PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 1, 1969

PROOF OF A CONJECTURE OF WHITNEY

W. S. MASSEY

Let M be a closed, connected, nonorientable surface of Euler characteristic χ which is smoothly embedded in Euclidean 4-space, R^4, with normal bundle ν. The Euler class of ν, denoted by $e(\nu)$, is an element of the cohomology group $H^2(M; \mathscr{Z})$ (the letter $\mathscr{Z}$ denotes twisted integer coefficients). Since the group $H^2(M; \mathscr{Z})$ is infinite cyclic, $e(\nu)$, is m times a generator for some integer m. In a paper presented to a Topology Conference held at the University of Michigan in 1940, H. Whitney studied the possible values that this integer m could take on for different embeddings of the given surface M. He gave examples to show that m can be nonzero (unlike the case for an orientable manifold embedded in Euclidean space) and proved that[1]

$$m \equiv 2\chi \pmod 4) .$$

Finally, he conjectured that m could only take on the following values:

$$2\chi - 4, 2\chi, 2\chi + 4, \cdots, 4 - 2\chi .$$

It is the purpose of the present paper to give a proof of this conjecture of Whitney. The proof depends on a corollary of the Atiyah-Singer index theorem.

This corollary is concerned with manifolds with an orientation preserving involution; an elementary proof of the corollary has recently been given by K. Jänich and E. Ossa, [5].

The author is grateful to G. Bredon and W. Browder for helpful discussions of the Atiyah-Singer theorem.

The precise statement of the theorem which was conjectured by Whitney is contained in the next section. In order to remove the ambiguity in the sign of the integer m, it is necessary to give a rather thorough discussion of some basic notions regarding questions of orientation, local coefficient systems, etc. Although this material is more or less known, it is nowhere published in a form convenient for our purposes; hence it has been relegated to the appendix of this paper.

2. Precise statement of the theorem. We will assume that M is a closed, connected, nonorientable surface which is embedded smoothly

[1] This result of Whitney was generalized by M. Mahowald in 1964. For a proof of Mahowald's theorem, see a recent paper of the author entitled "Pontryagin squares in the Thom space of a bundle" (Pacific J. Math.).

in the 4-sphere, S^4 (the one point compactification of R^4), and that S^4 has been given a definite orientation. Let ν denote the normal bundle of this embedding; by the Whitney duality theorem, we have equality of Stiefel-Whitney classes,

$$w_1(\nu) = w_1(M) \ .$$

Let $\mathscr{Z}$ denote the local system of integers on M with twisting determined by $w_1(\nu) = w_1(M)$. The local systems of orientations $O(\nu)$ and $O(M)$ (see Appendix 1) are both isomorphic to $\mathscr{Z}$, and in each case the isomorphism may be chosen in two different ways. Note that if $\tau(M)$ denotes the tangent bundle to M, we have

$$\nu \oplus \tau(M) = \tau'$$

where τ' denotes the restriction of the tangent bundle of S^4 to M. Therefore we have a natural isomorphism

$$(*) \qquad O(\nu) \otimes O(\tau(M)) \approx O(\tau') \ .$$

The choosing of an orientation of S^4 determines an isomorphism of $O(\tau')$ with the group of integers, Z. Assume that one also chooses isomorphism

$$O(\nu) \approx \mathscr{Z} \quad \text{and} \quad O(\tau(M)) \approx \mathscr{Z} \ .$$

Then the equation $(*)$ becomes

$$(**) \qquad \mathscr{Z} \otimes \approx \mathscr{Z}\, Z \ .$$

We will consistently assume that the isomorphisms $O(\nu) \approx \mathscr{Z}$ and $O(\tau(M)) \approx \mathscr{Z}$ are chosen so that at each point of M the isomorphism of $(**)$ *is that determined by ordinary multiplication of integers.* This implies that the choice of the isomorphism $O(\nu) \approx \mathscr{Z}$ determines the choice of $O(\tau(M)) \approx \mathscr{Z}$ and conversely. It also implies that $e(\nu)[M]$ (the Euler class of ν evaluated on the fundamental class of M) is a positive or negative integer whose sign is determined by the orientation of S^4. With these conventions, we can state our main theorem;

THEOREM. *Let M be a closed, connected, nonorientable surface of Euler Characteristic χ which is smoothly embedded in the oriented 4-sphere, S^4. Then the integer $e(\nu)[M]$ has one of the following values:*

$$2\chi - 4, 2\chi, 2\chi + 4, \cdots, 4 - 2\chi \ .$$

Moreover, any of these possible values can be attained by an appropriate embedding of M in S^4.

