hence S is idempotent. Obviously, (3) $\Rightarrow$ (2).

(3) $\Rightarrow$ (4). If S is idempotent, so is $\hat{S}$. Given $f \in \hat{S}$, let $J = \{s \in S \mid f(s) = 0\}$, We note that J is a prime ideal in S. If we let $N_f = \{\mu \in M \mid \mu_S$ is concentrated on $J\}$, then N_f is a prime L-ideal

(Theorem 3.2, [5]) with orthogonal complement $N_f^\perp$. Thus if $\mu = \mu_1 + \mu_2$ where $\mu_1 \in N_f$ and $\mu_2 \in N_f^\perp$, then

$$f(\mu) = \int_S f d\mu_s = \int_S fd(\mu_1 + \mu_2)_s = \int_J fd(\mu_1)_s + \int_S fd(u_2)_s = (\mu_2)_s(S) .$$

(4) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (5). Recall that the spectral radius of an element χ is a Banach algebra (written $\|\chi\|_{sp}$) is given by $\|\chi\|_{sp} = \lim_{n\to\infty} \|\chi^n\|^{1/n}$. If the algebra is commutative, then the spectral radius of χ is also the supremum norm of the Gelfand transform of χ. If (1) holds, then, by definition, each complex homomorphism takes positive measures to nonnegative numbers. Let μ be a positive measure in M of norm one. Then $0 \leqq f(\mu) = \hat{\mu}(f) \leqq 1$ for every $f \in \hat{S}$. Therefore,

$$1 \geqq \sup_{f \in \hat{S}} |(\mu - e)^\wedge(f)| = \|(\mu - e)^\wedge\|_\infty = \|\mu - e\|_{sp} .$$

(5) $\Rightarrow$ (6). Let μ be a positive measure in M. Then $\mu/\|\mu\|$ is a positive measure of norm one and hence

$$\|(\mu/\|\mu\| - e)^\wedge\|_\infty = \|\mu/\|\mu\| - e\|_{sp} \leqq 1 .$$

Clearly $\hat{\mu}$ can never assume the value -1. Thus $-1 \notin \sigma(\mu)$, and hence $\mu + e$ is invertible.

(6) $\Rightarrow$ (1). Let μ be a positive measure in M. If $\lambda > 0$, then $\mu/\lambda + e$ is invertible. Hence $\mu + \lambda e$ is invertible and $-\lambda \notin \sigma(\mu)$. Therefore, the spectrum of any positive measure in M contains no negative members. We claim that this fact ensures us that every positive measure will have real, nonnegative spectrum. For suppose μ is a positive measure whose spectrum is not real. Then there is an $f \in \hat{S}$ such that $\hat{\mu}(f) = \lambda = \lambda_1 + i\lambda_2$ where λ_1 and λ_2 are real and $\lambda_2 \neq 0$. We can choose a number $t > 0$ such that $\exp(t\lambda) < 0$. Thus $\exp(t\mu)$ is a positive measure with a negative number in its spectrum, a contradiction. The proof of the theorem is complete.

An L-subalgebra of a convolution measure algebra is again a convolution measure algebra. Since the spectral radius of a measure depends only upon the norms of the measure and its powers, statement (5) together with the above observation yields the following corollary.

COROLLARY. *Every L-subalgebra of a P-algebra is a P-algebra.*

3. A sufficient condition.

Our next theorem gives a sufficient condition for an algebra to be a P-algebra. We suspect that the condition is also necessary but have not been able to prove it.

THEOREM 2. *Let M be a semisimple convolution measure algebra* *with structure semigroup S. If for every positive (nonzero) measure μ in M there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ of positive measures in M such that*

(1) $\mu_n \to \mu$,

(2) $\mu_n * \nu' = \mu_n(S)\nu'$ *for all* $\nu' < \nu_n$ *and each* n,

(3) $\nu_n < \sum_{m=0}^{\infty} \mu^m/2^m$ *for all* n

then M is a P-algebra.

Proof. Throughout this proof, M is considered a subalgebra of $M(S)$. Let λ be a positive measure in M, let f be in $\hat{S}$, and let $J = \{s \in S \mid f(s) = 0\}$. Then J is a prime ideal in S, and hence

$$M_J = \{\nu \in M \mid \nu \text{ is concentrated on } J\}$$

is a prime L-ideal of M with orthogonal complement $M_J^{\perp}$. Define μ, μ' by $\mu'(E) = \lambda(E \cap J)$, for Borel sets E, and $\mu = \lambda - \mu'$. Then $\mu \in M_J$ and $\mu' \in M_J^{\perp}$.

