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BOUNDARY BEHAVIOR OF RANDOM VALUED HEAT POLYNOMIAL EXPANSIONS

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This paper is concerned with random series of the form $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ where the X_n's are random variables, the a_n's are real numbers, and the v_n's are heat polynomials as introduced by P. C. Rosenbloom and D. V. Widder. The sequences $\{a_n\}$ are assumed to satisfy $\limsup_{n\to\infty} |a_n|^{2/n}(2n/e) = 1$ which implies $\sum_{n=0}^{\infty} a_n v_n(x, t)$ has $|t| < 1$ as its strip of convergence, i.e., it converges to a C^2-solution of the heat equation in $|t| < 1$ and does not converge everywhere in any larger open strip. Associated with each sequence $\{a_n\}$ is its classification number from $[0, 1]$ which indicates how rapidly a_n tends to zero. Assumptions are placed on the random variables which imply that for almost every ω the series $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ has $|t| < 1$ as its strip of convergence.

The main results of the paper are two theorems. The first states that if $\{a_n\}$ has its classification number in $[0, 1/2)$, then for almost every ω the lines $t = 1$ and $t = -1$ form the natural boundary for $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$. The second is concerned with sequences having their classification numbers in $(1/2. 1]$. The conclusion implies that for almost every ω no interval of either of the lines $t = 1$ or $t = -1$ is part of the natural boundary for $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$.

The present work had it original motivation in the study of the boundary behavior of random power series. These are series of the form $\sum_{n=0}^{\infty} a_n(\omega)z^n$ where the a_n's are complex valued random variables and z is a complex number. Reference [1] contains a history of results in this area. One of the early results which helped to motivate the first part of the proof of our Theorem 1 is due to Paley and Zygmund in a 1932 paper [see 6, p. 220]. In this theorem it is assumed that $\sum_{n=0}^{\infty} a_n z^n$ is an ordinary power series with a finite radius of convergence. Letting $\{\phi_n\}$ be the sequence of Rademacher functions, the conclusion is that for almost every ω in $[0, 1]$ the series $\sum_{n=0}^{\infty} \phi_n(\omega)a_n z^n$ has its circle of convergence as its natural boundary.

More recently [see 3] V. L. Shapiro has considered series of the form $\sum_{n=0}^{\infty} X_n(\omega)H_n(x)$ where the X_n's are random variables and

$$\sum_{n=0}^{\infty} H_n(x)$$

is the spherical harmonic representation of a harmonic function in the unit ball. The harmonic continuability across the boundary of the unit ball of the functions $\sum_{n=0}^{\infty} X_n(\omega)H_n(x)$ was investigated. This work further motivated the first part of the proof of our Theorem 1 and influenced our choice of the class of random variables to be considered.

2. Definitions and preliminary comments. For a point (x_0, t_0) in the plane and a number $\rho > 0$ we let

$$S(x_0, t_0; \rho) = \{(x, t): |x - x_0| < \rho \text{ and } |t - t_0| < \rho\} .$$

If $u(x, t)$ is a C^2-solution to the heat equation in the strip $|t| < \sigma$ we say the line $t = -\sigma$ $(t = \sigma)$ is part of the natural boundary for u in case for every x_0 and every $\rho > 0$ there is no C^2-solution $v(x, t)$ in $S(x_0, -\sigma; \rho)$ $(S(x_0, \sigma; \rho))$ which agrees with $u(x, t)$ where u and v are both defined.

Let E_0 be the set of all sequences $\{a_n\}_{n=0}^{\infty}$ of real numbers. For $r > 0$ let

$$E_r = \{\{a_n\} \in E_0: |a_n| (2n/e)^{n/2} = O(e^{-n^r}) \text{ as } n \to \infty\} .$$

We call $\sup\{r: \{a_n\} \in E_r\}$ the classification number of $\{a_n\}$.

Explicitly, from [2, p. 222]

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!}, \quad n = 0, 1, \cdots . \tag{2.1}$$

In [2, Th. 5.3, p. 231] it was shown that the series $\sum_{n=0}^{\infty} a_n v_n(x, t)$ converges to a C^2-solution of the heat equation in the strip $|t| < \sigma$ where

$$\sigma = (\limsup |a_n|^{2/n}(2n/e))^{-1} \tag{2.2}$$

and that this strip is the largest open strip of convergence of the series. One easily shows that sequences $\{a_n\}$ satisfying

$$\limsup |a_n|^{2/n}(2n/e) = 1$$

have their classification numbers in $[0, 1]$.