REMARK. This theorem is actually true if S^4 is an oriented homology sphere; it is not necessary to assume that it is simply connected.

The rest of the paper is organized as follows: Section 3 contains an outline of the proof. The more tedious details are relegated to lemmas which are proved in §4 and 5. In §6 we prove the statement contained in the last sentence of the theorem; this part of the proof is completely independent of the rest.

3. Outline of the proof. We are assuming the surface M is smoothly embedded in the oriented 4-sphere, S^4. By the Alexander duality theorem,

$$H_1(S^4 - M; Z) \approx H^2(M, Z) = Z_2 .$$

Hence the space $S^4 - M$ has a unique 2-sheeted covering space, namely, that which corresponds to the commutator subgroup of the fundamental group $\pi_1(S^4 - M)$. This covering space can be "completed" to a branched covering space

$$p: S' \longrightarrow S^4$$

with M as the set of branch points (for the theory of branched covering spaces, see R. H. Fox, [2]. We orient S' so that its orientation agrees with that of S^4 under the map p. For the sake of convenience, we will identify M and $p^{-1}(M)$ by means of the map p. Note that S' is a 4-dimensional compact orientable manifold; we denote by

$$T: S' \longrightarrow S'$$

the obvious involution of S' which interchanges the two sheets of the covering. T is an orientation preserving smooth involution and its fixed point set is precisely the surface M.

We will denote by ν' the normal bundle of the imbedding of M in S'. Let $S(\nu)$ and $S(\nu')$ denote the associated 1-sphere bundles of the bundles ν and ν' respectively; $S(\nu)$ and $S(\nu')$ can be realized as the boundaries of smooth tubular neighborhoods of M in S^4 and S' respectively. The projection $p: S' \to S^4$ induces a fibre-preserving map $S(\nu') \to S(\nu)$ which has degree ± 2 on each fibre. We can now apply Lemma 1 (see §4) to this fibre preserving map and conclude that the Euler classes of the bundles ν and ν' are related by the following equation:

$$e(\nu) = \pm\, 2 \cdot e(\nu') . \tag{1}$$

Lemma 1 is applicable here, because the Euler class is the first obstruction to a cross section of a sphere bundle. We may assume that the local orientations, etc., are chosen so that $e(\nu) = 2 \cdot e(\nu')$.

Next, we will apply equation (6.14) of Atiyah and Singer [1] to the involution T of the 4-manifold S'. The result is the following equation:

$$\mathrm{Sign}\,(T, S') = \{\mathscr{L}(M) \cdot \mathscr{L}(\nu')^{-1} e(\nu')\}[M] \,. \tag{2}$$

In this equation, we have used the notation of Atiyah and Singer. Here Sign (T, S') denotes the signature of the involution T; for a simplified definition, see Hirzebruch, [4], or Jänich and Ossa, [5]. This simplified definition is repeated below. $\mathscr{L}(M)$ and $\mathscr{L}(\nu')$ are certain polynomials in the Pontrjagin classes of M and ν' respectively. Since M is a 2-dimensional manifold,

$$\mathscr{L}(M) = \mathscr{L}(\nu') = 1 \,. \tag{3}$$

In view of (3) equation (2) simplifies to the following:

$$\mathrm{Sign}\,(T, S') = e(\nu')[M] \,. \tag{4}$$

Thus to determine the possible values of the integer $e(\nu')[M]$ (and hence $e(\nu)[M]$, by equation (1)), we must determine the possible values of the signature, Sign (T, S').