If $\mu = 0$, then $f(\lambda) = 0$. If $\mu \neq 0$, then choose sequences $\{\mu_n\}$ and $\{\nu_n\}$ guaranteed by the hypothesis of the theorem. We claim that there is a measure $\nu_n' < \nu_n$ such that $f(\nu_n') \neq 0$ for each n. If not, then for some n, $f(\nu_n') = \int f d\nu_n' = 0$ for all $\nu_n' < \nu_n$ and $f = 0$ a.e. $[\nu_n]$. Thus there is a Borel set $E \subset S$ of ν_n-measure zero such that $f = 0$ on E. Hence $\nu_n \in M_J$. But since $\mu \in M_J^{\perp}$ and $M_J^{\perp}$ is an L-subalgebra, $\sum_{m=0}^{\infty} \mu^m/2^m$ is in $M_J^{\perp}$. Since $\nu_n < \sum_{m=0}^{\infty} \mu^m/2^m$, ν_n is in $M_J^{\perp}$. Therefore, $\nu_n \perp \nu_n$ and so $\nu_n = 0$, a contradiction. This establishes our claim.

Choose measures $\nu_n' < \nu_n$ such that $f(\nu_n') \neq 0$. Note that since $\mu_n * \nu_n' = \mu_n(S)\nu_n'$ and $f(\nu_n') \neq 0$, then, $f(\mu_n) = \mu_n(S) \geqq 0$. But $\mu_n \to \mu$; thus, $f(\mu_n) \to f(\mu)$. Therefore, $f(\mu) = \mu(S) \geqq 0$ and $f(\lambda) = f(\mu + \mu') = f(\mu) + f(\mu') = f(\mu) \geqq 0$. Hence M is a P-algebra.

4. An application of Theorem 2. If T is an idempotent semigroup we can introduce a partial ordering "$\leqq$" in T by defining $x \leqq y$ if and only if $x \cdot y = y$ for all x any y in T. A totally ordered idempotent semigroup is one in which the above partial ordering is a total ordering. Our goal in this section is to show that the measure algebra on a finite product of totally ordered, locally compact, idempotent semigroups is a P-algebra. This result follows trivially from a theorem of Baartz (Theorem 3.5, [1]); however, we shall give an independent development using Theorem 2. We will need the three lemmas that follow.

LEMMA 1. *Let T be a locally compact idempotent semigroup and let μ and ν be in $M(T)$. Suppose* $\operatorname{supp} \mu \leqq \operatorname{supp} \nu$ *in the sense that* *for any* $s \in \operatorname{supp} \mu$ *and* $t \in \operatorname{supp} \nu$, $s \leqq t (s \cdot t = t)$. *Then* $\mu * \nu = \mu(T)\nu$.

Proof. Supp μ denotes the support of the measure μ. Let $A = \operatorname{supp} \mu$, $B = \operatorname{supp} \nu$, and E be a Borel subset of T. Then

$$\mu * \nu(E) = \iint \chi_E(x \cdot y) d\mu(x) d\nu(y) = \int \mu(E_y) d\nu(y) \text{ where } E_y = \{x \in T \mid x \cdot y \in E\} .$$

But $\int_T \mu(E_y) d\nu(y) = \int_B \mu(E_y) d\nu(y)$ since ν is concentrated on B. Furthermore,

$$\mu(E_y) = \begin{cases} 0 & y \in B \backslash E \\ \mu(T) & y \in B \cap E \end{cases}$$

since for $y \in B \cap E$, $A \subset E_y$ and for $y \in B \backslash E$, $A \cap E_y = \phi$. Thus

$$\int_B \mu(E_y) d\nu(y) = \mu(T)\nu(E)$$

and hence $\mu * \nu = \mu(T)\nu$.

LEMMA 2. *Let T be a totally ordered, locally compact, idempotent semigroup and let μ be a positive measure in $M(T)$. Then given $\varepsilon > 0$, there is an $x \in T$ such that $\mu(\{y \in T \mid y \geqq x\}) > 0$ and $\mu(\{y \in T \mid y > x\}) < \varepsilon$.*

Proof. Since μ is a bounded regular measure, there is a compact set $K \subset \operatorname{supp} \mu$ such that $\mu(T \backslash K) < \varepsilon$. Let $z = \sup \{y \mid y \in K\}$. If $\mu(\{y \mid y \geqq z\}) > 0$, then the choice of $x = z$ completes the proof. If $\mu(\{y \mid y \geqq z\}) = 0$, we again apply the regularity of μ to obtain an $x < z$ such that $\mu(\{y \mid y > x\}) < \varepsilon$. The choice of z forces $\mu(\{y \mid y \geqq x\}) > 0$. The proof of the lemma is complete.