We will make repeated use of the following bounds which appear in [4] by S. Täcklind. Assume $u(x, t)$ is continuous on the rectangle $R = \{(x, t): |x| \leqq \mathscr{L}, 0 \leqq t \leqq T\}$, is a C^2-solution to the heat equation in the interior of R, and μ is an upper bound for $|u(x, t)|$ on R; then $u(x, t)$ is in class C^∞ on the interior of R and for $n = 2, 3, \cdots, |x| < \mathscr{L}$, and $0 < t \leqq T$

$$\begin{aligned} \left|\frac{\partial^n u}{\partial x^n}(x, t)\right| &\leqq \frac{\mu}{2\sqrt{\pi}} \frac{2^{(n+3)/2}}{t^{n/2}} \Gamma((n+1)/2) \\ &\quad + \frac{\mu}{\sqrt{\pi}}\left(\frac{\pi}{2}\right)^{5/2} \frac{2^{3n/2}}{(\mathscr{L} - |x|)^n} \Gamma(n+1) . \end{aligned} \tag{2.3}$$

3. THEOREM 1. *Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of symmetric independent random variables defined on the complete probability space $(\Omega, \mathscr{F}, P)$ and satisfying*

(i) *there exists a number M such that*

$$\int_{\Omega} | X_n(\omega) |^2 dP(\omega) \leqq M \text{ for } n = 0, 1, \cdots, \text{ and}$$

(ii) *there exists $N > 0$ such that*

$$N \leqq \int_{\Omega} | X_n(\omega) | \, dP(\omega), \, n = 0, 1, \cdots .$$

Assume $\{a_n\}$ satisfies $\limsup | a_n |^{2/n}(2n/e) = 1$ and has its classification number in $[0, 1/2)$. Then for almost every ω in Ω the lines $t = 1$ and $t = -1$ form the natural boundary for

$$u_\omega(x, t) = \sum_{n=0}^{\infty} X_n(\omega) a_n v_n(x, t) .$$

Proof. Letting $\Omega' = \{\omega \varepsilon \Omega \colon \sum_{n=0}^{\infty} X_n(\omega) a_n v_n(x, t)$ converges in the strip $|t| < 1\}$, we will first show $P(\Omega') = 1$. Clearly

$$[\limsup | X_n |^{2/n} \leqq 1] \supseteq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [| X_n | \leqq n M^{1/2}]$$

and by the Borel-Cantelli Lemma the last set has probability 1 since $P[| X_n | > n M^{1/2}] \leqq 1/n^2$ from (i). Hence

$$P\{\omega \colon \limsup | X_n(\omega) a_n |^{2/n}(2n/e) \leqq 1\} = 1$$

which by (2.2) shows $P(\Omega') = 1$.

The following fact is essentially a merger of Lemma 1 from [3] and a special case of Lemma 2 from [3]. There exist numbers Λ in $(0, 1)$ and $B > 0$ with the following property: for $E \in \mathscr{F}$ with $P(E) > \Lambda$ there is a positive integer n_0 such that for $n \geqq n_0$, every sequence $\{c_j\}_{j=0}^{\infty}$ of real numbers, and $k \geqq 1$ we have

$$\sum_{j=n}^{n+k} c_j^2 \leqq B \int_E \left\{ \sum_{j=n}^{n+k} c_j X_j(\omega) \right\}^2 dP(\omega) . \tag{3.1}$$

We will show that for almost every ω the line $t = -1$ is part of the natural boundary for u_ω and will use this in the proof for the line $t = 1$.

Assume it is false that for a.e. ω in Ω the line $t = -1$ is part of the natural boundary for u_ω. The first part of the argument we give in order to obtain a contradiction is analogous to parts of the proof of Theorem 1 in [3] by V. L. Shapiro. We will employ (2.3), (3.1), and an asymptotic estimate for heat polynomials from [2] in order to obtain conditions on the sequence $\{a_n\}$ which contradict the fact that the classification number of $\{a_n\}$ is in $[0, 1/2)$.

Let $E = \{\omega \in \Omega' : t = -1$ is not part of the natural boundary for $u_\omega\}$. Then either (i) $E \notin \mathscr{F}$, or (ii) $E \in \mathscr{F}$ and $P(E) > 0$. Using the fact that the real line is separable and the countable additivity of the probability P, it follows that there exist a real number x_0 and $\rho_0 > 0$ such that $E_1 = \{\omega \in E$: there is a C^2-solution to the heat equation in $S(x_0, -1; \rho_0)$ which agrees with u_ω where they are both defined$\}$ satisfies either (i) $E_1 \notin \mathscr{F}$, or (ii) $E_1 \in \mathscr{F}$ and $P(E_1) > 0$. For $i = 1, 2, \cdots$ define

$$E_{2,i} = \left\{\omega \in \Omega' : \left|\frac{\partial^m u_\omega}{\partial x^m}(x, t)\right| \leqq i^m m^m \text{ for } (x, t) \text{ in } S\left(x_0, -1; \frac{\rho_0}{2}\right),\right.$$
$$\left. |t| < 1, \text{ and } m = i, i+1, \cdots\right\}$$