We recall that Sign (T, S') may be defined as the signature of a quadratic form defined on the real cohomology group $H^2(S', R)$ as follows:

$$(x, y) = (x \cup T^*y)[S'],\ x, y \in H^2(S', R) \,.$$

Now by Lemma 2, $T^*(y) = -y$ for any $y \in H^2(S' \colon R)$, hence (x, y) is the negative of the usual quadratic form of the oriented 4-manifold S'; it follows that (x, y) is a nonsingular quadratic form. It is also proved in Lemma 2 than $H^2(S', R)$ has rank n, where $n = 2 - \chi$ is the (nonorientable) genus of the surface M (i.e., M is the connected sum of n projective planes). Therefore the possible values of Sign (T, S') are the following:

$$-n, -n+2, \cdots, n-2, n \,.$$

Whitney's conjecture now follows by making use of equation (1) and the equation $n = 2 - \chi$. To complete the proof, it remains to prove Lemmas 1 and 2; this is done in the following sections. We will also show that all the possible values of the integer $e(\nu)[M]$ can be attained by actual embeddings.

4. Statement of Lemma 1. Let B be a CW-complex, $p \colon E \to B$ and $p' \colon E' \to B$ locally trivial fibre spaces over B with fibres F and

F' respectively, and assume that $f: E \rightarrow E'$ is a fibre preserving map, i.e., the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & E' \arrow[d] \\ & B \end{tikzcd}$$

is commutative. Finally, let us assume that the fibres F and F' are $(n-1)$-connected, $n \geqq 1$. Then the first obstructions to cross sections of these bundles are well-defined cohomology classes

$$c \in H^{n+1}(B, \pi_n(F)),\ c' \in H^{n+1}(B, \pi_n(F'))\ ,$$

(these are cohomology groups with local coefficients in general). The map f induces a coefficient homomorphism of cohomology groups,

$$f^{\#}: H^{n+1}(B, \pi_n(F)) \longrightarrow H^{n+1}(B, \pi_n(F'))$$

in an obvious way.

LEMMA 1. *Under the above hypotheses, the first obstructions satisfy the following naturality condition*:

$$f^{\#}(c) = c'\ .$$

The proof of this lemma is an easy consequence of the definition of obstructions. The details may be left to the reader.

5. Statement and proof of Lemma 2. In this section, we will use the same notation as in §2: $p: S' \rightarrow S^4$ is a 2-sheeted branched covering with the nonorientable surface M as the set of branch points, and $T: S' \rightarrow S'$ is the involution or covering transformation which interchanges the two sheets of the covering. The surface M is the connected sum of n projective planes, where $n = 2 - \chi$.

LEMMA 2. *The cohomology group $H^2(S', R)$ is a vector space over the reals of rank n and the homomorphism $T^*: H^2(S', R) \rightarrow H^2(S', R)$ induced by T satisfies the equation*

$$T^*(x) = -x,\ x \in H^2(S', R)\ .$$

The proof of this lemma involves several steps; as a first step, we will prove the following lemma which may be of independent interest:

LEMMA 3. *Let X be a finite, connected CW complex such that $H_1(X, Z)$ is cyclic of order 2, and let $\pi: \widetilde{X} \rightarrow X$ denote the covering* *space corresponding to the commutator subgroup of* $\pi_1(X)$. *Then* $H_1(\tilde{X}, Z)$ *is a finite abelian group of odd order.*

Proof. Since $\pi\colon \tilde{X} \to X$ is a 2-sheeted covering space, it may be considered as a nonorientable 0-sphere bundle. Hence there is a Gysin sequence for this situation with Z_2-coefficients (see Thom [7]). We will make use of the following portion of this Gysin sequence:

$$H^{\circ}(X) \xrightarrow{\mu} H^1(X) \xrightarrow{\pi^*} H^1(\tilde{X}) \xrightarrow{\psi} H^1(X) \xrightarrow{\mu} H^2(X) .$$

Here the homomorphism $\mu\colon H^m(X, Z_2) \to H^{m+1}(X, Z_2)$ is cup product with the characteristic class, $w_1 \in H^1(X, Z_2)$. The hypothesis of the lemma implies that $H^1(X, Z_2)$ is cyclic of order 2; since $\tilde{X}$ is a nontrivial covering space, w_1 must be the unique nonzero element of $H^1(X, Z_2)$. From this it follows that $\mu\colon H^{\circ}(X) \to H^1(X, Z_2)$ is an isomorphism onto. We assert that $\mu\colon H^1(X) \to H^2(X)$ is a monomorphism; it then follows by exactness that $H^1(\tilde{X}, Z_2) = 0$. Since $H^1(\tilde{X}, Z_2) = \operatorname{Hom}[H_1(\tilde{X}, Z), Z_2]$, and $H_1(\tilde{X}, Z)$ is a finitely generated abelian group, the conclusion of the lemma follows. It remains to prove the assertion. To do this, it suffices to prove that $\mu(w_1) \neq 0$. Now