LEMMA 3. *Let $T = P_{i=1}^m T_i$ be a finite product of totally ordered, locally compact, idempotent semigroups T_i. Let μ be a positive measure in $M(T)$. Then given $\varepsilon > 0$, there is an $x = (x_1, \cdots x_m)$ in T such that $\mu^m(\{y \in T \mid y \geqq x\}) > 0$ and $\mu(T \backslash \{y \in T \mid y \leqq x\}) < \varepsilon$.*

Proof. Let π_i be the projection map of T onto T_i for

$$i = 1, 2, \cdots m .$$

The measure $\mu_i = \mu \circ \pi_i^{-1}$ is a positive measure in $M(T_i)$, Therefore, by Lemma 2, there is an x_i in T_i such that $\mu_i(\{y \in T_i \mid y > x_i\}) < \varepsilon/m$ and $\mu_i(\{y \in T_i \mid y \geqq x_i\}) > 0$. For notational convenience, let

$$J_i = \{y \in T_i \mid y > x_i\} \quad \text{and} \quad K_i = \{y \in T_i \mid y \geqq x_i\} \ .$$

Then the above statement becomes $\mu_i(J_i) < \varepsilon/m$ and $\mu_i(K_i) > 0$.

Let $x = (x_1, x_2, \cdots x_m)$. We first note that

$$T \backslash \{y \in T \mid y \leqq x = \bigcup_{i=1}^{m} \pi_i^{-1}(J_i) \ .$$

Thus $\mu(T\backslash\{y \in T \mid y < x\}) = \mu(\cup_{i=1}^{m} \pi_i^{-1}(J_i)) \leqq \sum_{i=1}^{m} \mu \circ \pi_i^{-1}(J_i) < \sum_{i=1}^{m} \varepsilon/m = \varepsilon$. We next note that

$$\pi_1^{-1}(K_1)\cdot\pi_2^{-1}(K_2) \cdots \pi_m^{-1}(K_m) = \bigcap_{i=1}^{m} \pi_i^{-1}(K_i) = \{y \in T \mid y \geqq x\} \ .$$

Since μ has mass on each of the sets $\pi_i^{-1}(K_i)$, μ^m has mass on

$$\pi_1^{-1}(K_1)\cdot\pi_2^{-1}(K_2) \cdots \pi_m^{-1}(K_m); \text{ i.e., } \mu^m(\{y \in T \mid y \geqq x\}) > 0 \ .$$

This establishes the lemma.

THEOREM 3. *Let $T = P_{i=1}^{m} T_i$ be a finite product of totally ordered, locally compact, idempotent semigroups T_i. Then the measure algebra $M(T)$ is a P-algebra.*

Proof. Let μ be a positive measure in $M(T)$. Lemma 3 guarantees the existence of a sequence $\{x_n\}_{n=1}^{\infty}$ in T such that

$$\mu(T\backslash\{y \in T \mid y \leqq x_n\} < 1/n$$

and $\mu^m(\{y \in T \mid y \geqq x_n\} > 0$.

Let $\mu_n = \mu \mid \{y \in T \mid y \leqq x_n\}$ and let $\nu_n = \mu^m \mid \{y \in T \mid y \geqq x_n\}$. Here we denote the measure μ restricted to a set A by $\mu \mid A(\mu \mid A(E) = \mu(A \cap E)$ for any Borel set E). We claim that the sequences $\{\mu_n\}$ and $\{\nu_n\}$ satisfy the hypothesis of Theorem 2. Clearly, $\mu_n \to \mu$. Since $\operatorname{supp} \mu_n \leqq \operatorname{supp} \nu_n$, and since $\operatorname{supp} \nu_n' \subset \operatorname{supp} \nu_n$ for any $\nu_n' < \nu_n$, Lemma 1 assures us that $\mu_n * \nu_n' = \mu_n(T)\nu_n'$. Finally, ν_n is a nonzero measure such that $\nu_n < \mu^m < \sum_{k=0}^{\infty} \mu^k/2^k$. Since $M(T)$ is semisimple [1], $M(T)$ is a P-algebra by Theorem 2.