and let $E_2 = \bigcup_{i=1}^\infty E_{2,i}$. E_2 is in the tail σ-field generated by the independent X_n's. From (2.3) it follows that $E_1 \subseteq E_2$. By Kolmolgorov's zero-one law $P(E_2) = 1$. Let Λ and B be as in (3.1). Take i_0 sufficiently large that $P(E_{2,i_0}) > \Lambda$ and let n_0 correspond to E_{2,i_0} as in (3.1). Now let $m \geqq \max\{n_0, i_0\}$ and let (x, t) be in $S(x_0, -1; \rho_0/2)$ with $|t| < 1$. Then by (3.1) for $k = 1, 2, \cdots$

$$\sum_{n=m}^{m+k}\left[\frac{n!}{(n-m)!}a_n v_{n-m}(x, t)\right]^2$$
$$\leqq B\int_{E_{2,i_0}}\left[\sum_{n=m}^{m+k}\frac{n!}{(n-m)!}a_n v_{n-m}(x, t)X_n(\omega)\right]^2 dP(\omega)\ .$$

Making use of the independence and symmetry of the random variables and of condition (i) we see that the integrand of the last integral is Cauchy in the variable k in $L^1(\Omega)$ and thus in $L^1(E_{2,i_0})$. Hence

$$\sum_{n=m}^{\infty}\left[\frac{n!}{(n-m)!}a_n v_{n-m}(x, t)\right]^2$$
$$\leqq B\int_{E_{2,i_0}}\left[\sum_{n=m}^{\infty}\frac{n!}{(n-m)!}a_n v_{n-m}(x, t)X_n(\omega)\right]^2 dP(\omega)$$
$$= B\int_{E_{2,i_0}}\left|\frac{\partial^m u_\omega}{\partial x^m}(x, t)\right|^2 dP(\omega) \leqq B i_0^{2m} m^{2m}$$

with the last inequality following from the definition of E_{2,i_0}. We conclude that for every $m \geqq \max\{n_0, i_0\}$, every $n \geqq m$, and every (x, t) in $S(x_0, -1; \rho_0/2)$ with $|t| < 1$; we have

$$\text{(3.2)} \qquad \frac{n!}{(n-m)!}|a_n||v_{n-m}(x, t)| \leqq B^{1/2} i_0^m m^m\ .$$

It follows from Theorem 3.1 of [2] that there exist numbers A and l_0 such that for $n \geqq l_0$

$$\sup_{|x-x_0|<\rho_0/2} |v_n(x, -1)| \geqq A[2n/e]^{n/2} .$$

Thus from (3.2) we have for $n > m + l_0 > m \geqq \max\{n_0, i_0\}$

$$|a_n| \frac{n!}{(n-m)!} A[2(n-m)/e]^{(n-m)/2} \leqq B^{1/2} i_0^m m^m .$$

Employing Stirling's theorem we see there is a number C such that for $n > m + l_0 > m \geqq \max\{n_0, i_0\}$

$$(3.3) \qquad |a_n| (2n/e)^{n/2} \leqq \left[\frac{Cm}{\sqrt{n-m}}\right]^m \cdot ((n-m)/n)^{(n+1)/2} .$$

Let r be a number which is strictly greater than the classification number of $\{a_n\}$ and strictly less than 1/2. Let m be related to n by $m = [4n^r] + 1$ where the brackets denote the greatest integer function. Then from (3.3), for sufficiently large n,

$$(3.4) \qquad |a_n| (2n/e)^{n/2} \leqq (1 - 4/n^{1-r})^{(n^{1-r}/4)\cdot 2\cdot n^r} .$$

For large enough n, $(1 - 4/n^{1-r})^{(n^{1-r}/4)\cdot 2} \leqq 1/e$ and thus from (3.4) we have for such n, $|a_n| (2n/e)^{n/2} \leqq 1/e^{n^r}$. Hence $\{a_n\} \in E_r$ which contrandicts the fact that r is strictly greater than the classification number of $\{a_n\}$ and concludes the proof for the line $t = -1$.

For the last part of the proof we find it convenient to introduce the probability space $(R^\omega, \mathscr{A}', \mu')$ which we now describe.

$$R^\omega = \prod_{n=0}^{\infty} R_n$$

where each R_n is the set of real numbers. Let $\mathscr{A}_0$ be the field of all subsets of R^ω of the form $B \times (\prod_{n=n_0+1}^{\infty} R_n)$ where n_0 is a positive integer and B is a Borel set in $\prod_{n=0}^{n_0} R_n$. Let $\mathscr{A}$ be the σ-field generated by $\mathscr{A}_0$. Let μ be the probability on $(R^\omega, \mathscr{A})$ which is induced by the X_n's. Then $(R^\omega, \mathscr{A}', \mu')$ is the completion of $(R^\omega, \mathscr{A}, \mu)$.