$$\mu(w_1) = w_1 \cup w_1 = Sq^1(w_1) ,$$

and the homomorphism Sq^1 is well-known to be the composition of the Bockstein homomorphism (associated with the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$) and reduction mod 2. The hypothesis that $H_1(X, Z)$ is cyclic of order 2 enables one to prove that $Sq^1(w_1) \neq 0$; the details are left to the reader.

REMARK. Professor E. Schenkman has communicated to the author a purely group-theoretic proof of the following generalization of Lemma 3. Assume that X is a finite, connected CW-complex and $\pi; \tilde{X} \to X$ is the covering space corresponding to the commutator subgroup of $\pi_1(X)$, exactly as in the lemma. The generalization consists in assuming that $H_1(X, Z)$ is cyclic *of prime power order*. The conclusion is that $H_1(\tilde{X}, Z)$ is a finite abelian group, and the orders of $H_1(X, Z)$ and $H_1(\tilde{X}, Z)$ are relatively prime. Professor Schenkman also has an example to show that this conclusion does *not* necessarily hold if $H_1(X, Z)$ is a cyclic group of order 6.

We will now continue with the proof of lemma 2. Let A be a smooth closed tubular neighborhood of M in S^4, C = closure of $S^4 - A$, and $E = A \cap C$. Then E is a closed, orientable 3-manifold which is the common boundary of A and C; also, E is a realization of the normal 1-sphere bundle $S(\nu)$. In general, we will denote the corresponding subsets of S' by means of primes, i.e.,

$$A' = p^{-1}(A)\ ,$$
$$C' = p^{-1}(C), \text{ and}$$
$$E' = p^{-1}(E)\ .$$

Then A' is a closed tubular neighborhood of M in S', $A' \cup C' = S'$, and E' is the common boundary of A' and C'. Note that C is a deformation retract of $S^4 - M$, C' is a deformation retract of $S' - M$, and C' is a 2-fold (unbranched) covering of C. We can apply Lemma 3 with $X = C$, $\tilde{X} = C'$ to conclude that $H_1(C', Z)$ is a finite group of odd order. It follows immediately that

$$H^1(C', R) = 0\ . \tag{5}$$

Next, we wish to compute the real cohomology of the space E'. Since $E' = S(\nu')$ is a nonorientable 1-sphere bundle over M, we can use the Gysin sequence for this purpose:

$$\cdots \xrightarrow{\psi} H^{q-2}(M, \mathscr{R}) \xrightarrow{\mu} H^q(M, R) \xrightarrow{p^*} H^q(E', R)$$
$$\xrightarrow{\psi} H^{q-1}(M, \mathscr{R}) \xrightarrow{\mu} \cdots\ .$$

Here $H^m(M, \mathscr{R})$ means the m-dimensional cohomology group of M with local coefficient group the twisted real numbers. By the Poincaré duality theorem for nonorientable manifolds,

$$H^{q-2}(M, \mathscr{R}) \approx H_{4-q}(M, R)\ .$$

From this it follows readily that for any value of q, $H^{q-2}(M, \mathscr{R}) = 0$ or $H^q(M, R) = 0$. Therefore $\mu = 0$, and

$$\begin{aligned} \operatorname{rank} H^q(E', R) &= \operatorname{rank} H^q(M, R) + \operatorname{rank} H^{q-1}(M, \mathscr{R}) \\ &= \operatorname{rank} H^q(M, R) + \operatorname{rank} H_{3-q}(M, R)\ . \end{aligned}$$

From this we conclude that

$$\operatorname{rank} H^\circ(E', R) = \operatorname{rank} H^3(E', R) = 1\ , \tag{6}$$

$$\operatorname{rank} H^1(E', R) = \operatorname{rank} H^2(E', R) = n - 1\ . \tag{7}$$

Of course, (6) also follows from the fact that E' is a closed, connected, orientable 3-manifold.