5. A counterexample. Theorem 1 shows that any P-algebra may be considered as an L-subalgebra of the measure algebra on a compact idempotent semigroup. Each of the examples given in § 1 is a measure algebra on a locally compact idempotent semigroup (not the structure semigroup). It is therefore natural to ask whether or not the measure algebra on *any* locally compact, idempotent semigroup is a P-algebra. The answer to this question is "no" as the counterexample of this section will show. We first make the following definition.

DEFINITION 4. A subset Q of an idempotent semigroup S will be called independent if whenever $x_1 \cdot x_2 \cdots x_n = y_1 \cdot y_2 \cdots y_m$ for $\{x_i\}_{i=1}^n \cup \{y_j\}_{j=1}^m \subset Q$ and $m < n$, then $x_i = y_j$ for some i and j ($1 \leqq i \leqq n$ and $1 \leqq j \leqq m$).

Let C denote the Cantor set on the interval $[0, 1]$. Let S denote the collection of all finite subsets of C and let union be the semigroup operation in S. Note that the one point sets form an independent subset of S in the sense of Definition 4. For an open-compact subset $U \subset C$, $X \in S$, define

$$\chi_U(X) = \begin{cases} 1 & \text{if} \quad X \subset U \\ 0 & \text{if} \quad X \not\subset U . \end{cases}$$

Give S the weak topology generated by the functions $\{\chi_U\}$ (U open-compact). Observe that each χ_U is a continuous semicharacter on S. Let V be a countable open-compact base for C and let $\tilde{V} = \{\tilde{U} \mid U \in V\}$. Finally, let $\{U_i\}_{i=1}^\infty = V \cup \tilde{V}$ and note that the family $\{\chi_{U_i}\}_{i=1}^\infty$ separates points in S.

Let T be the countable topological product of the two-point semigroup $\{0, 1\}$, under multiplication. Thus T is a compact idempotent semigroup. We now define a map $\alpha: S \to T$ by $[\alpha(X)]_i = \chi_{U_i}(X)$ for any $X \in S$. Note that α is a continuous one-to-one homomorphism from S into T. We further observe that C is embedded in an obvious way in S, and hence in T, as an independent set.

The concluding argument is similar to the one given in the Hewitt-Kakutani paper on $M(G)$ [2]. There is a positive continuous measure μ of norm one concentrated on C. Using Fubini's theorem and the fact that C is independent, it can be shown that μ and all its powers are mutually singular [2]. Now let $\sigma = \delta_e - \mu$ (e is the identity in T). Then $\| \sigma^n \| = \| \sum_{k=0}^n C_{n,k}(-1)^k \mu^k \| = \sum_{k=0}^n C_{n,k} = 2^n$. Hence

$$\| \hat{\sigma} \|_\infty = \lim_{n \to \infty} \| \sigma^n \|^{1/n} = 2 .$$

Thus there is a complex homomorphism h of $M(T)$ such that $| h(\sigma) | = 2$. This forces $h(\mu) = -1$. Therefore, $M(T)$ is not a P-algebra.

The countable product of the two point semigroup, with the operation of coordinatewise multiplication, is a sub-semigroup of the countable product of unit intervals, with the operation of coordinatewise minimum. Thus although the measure algebra on a finite product of intervals with coordinatewise maximum multiplication is a P-algebra, this is not the case for an infinite product of intervals. We are therefore led to conjecture that a measure algebra $M(T)$ is a P-algebra if and only if T is an idempotent semigroup which satisfies a certain "finite dimensionality" condition.

BIBLIOGRAPHY

1. A. P. Baartz, *The measure algebra of a locally compact semigroup*, Pacific J. Math. **21** (1967), 199-213.
2. E. Hewitt and S. Kakutani, *A class of multiplicative linear functionals on the measure algebra of a locally compact abelian group*, Illinois J. Math. **4** (1960), 553-574.
3. E. Hewitt and H. S. Zuckerman, *Structure theory for a class of convolution algebras*, Pacific J. Math. **7** (1957), 913-941.
4. K. A. Ross, *The structure of certain measure algebras*, Pacific J. Math. **11** (1961), 723-736.
5. J. L. Taylor, *The structure of convolution measure algebras*, Trans. Amer. Math. Soc. **119** (1965), 150-166.

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