Let $\{\eta_i\}_{i=0}^{\infty}$ be a sequence of ± 1's. Define $T: R^\omega \to R^\omega$ by

$$T((\xi_0, \xi_1, \cdots)) = (\eta_0\xi_0, \eta_1\xi_1, \cdots) .$$

Notice that

$$\mu\Big(\prod_{n=0}^{n_0} (a_n, b_n] \times \prod_{n=n_0+1}^{\infty} R_n\Big) = \prod_{n=0}^{n_0} P[X_n \in (a_n, b_n]]$$
$$= \prod_{n=0}^{n_0} P[X_n \in \eta_n(a_n, b_n]] = \mu\Big(T\Big(\prod_{n=0}^{n_0} (a_n, b_n] \times \prod_{n=n_0+1}^{\infty} R_n\Big)\Big)$$

where we have used both the independence and symmetry of the X_n's. From this it follows that for $A \in \mathscr{S}'$, $\mu'(A) = \mu'(T(A))$. We will make use of this fact twice in the remainder of this proof.

To finish the proof it suffices to show that for a.e. $p \in R^\omega$ the line $t = 1$ is part of the natural boundary for

$$u_p(x, t) = \sum_{n=0}^{\infty} \pi_n(p) a_n v_n(x, t)$$

where the π_n's are the projection random variables. Suppose this is false. From the first paragraph of the present proof we know $R^{\omega\prime} = \{p \in R^\omega : \sum_{n=0}^{\infty} \pi_n(p) a_n v_n(x, t)$ converges in $|t| < 1\}$ has μ'-measure 1. Now let $F = \{p \in R^{\omega\prime} : t = 1$ is not part of the natural boundary for $u_p\}$. Then either (i) $F \notin \mathscr{S}'$, or (ii) $F \in \mathscr{S}'$ and $\mu'(F) > 0$. It follows that there exist numbers a, b, ρ with $a < b$ and $\rho > 0$ such that $F_1 = \{p \in R^{\omega\prime}$: there is a function $v_p(x, t)$ which is continuous on $a \leqq x \leqq b$, $0 \leqq t \leqq 1 + \rho$; is a C^2-solution to the heat equation for $a < x < b$, $0 < t < 1 + \rho$; and agrees with $u_p(x, t)$ in $a \leqq x \leqq b$, $0 \leqq t < 1\}$ satisfies either (i) $F_1 \notin \mathscr{S}'$, or (ii) $F_1 \in \mathscr{S}'$ and $\mu'(F_1) > 0$. But $F_1 = \{p \in R^{\omega\prime}$: $\lim_{t \uparrow 1} u_p(a, t)$ and $\lim_{t \uparrow 1} u_p(b, t)$ both exist$\}$. F_1 is in the tail σ-field generated by the independent π_n's. From the zero-one law, $\mu'(F_1) = 1$.

Either $a \neq 0$ or $b \neq 0$ and for definiteness we assume $a \neq 0$. Then $F_2 = \{p \in R^{\omega\prime} : \lim_{t \uparrow 1} u_p(a, t)$ exists$\}$ has $\mu'(F_2) = 1$. Let $T: R^\omega \to R^\omega$ be defined by $T((\xi_0, \xi_1, \cdots)) = (\xi_0, -\xi_1, \xi_2, -\xi_3, \cdots)$. By our earlier comments concerning such mappings we have $\mu'(F_2 \cap T(F_2)) = 1$. For $p \in R^{\omega\prime}$ and $|t| < 1$ one checks that $u_{T(p)}(-a, t) = u_p(a, t)$. Hence for $p \in F_2 \cap T(F_2)$, $\lim_{t \uparrow 1} u_p(-a, t)$ and $\lim_{t \uparrow 1} u_p(a, t)$ both exist. Thus for $p \in F_2 \cap T(F_2)$ there is a function $w_p(x, t)$ which is continuous in $|x| \leqq a$, $0 \leqq t \leqq 2$; is a C^2-solution to the heat equation in $|x| < a$, $0 < t < 2$; and agrees with u_p in $|x| \leqq a$, $0 \leqq t < 1$. For $p \in F_2 \cap T(F_2)$ and $0 \leqq t \leqq 2$ let $\phi_p(t) = w_p(0, t)$ and $\psi_p(t) = (\partial w_p / \partial x)(0, t)$. Then, employing (2.3), we see that ϕ_p and ψ_p are in class $C\{(2n)!\}$ on $[0, 2]$ (a function f is in class $C\{(2n)!\}$ on an interval I if f is in class C^∞ on I and there exist constants β and B such that for every t in I, $|f^{(n)}(t)| \leqq \beta B^n (2n)!$, $n = 0, 1, \cdots$).