Next, we consider the real cohomology sequence of the pair (C', E'). By making use of (5) and the fact that

$$\operatorname{rank} H^q(C', E', R) = \operatorname{rank} H^{4-q}(C', R)$$

(which is a consequence of the Lefschetz-Poincaré duality theorem for orientable manifolds with boundary) we conclude that

$$H^3(C', R) = H^4(C', R) = 0 \; . \tag{8}$$

Next, since C' is a 2-sheeted covering of C, we have the following obvious relation between the Euler characteristics:

$$\chi(C') = 2 \cdot \chi(C) \; .$$

Now one readily computes that $\chi(C) = n$ (use the Alexander duality theorem). Hence $\chi(C') = 2n$; then making use of (5) and (8) we conclude that

$$\operatorname{rank} H^2(C', R) = 2n - 1 \; . \tag{9}$$

Next, we will use the information we have already obtained about $H^*(C', R)$ and the real cohomology sequence of the pair (S', C') to determine $H^*(S', R)$. For this purpose, note that by the excision property,

$$H^q(S', C') \approx H^q(A', E') \; .$$

Now the pair (A', E') is the Thom space of the normal bundle ν'; therefore, we can apply the Thom isomorphism theorem for (non-orientable) vector bundles to conclude that

$$H^q(A', E', R) \approx H^{q-2}(M, \mathscr{R}) \; .$$

Also $H^{q-2}(M, \mathscr{R}) \approx H_{4-q}(M, R)$, as was noted above. Combining these isomorphisms, we see that

$$\operatorname{rank} H^4(S', C', R) = 1 \; , \tag{10}$$

$$\operatorname{rank} H^3(S', C', R) = n - 1 \; , \tag{11}$$

$$H^q(S', C', R) = 0 \text{ for } q \neq 3 \text{ or } 4 \; . \tag{12}$$

If we incorporate all the information we have obtained about $H^*(S', C', R)$ and $H^*(C', R)$ together with the fact that

$$\operatorname{rank} H^3(S', R) = \operatorname{rank} H^1(S', R)$$

(which is a consequence of the Poincaré duality theorem) into the cohomology sequence of the pair (S', C'), we see that

$$\operatorname{rank} H^2(S', R) = n \; ,$$

as was to be proved. We note that it also follows that

$$H^1(S', R) = H^3(S', R) = 0 \; .$$

It remains to prove the last statement of Lemma 2. For this purpose, note that the projection $p: S' \to S^4$ induces a map of the real cohomology sequence of the pair (S^4, C) into that of the pair (S', C'); hence we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^2(C) & \xrightarrow{\delta} & H^3(S^4, C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_2^*} & & \\
0 & \longrightarrow & H^2(S') & \xrightarrow{i^*} & H^2(C') & \xrightarrow{\delta'} & H^3(S', C') & \longrightarrow & 0
\end{array} \tag{13}
$$

The involution T^* operates on each of the real vector spaces in the bottom line of this diagram, and the homomorphisms i^* and δ' commute with T^*. Each of these three vector spaces decomposes into the direct sum of two subspaces corresponding to the eigenvalues $+1$ and -1 respectively of the involution T^*. These subspaces are respectively the subspace of elements left fixed by T^*, and the subspace consisting of those elements x such that $T^*(x) = -x$. The homomorphisms i^* and δ' respect these direct sum decompositions. Furthermore, it is clear that the images of p_1^* and p_2^* are contained in the subspaces of elements left fixed by T^*.

Next, we assert that p_1^* is a monomorphism, and its image is the entire subspace of elements of $H^2(C')$ which are left fixed by T^*. To prove this, note that C' is a covering space of C; hence we can apply the results of appendix No. 2. By equation (V) we see that

$$H^2(C') = \text{image } p_1^* \oplus \text{kernel } t^* \tag{14}$$

and by (VI) the elements of kernel t^* satisfy the equation

$$x + T^*(x) = 0 ,$$

i.e., $T^*(x) = -x$. Thus the direct sum decomposition in (14) is the same as that corresponding to the eigenvalues of T^*.

Finally, we assert that p_2^* is an isomorphism. This follows from consideration of the following diagram:

$$
\begin{array}{ccccc}
H^3(S^4, C) & \xrightarrow{j} & H^3(A, E) & \xleftarrow{\varphi} & H^1(A, \mathscr{R}) \\
\downarrow{\scriptstyle p_2^*} & & \downarrow{\scriptstyle p_3^*} & & \downarrow{\scriptstyle p_4^*} \\
H^3(S', C') & \xrightarrow{j'} & H^3(A', E') & \xleftarrow{\varphi'} & H^1(A', \mathscr{R})
\end{array} \tag{15}
$$

The left hand square of this diagram is commutative, and j and j' are isomorphisms by the excision property. (A, E) and (A', E') are the Thom spaces of the bundles ν and ν' respectively, and φ and φ' are the Thom isomorphisms defined by

$$\varphi(x) = x \cup U ,$$
$$\varphi'(y) = y \cup U' ,$$

where $U \in H^2(A, E; \mathscr{R})$ and $U' \in H^2(A', E', \mathscr{R})$ are the Thom classes (with twisted coefficients) of the bundles ν and ν' respectively. Note that $p_4: A' \to A$ is a homotopy equivalence, hence p_4^* is a isomorphism. The Thom classes are related by the following equation

$$p_3^*(U) = \pm 2U'$$

since the projection $p_3: (A', E') \to (A, E)$ is a fibre preserving map having degree ± 2 on each fibre (cf. Spanier [6], Chapter V, §7). Thus the right hand square of the diagram (15) is commutative up to a factor of ± 2. Putting all these facts together, we see that p_2^* is an isomorphism, as asserted.

It follows that every element of $H^3(S', C')$ is left fixed by T^*. Therefore every element of the subspace (kernel t^*) of $H^2(C')$ (i.e., those corresponding to the eigenvalue -1) is contained in kernel δ' = image i^*. But it is readily seen that

$$\begin{aligned} &\text{rank (kernel } t^*) = n, \text{ and} \\ &\text{rank (image } i^*) = n\,. \end{aligned}$$

Therefore

$$\text{image } i^* = \text{kernel } t^*\,.$$

Since i^* is a monomorphism, it follows that on the vector space $H^2(S')$ the only eigenvalue of T^* is -1. This completes the proof of Lemma 2.

6. **Proof that all possible values of the integer $e(\nu)[M]$ can actually be realized.** It follows readily from our conventions that changing the orientation of the 4-sphere, S^4, changes the sign of the integer $e(\nu)[M]$. Alternatively, we could achieve the same result by keeping the orientation of S^4 fixed and replacing the given embedding $i: M \to S^4$ by the composite

$$M \xrightarrow{i} S^4 \xrightarrow{h} S^4$$

where h is an orientation reversing diffeomorphism of S^4.

If we are given two pairs (S_i^4, M_i), $i = 1, 2$, consisting of an oriented 4-sphere and a smoothly embedded nonorientable surface, we can form the connected sum

$$(S^4, M) = (S_1^4, M_1) \# (S_2^4, M_2)$$

as defined by Haefliger [3]. Denote the normal bundles of M, M_1, and M_2 by ν, ν_1, and ν_2 respectively. We then have the following equation:

$$e(\nu)[M] = e(\nu_1)[M_1] + e(\nu_2)[M_2]\,.$$

The proof of this equation is not difficult; we leave it to the reader.

Let P be a real projective plane imbedded smoothly in an oriented 4-sphere, S^4 with normal bundle ν. It is a consequence of the theorem proved so far that

$$e(\nu)[P] = \pm 2 ,$$

the sign depending on the orientation of S^4. Let us assume that the orientation is chosen so that

$$e(\nu)[P] = 2 .$$

If we now form the connected sum of i copies of the pair (S^4, P) and $(n - i)$ copies of the pair $(-S^4, P)$, we obtain a pair (S^4, M) such that

$$e(\nu_M)[M] = 4i - 2n ,$$

and $\chi(M) = 2 - n$. By choosing $i = 0, 1, 2, \cdots, n$ we obtain all possible values for the Euler class of the normal bundle of a surface M with $\chi(M) = 2 - n$.