Now let $T': R^\omega \to R^\omega$ be defined by

$$T'((\xi_0, \xi_1, \cdots)) = (\xi_0, \xi_1, -\xi_2, -\xi_3, \xi_4, \xi_5, -\xi_6, -\xi_7, \cdots) .$$

Then for $p \in R^{\omega\prime}$ and $|t| < 1$, $u_p(0, t) = u_{T'(p)}(0, -t)$ and

$$\frac{\partial u_p}{\partial x}(0, t) = \frac{\partial u_{T'(p)}}{\partial x}(0, -t) .$$

For p in the almost sure set $T'(F_2 \cap T(F_2))$ we have $T'(p) \in F_2 \cap T(F_2)$ and we define ϕ'_p and ψ'_p on $[-2, 0]$ by $\phi'_p(t) = \phi_{T'(t)}(-t)$ and

$$\psi'_p(t) = \psi_{T'(p)}(-t)$$

thereby obtaining class $C\{(2n)!\}$ extensions of $u_p(0, t)$ and $(\partial u_p/\partial x)(0, t)$ on $[-1, 0]$. Thus for $p \in T'(F_2 \cap T(F_2))$

$$u_p'(x, t) = \sum_{n=0}^{\infty} \frac{\phi_p'^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\psi_p'^{(n)}(t)x^{2n+1}}{(2n+1)!}$$

provides a solution to the heat equation which is a C^2-extension of u_p into some rectangle $|x| < r$, $-2 < t < 0$ which contradicts the first part of the proof.

4. THEOREM 2. *Let* $\{X_n\}$ *be a sequence of independent random variables over a probability space* $(\Omega, \mathscr{F}, P)$ *which satisfies* (i) *and* (ii) *of Theorem* 1. *Assume* $\{a_n\}$ *satisfies* $\limsup |a_n|^{2/n}(2n/e) = 1$ *and has its classification number in* $(1/2, 1]$. *Then for almost every* ω *in* Ω *the following holds*: $|t| < 1$ *is the strip of convergence of* $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ *which for every* $\mathscr{L} > 0$ *can be extended as a* C^2-*solution of the heat equation into* $\{|t| < 1\} \cup \{|x| < \mathscr{L}\}$.

Proof. We will first show for almost every ω in Ω that $|t| < 1$ is the strip of convergence of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$. By (2.2) we must show that almost surely $\limsup |X_n(\omega)a_n|^{2/n}(2n/e) = 1$. The argument given in the first part of the proof of Theorem 1 shows that almost surely the last limit superior does not exceed 1. Let $\{n_j\}$ be a strictly increasing sequence of positive integers such that

$$\lim_{j\to\infty} |a_{n_j}|^{2/n_j}(2n_j/e) = 1 .$$

Then $\limsup |X_n(\omega)a_n|^{2/n}(2n/e) \geqq \limsup_{j\to\infty} |X_{n_j}(\omega)a_{n_j}|^{2/n_j}(2n_j/e) \geqq \limsup_{j\to\infty} |X_{n_j}(\omega)|^{2/n_j}$ which by the zero-one law is equal to some number c almost surely. Suppose $c < 1$. Then $X_{n_j} \to 0$ almost surely. By (ii) for $A > 0$ and $j = 0, 1, \cdots$

$$N \leqq \int_{[|X_{n_j}| \leqq A]} |X_{n_j}(\omega)|\, dP(\omega) + A^{-1}\int_{[|X_{n_j}| > A]} |X_{n_j}(\omega)|^2 dP(\omega) .$$

By the Lebesgue dominated convergence theorem the next to the last integral tends to 0 as j tends to ∞. From (i) the last term is uniformly bounded by $A^{-1}M$. Thus for every $A > 0$, $N \leqq A^{-1}M$ which is a contradiction. We conclude that $c \geqq 1$. Thus almost surely

$$\limsup |X_n(\omega)a_n|^{2/n}(2n/e) \geqq 1$$

which concludes the proof that almost surely this limit superior is 1.

In order to establish Theorem 2 for the line $t = 1$ we first construct a function which is C^∞ on the closed strip $|t| \leqq 1$ and has a heat polynomial expansion in $|t| < 1$. Let r be a number which is strictly greater than 1/2 and strictly less than the classification number of $\{a_n\}$. For $n = 0, 1, \cdots$ define $\alpha_n = (2n)e^{-n^r}$. Define f on $[-1, 1]$ by $f(t) = \sum_{k=0}^{\infty} \alpha_k t^k$. We will show this definition makes sense and obtain some bounds on the derivatives of f.