Appendix 1. Generalities on orientations of vector bundles and local coefficients. If $E \to B$ is an n-dimensional real vector bundle over the space B, we will consistently use the notation $S(E) \to B$ and $D(E) \to B$ to denote the associated $(n - 1)$ -sphere bundle and the associated n-dimensional disc bundle respectively. For any point $b \in B$, the fibres of these bundle will be denoted by E_b, $S(E)_b$, and $D(E)_b$ respectively. Associated with the bundle $E \to B$ is a certain local system of groups $O(E)$, called "the local coefficient system of orientations of E". This local system of groups associates with each point $b \in B$ the group $H_n(D(E)_b, S(E)_b; Z)$ (or alternatively, the group $H_{n-1}(S(E)_b; Z)$ or $\pi_{n-1}(S(E)_b)$; these different groups are related by obvious canonical isomorphisms). The Euler class, $e(E)$, is an n-dimensional cohomology class with coefficients in $O(E)$. Note that the local system $O(E)$ is determined up to isomorphism by the first Stiefel-Whitney class, $w_1(E)$.

If M is a (possibly nonorientable) differentiable closed, connected n-manifold, the local coefficient system of orientations of M is, by definition, the local coefficient system of orientations of the tangent bundle of M; it is denoted by $O(M)$. The "fundamental homology class of M" is a uniquely defined homology class, $[M] \in H_n(M, O(M))$. If M is triangulated, it is represented by an n-cycle which assigns to each *oriented* n-simplex the corresponding "local orientation".

Let E and E' be vector bundles over B, and let $E \oplus E$ denote their Whitney sum. There is a natural isomorphism

$$O(E) \otimes O(E') \approx O(E \oplus E'))$$

which is determined at each point $b \in B$ by the natural isomorphism

$$H_n(D(E)_b, S(E)_b) \otimes H_{n'}(D(E')_b, S(E')_b) \\ \approx H_{n+n'}(D(E \oplus E')b, S(E \oplus E')_b) .$$

This natural isomorphism can also be looked on as a bilinear pairing

$$O(E) \times O(E') \longrightarrow O(E \oplus E') ,$$

which can be used to define cup products, cap products, etc.

Given the bundle E over a connected space B, the local system of groups $O(E)$ is isomorphic to a local system of groups in B which assigns to each point $b \in B$ the additive group of integers, Z, with the "twisting" of this local system of integers determined by $w_1(E)$. We will denote this local system of integers by $\mathscr{Z}$. As a matter of fact, there are actually two distinct isomorphisms between the local systems $O(E)$ and $\mathscr{Z}$; to choose one of them as a preferred isomorphism amounts to "orienting" the bundle E in some sense, even though the bundle E may be nonorientable in the usual sense.

Appendix 2. The transfer homomorphism in a covering space. Let X be an arcwise connected topological space and $p: \tilde{X} \rightarrow X$ a regular covering space of X with *finitely many* sheets. In this section, we shall consider some relations between the homology and cohomology groups of X and $\tilde{X}$. These relations are well known, but do not seem to have been published anywhere.

We will simultaneously consider the following two situations:

(a) X and $\tilde{X}$ are simplicial polyhedra, p is a simplicial map, and we use simplicial chains and cochains.

(b) X and $\tilde{X}$ are not assumed triangulated, and we use singular chains and cochains.

In either case, the projection p induces a chain transformation

$$p_\#: C_*(\tilde{X}) \longrightarrow C_*(X) .$$

Since the covering is assumed to have only finitely many sheets, the so called "transfer homomorphism" is defined in the opposite direction:

$$t: C_*(X) \longrightarrow C_*(\tilde{X}) .$$

The definition of t is as follows: for any n-simplex σ of X, $t(\sigma)$ is defined to be the sum of all n-simplexes σ' of $\tilde{X}$ such that $p_\#(\sigma') = \sigma$. It is readily verified that t is a chain transformation. One can also easily verify the following two relations:

$$\text{(I)} \qquad tp_\#(u) = \sum_{g \in G} g_\#(u),\ u \in C_*(\tilde{X}) ,$$

$$p_{\#}t(v) = mv, \qquad v \in C_*(X) \,. \tag{II}$$

Here G denotes the group of covering transformations of $\tilde{X}$, and m denotes the number of sheets of the covering.