Let n be a nonnegative integer. Differentiating $\sum_{k=0}^{\infty} \alpha_k t^k$ term by term n times yields $\sum_{k=n}^{\infty} k!/(k-n)!\, \alpha_k t^{k-n}$. For $|t| \leqq 1$ the k^{th} term of this series is dominated by $2\, k^{n+1}e^{-k^r}$. One checks that

$$g_n(x) = x^{n+1}e^{-x^r}$$

is increasing on $(0, (n + 1/r)^{1/r})$ and decreasing on $((n + 1/r)^{1/r}, \infty)$. Hence

$$\sum_{k=n}^{\infty} k^{n+1}e^{-k^r} \leqq \int_n^{\infty} g_n(x)dx + g_n\left(\left(\frac{n+1}{r}\right)^{1/r}\right) \leqq 3\Gamma((n+2)/r)/r \; .$$

We conclude that f is a C^∞-function with $|f^{(n)}(t)| \leqq 6\Gamma((n + 2/r)/r$ for $n = 0, 1, \cdots$ and $|t| \leqq 1$.

Now define

$$u(x, t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{f^{(n+1)}(t)x^{2n+1}}{(2n+1)!} \; . \tag{4.1}$$

Because of the bounds obtained in the preceding paragraph it can be shown that the series of (4.1) can be differentiated term by term and that $u(x, t)$ is a C^∞-solution to the heat equation in the closed strip $|t| \leqq 1$. Since both $u(0, t)$ and $\partial u/\partial x(0, t)$, as functions of t on $(-1, 1)$, are given by their Maclaurin expansions, u has a heat polynomial expansion in $|t| < 1$ (see [5]). Thus

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} b_n v_n(x, t) \; , \\ b_{2n} &= f^{(n)}(0)/(2n)! \; , \\ b_{2n+1} &= f^{(n+1)}(0)/(2n+1)! \; . \end{aligned} \tag{4.2}$$

According to the first paragraph of the proof of Theorem 1, $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [|X_n| \leqq nM^{1/2}]$ has probability 1. Let ω be in this almost sure set. Let k_0 be a positive integer such that for $n \geqq k_0$, $|X_n(\omega)| \leqq nM^{1/2}$. Since r is less than the classification number of $\{a_n\}$, there is a number K such that $|a_n|(2n/e)^{n/2} \leqq Ke^{-n^r}$, $n = 1, 2, \cdots$. Using Stirling's theorem we have for $2n \geqq k_0$

$$b_{2n}(4n/e)^n \geqq |X_{2n}(\omega)a_{2n}|\, (4n/e)^n(1/2)^{3/2}/KM^{1/2} \; .$$

Similarly for $2n + 1 \geqq k_0$

$$b_{2n+1}(2(2n+1)/e)^{(2n+1)/2} \geqq |X_{2n+1}(\omega)a_{2n+1}|\, (2(n+1)/e)^{(2n+1)/2}e^{-1/2}/KM^{1/2} \; .$$

Letting $K' = K(Me)^{1/2}$ we have

$$| X_n(\omega)a_n | \leqq K'b_n \text{ for } n \geqq k_0 .$$

Let $\mathscr{L} > 0$. Then for $0 < t < 1$ we have

$$\left| \frac{\partial}{\partial t} \sum_{n=k_0}^{\infty} X_n(\omega)a_n v_n(\pm \mathscr{L}, t) \right| = K' \sum_{n=k_0}^{\infty} b_n n(n-1) | v_{n-2}(\pm \mathscr{L}, t) |$$

$$\leqq K' \sum_{n=k_0}^{\infty} b_n n(n-1) v_{n-2}(\mathscr{L}, t) \leqq K' \frac{\partial u}{\partial t}(\mathscr{L}, 1) < \infty .$$

Thus $\lim_{t \uparrow 1} \sum_{n=k_0}^{\infty} X_n(\omega)a_n v_n(\pm \mathscr{L}, t)$ both exist as is easily seen from the mean value theorem and the Cauchy criterion. Hence we can obtain an extension of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ into

$$\{(x, t): |t| < 1\} \cup \{(x, t): |x| < \mathscr{L}, 0 < t\}$$

which is a C^2-solution of the heat equation. (Notice at this point that we can also obtain an extension which is a bounded C^2-solution in $\{(x, t): |x| < \mathscr{L}, 0 \leqq t\}$.) Since ω was from the almost sure set

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [| X_n | \leqq nM^{1/2}] ,$$

this establishes the result for the line $t = 1$.

We now turn to the line $t = -1$. Define $\{Y_n\}_{n=0}^{\infty}$ on Ω by $Y_{2n} = (-1)^n X_{2n}$ and $Y_{2n+1} = (-1)^n X_{2n+1}$. Then, applying the first part of the proof, there is a set F in $\mathscr{F}$ with $P(F) = 1$ such that for ω in F and $\mathscr{L} > 0$ the solution $v_\omega(x, t) = \sum_{n=0}^{\infty} Y_n(\omega)a_n v_n(x, t)$ can be extended into $\{|t| < 1\} \cup \{|x| < \mathscr{L}$ and $0 < t\}$ so as to be a bounded C^2-solution of the heat equation in $\{(x, t): |x| < \mathscr{L}$ and $0 < t\}$. One easily checks that for ω in F,

$$\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(0, t) = \sum_{n=0}^{\infty} Y_n(\omega)a_n v_n (0, -t)$$

and $\sum_{n=1}^{\infty} X_n(\omega)a_n n v_{n-1}(0, t) = \sum_{n=1}^{\infty} Y_n(\omega)a_n n v_{n-1}(0, -t)$. Using these facts and (2.3) we see that for ω in F and $\mathscr{L} > 0$ the functions $\phi(t) = \sum_{n=0}^{\infty} X_n(\omega)a_n v_n(0, t)$ and $\psi(t) = \sum_{n=1}^{\infty} X_n(\omega)a_n n v_{n-1}(0, t)$ on $(-1, 1)$ possess sufficiently well behaved extensions ϕ' and ψ' to $(-\infty, 1)$ that

$$\sum_{n=0}^{\infty} \frac{\phi'^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\psi'^{(n)}(t)x^{2n+1}}{(2n+1)!}$$

is an extension of $\sum_{n=0}^{\infty} X_n(\omega)a_n v_n(x, t)$ in $|t| < 1$ to

$$\{(x, t): |t| < 1\} \cup \{(x, t): |x| < \mathscr{L} \text{ and } -\infty < t < 1\} .$$

5. Examples. The first example will show that our two theorems are best possible with respect to the allowable values of the classification number.

EXAMPLE 1. We will take $[0, 1]$ with Lebesgue measure as the probability space and the sequence of Rademacher functions, $\{\phi_n\}_{n=0}^{\infty}$, for the random variables.

For $k = 0, 1, \cdots$ define $\alpha_k = e^{-\sqrt{k}}$. Then, as in the proof of Theorem 2, defining f on $[-1, 1]$ by $f(t) = \sum_{k=0}^{\infty} \alpha_k t^k$ yields a C^∞-function whose n^{th} derivative on $[-1, 1]$ is bounded in absolute value by $6\Gamma(2(2n + 1))$. In the strip $|t| < 1$ define $u(x, t) = \sum_{n=0}^{\infty} (f^{(n)}(t)x^{2n})/(2n)!$. To see that this definition makes sense and that term by term partial differentiation is permitted, we note that for every closed interval $I \subsetneqq (-1, 1), f$ is in class $C\{n!\}$ on I. Because of the bounds on the derivatives of f we see from the defining series for u that u may be extended as a C^∞-solution of the heat equation to

$$\{|t| < 1\} \cup \{(x, 1): |x| < 1\} .$$

Since $u(0, t)$ and $\partial u/\partial x(0, t)$ are both given by their Maclaurin expansions in $|t| < 1, u$ possesses a heat polynomial expansion in the strip $|t| < 1$ (see [5]). Thus for $|t| < 1, u(x, t) = \sum_{n=0}^{\infty} a_n v_n(x, t)$; $a_{2n} = (e^{-\sqrt{n}} n!)/(2n)!$, $a_{2n+1} = 0$. One checks that $\limsup |a_n|^{2/n}(2n/e) = 1$. Also it is easily seen that limit $|a_{2n}|(4n/e)^n e^{\sqrt{2n}} = \infty$ which implies $\{a_n\} \notin E_{1/2}$ and thus the classification number of $\{a_n\}$ is in $[0, 1/2]$. As in the proof of Theorem 2, $\lim_{t \uparrow 1} u_\omega(\pm 1/2, t)$ both exist for every ω in $[0, 1]$. Thus for every $\omega \in [0, 1]$ the line $t = 1$ is not part of the natural boundary for $u_\omega(x, t)$. Using Theorem 1, we conclude that the classification number of $\{a_n\}$ is 1/2 and that in Theorem 1 we cannot replace $[0, 1/2)$ by $[0, 1/2]$ as the allowable range for the classification number.

We will next show that the conclusion of Theorem 2 does not hold for $\sum_{n=0}^{\infty} \phi_n(\omega) a_n v_n(x, t)$. Assume there is a set A in $[0, 1]$ with $m(A) = 1$ such that for each ω in A no interval of the line $t = 1$ is part of the natural boundary for $u_\omega(x, t)$. Thus for ω in A, $g_\omega(x) = \lim_{t \uparrow 1} u_\omega(x, t)$ is well defined and is the restriction of an entire function to the real axis (this last assertion can be seen by employing (2.3)). Thus for ω in A, $\limsup (|g_\omega^{(n)}(0)|/n!)^{1/n} = 0$. For ω in A, $|g_\omega^{(2n+1)}(0)| = 0$ and $|g_\omega^{(2n)}(0)| = |\sum_{k=2n}^{\infty} \phi_k(\omega) a_k (k!/(k - 2n)!) v_{k-2n}(0, 1)| = |\sum_{k=n}^{\infty} \phi_{2k}(\omega)(k!/(k - n)!) e^{-\sqrt{k}}|$. Thus for ω in A,

$$\limsup \left[\frac{\left| \sum_{k=n}^{\infty} \phi_{2k}(\omega) \dfrac{k!}{(k - n)!} e^{-\sqrt{k}} \right|}{(2n)!} \right]^{1/n} = 0 .$$