If we pass to cohomology with any coefficients, we have induced homomorphisms.

$$p^*: H^*(X) \longrightarrow H^*(\tilde{X})$$
$$t^*: H^*(\tilde{X}) \longrightarrow H^*(X)$$

and the relations (I) and (II) lead to the following relations:

$$p^*t^*(x) = \sum_{g \in G} g^*(x), \; x \in H^*(\tilde{X}) \tag{III}$$

$$t^*p^*(y) = my, \qquad y \in H^*(X) \,. \tag{IV}$$

Let us assume that we use a field for coefficients *whose characteristic does not divide the number of sheets, m.* Then from (IV) we easily deduce that p^* is a monomorphism, t^* is an epimorphism, and $H^*(\tilde{X})$ breaks up into a direct sum,

$$H^*(\tilde{X}) = \text{image } p^* \oplus \text{kernel } t^* \,. \tag{V}$$

The elements of the direct summand image p^* are obviously left fixed by the homomorphisms g^* for all $g \in G$. It follows from equation (III) that the elements of the direct summand kernel t^* satisfy the following condition:

$$\sum_{g \in G} g^*(x) = 0 \,. \tag{VI}$$

Appendix 3 (*added in proof, August,* 1969). Glen Bredon has pointed out in a letter to the author (dated May 14, 1969) that the proof of Lemma 2 can be considerably shortened, as follows:

The last statement of Lemma 2 is an immediate consequence of the known fact that the homomorphism $p^* : H^*(S^4; R) \rightarrow H^*(S'; R)$ is an isomorphism onto the set of invariant elements of T^*; see Theorem 19.1 on page 85 of Bredon's book (*Sheaf Theory*, McGraw-Hill, 1967). The proof of this theorem depends on the fact that the notion of the transfer homomorphism (see Appendix 2) can be generalized to cover the case of an arbitrary action of a finite group on a Hausdorff space, provided, one uses a Čech type cohomology theory; see Bredon's book (*loc. cit.*) or *Annals of Mathematics Study No.* 46, *Seminar on Transformation Groups*, Chapter III, § 2 (by E. E. Floyd).

The first statement of Lemma 2 can be proved more directly by use of the exact sequences of P. A. Smith (see Bredon, *loc. cit.*, page 86, or Floyd, *loc. cit.*, Chapter III, § 4). In the case at hand this gives the following exact sequence (Z_2 coefficients):

$$\cdots \longrightarrow H^i(S^4, M) \longrightarrow H^i(S') \longrightarrow H^i(S^4, M) \oplus H^i(M) \longrightarrow H^{i+1}(S^4, M) \longrightarrow \cdots$$

Also, the part $H^i(M) \to H^{i+1}(S^4, M)$ of the connecting homomorphism is just the coboundary for the pair (S^4, M). From this it follows immediately that $H^1(S'; Z_2) = 0 = H^3(S'; Z_2)$ and the vector space $H^2(S'; Z_2)$ has rank n. Then one applies the universal coefficient theorem to conclude that $H^2(S'; R)$ is also a vector space of rank n.

BIBLIOGRAPHY

1. M. F. Atiyah and I. M. Singer, *The index of elliptic operators*: *III*, Ann. of Math. **87** (1968), 546-604.

2. R. H. Fox, "Covering spaces with singularities," *Algebraic geometry and topology*: *A symposium in honor of S. Lefschetz*, Princeton University Press, 1957.

3. A. Haefliger, *Knotted* (4*k*-1)-*spheres in* 6*k*-*space*, Ann. of Math. **75** (1962), 452-466.

4. F. Hirzebruch, *Involutionen auf mannigfaltigkeiten*, Proceedings of Conference on Transformation Groups, Tulane, 1967.

5. K. Jänich and E. Ossa, *On the signature of an involution*, Topology, **8** (1969), 27-30.

6. E. Spanier, *Algebraic topology*, McGraw-Hill, 1966.

7. R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Ecole Norm. Sup. **69** (1952), 109-182.

8. H. Whitney, "On the topology of differentiable manifolds," *Lectures in topology*, Michigan Univ. Press, 1940.

Received May 15, 1969. An abstract of this paper was submitted to the American Mathematical Society in April, 1969; see the *Notices of the American Mathematical Society*, June 1969, page 697.

During the preparation of this paper the author was on leave of absence at Princeton University. It is a pleasure to acknowledge the hospitality and kindness of the Princeton Mathematics Department.

YALE UNIVERSITY
NEW HAVEN, CONNECTICUT