Let $\delta > 0$. For $m = 0, 1, \cdots$ let

$$F_m = \left\{ \omega \in A: \left(\left| \sum_{k=n}^{\infty} \phi_{2k}(\omega) \frac{k!}{(k - n)!} e^{-\sqrt{k}} \right| \Big/ (2n)! \right)^{1/n} \leq \delta \text{ for } n = m, m + 1, \cdots \right\}$$

and note $F_m \uparrow A$. Let A and B be two numbers associated with the sequence $\{\phi_{2n}\}_{n=0}^{\infty}$ as in (3.1). Let m_0 be sufficiently large that $m(F_{m_0}) > A$. Let n_0 be an integer larger than m_0 with n_0 corresponding to F_{m_0} as in (3.1). Thus for $n \geqq n_0$ and $k \geqq 1$

$$(5.1) \quad \sum_{j=n}^{n+k}\left[\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right]^2 \leqq B\int_{F m_0}\left(\sum_{j=n}^{n+k}\phi_{2j}(\omega)\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right)^2 dm(\omega) .$$

As in the proof of Theorem 1, letting k tend to ∞ yields (5.1) with $n+k$ replaced by ∞. Using the definition of F_{m_0}, we have

$$\sum_{j=n}^{\infty}\left[\frac{j!}{(j-n)!}e^{-\sqrt{j}}\right]^2 \leqq B((2n)!\,\delta^n)^2 ,$$

for $n \geqq n_0$. From this we conclude that

$$\limsup\left[\frac{\left[\sum_{k=n}^{\infty}\left(\frac{k!}{(k-n)!}e^{-\sqrt{k}}\right)^2\right]^{1/2}}{(2n)!}\right]^{1/n} = 0 .$$

On the other hand, letting L denote this last limit superior, we have

$$L \geqq \limsup \left[\frac{\left[\sum_{k=n}^{\infty}(k-n)^{2n}\exp(-2\sqrt{k-n})\exp(-(2\sqrt{k}-2\sqrt{k-n}))\right]^{1/2}}{(2n)!}\right]^{1/n} .$$

But $\exp(-(2\sqrt{k}-2\sqrt{k-n})) \geqq e^{-2\sqrt{n}}$ for $k \geqq n$ and $\lim(e^{-\sqrt{n}})^{1/n} = 1$. Hence $L \geqq \limsup((\sum_{k=0}^{\infty}k^{2n}e^{-2\sqrt{k}})^{1/2}/(2n)!)^{1/n}$. Define h_n on $(0, \infty)$ by $h_n(x) = x^{2n}e^{-2\sqrt{x}}$. One checks that h_n is increasing on $(0, (2n)^2)$ and decreasing on $((2n)^2, \infty)$. Thus $\sum_{k=0}^{\infty}k^{2n}e^{-2\sqrt{k}} \geqq \int_0^{\infty}h_n(x)dx - h_n((2n)^2) = (\Gamma(4n+2) - 2(4n)^{4n}e^{-4n})/(2\cdot 4^{2n})$. Thus

$$L \geqq \frac{1}{4}\limsup\left[\left(\frac{\Gamma(4n+2)}{(4n)!} - \frac{2(4n)^{4n}e^{-4n}}{(4n)!}\right)((4n)!/((2n)!)^2)\right]^{1/2n} > 0 .$$

This is a contradiction. Hence in Theorem 2 we cannot replace (1/2, 1] by [1/2, 1] as the allowable range for the classification number.

The next example shows that in Theorem 1 we cannot omit the symmetry of the random variables.

EXAMPLE 2. Let $k(x, t) = e^{-x^2/4t}/(4\pi t)^{1/2}$ for $t > 0$ and define

$$u(x, t) = k(x, t+1)$$

in the strip $|t| < 1$. Then [2, Th. 4.2, p. 227]

$$u(x, t) = (4\pi)^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 4^n} v_{2n}(x, t)\,.$$

Let $\{a_n\}_{n=0}^{\infty}$ be defined by $a_{2n} = (-1)^n/n!\,4^n$ and $a_{2n+1} = 0$. One easily checks that $\limsup |a_n|^{2/n}(2n/e) = 1$ and that the classification number of $\{a_n\} = 0$. Let $X_n = 1, n = 0, 1, \cdots$ on some complete probability space. Then for every ω, u_ω can be continued above the line $t = 1$.

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