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TENSOR PRODUCTS OF COMPACT CONVEX SETS

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Suppose that K_1 and K_2 are compact convex subsets of locally convex spaces E_1 and E_2 respectively. There are several definitions of new compact convex sets associated with K_1 and K_2, each of which may reasonably be called a "tensor product" of K_1 and K_2. We compare these different tensor products and their extreme points; in doing so, we obtain some new characterizations of Choquet simplexes, another formulation of Grothendieck's approximation problem and much simpler proofs of known characterizations of the extreme points of these tensor products. Most of these results are obtained as special cases of theorems in the first half of the paper which deal with the state spaces of tensor products of partially ordered linear spaces with order unit.

1. Tensor products of partially ordered spaces. *A partially ordered linear space with order unit* is a triple (E, P, u), where the linear space E is given the partial ordering induced by the cone P, where $P \cap (-P) = \{0\}$, and where u is an order unit for P, i. e., $P - u$ absorbs E. Given a partially ordered linear space (E, P), the *dual cone* $P^\star$ is the space of all linear functionals on E which are nonnegative on P. The subspace of the algebraic dual of E which is generated by $P^\star$ is denoted by $E^\star$; it is clear that $E^\star = P^\star - P^\star$. The partially ordered linear space $(E^\star, P^\star)$ is called the *order dual* of (E, P).

If (E_1, P_1, u_1) and (E_2, P_2, u_2) are two partially ordered linear spaces with order units, then in the tensor product $E_1 \otimes E_2$ the cone generated by elements of the form $x_1 \otimes x_2$ $(x_i \in P_i)$ will be denoted by $P_1 \otimes P_2$. The triple $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ is a partially ordered linear space with order unit [3, 8].

Given a partially ordered linear space with order unit (E, P, u), its *state space* S is the set of all f in $P^\star$ such that $\langle f, u\rangle = 1$, provided with the weak* $(=w(E^\star, E))$ topology. Clearly, S is convex compact and Hausdorff. It is possible for S to be empty (cf. [7, p. 26]). If S is the state space of $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ and $s \in S$, then there exists a related functional s_1 on E_1 defined by

$$\langle s_1, x_1\rangle = \langle s, x_1 \otimes u_2\rangle , \qquad x_1 \in E_1 .$$

It is clear that s_1 is in the state space S_1 of (E_1, P_1, u_1), and it is clear how to define the analogous state s_2 in S_2. In the reverse direction, suppose that t_i is in the state space S_i of (E_i, P_i, u_i) and define the functional $t_1 \otimes t_2$ on $E_1 \otimes E_2$ by setting

$$\langle t_1 \otimes t_2, x_1 \otimes x_2 \rangle = \langle t_1, x_1 \rangle \langle t_2, x_2 \rangle , \qquad x_i \in E_i ,$$

and extending linearly. The functional $t_1 \otimes t_2$ is clearly in the state space S of $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$.

It is known [1, 3, 4] that if S_1 and S_2 are simplexes (see below), then a state s of S is an extreme point of S if and only if $s = s_1 \otimes s_2$, and each s_i is extreme in S_i. Our first few results show the extent to which this remains true without making any assumptions about the sets S_1 and S_2.

LEMMA 1.1. *Let S_1, S_2 and S be the state spaces of the partially ordered linear spaces with order unit (E_1, P_1, u_1), (E_2, P_2, u_2) and $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ respectively. If $s \in S$ and if s_1 is an extreme point of S_1, then $s = s_1 \otimes s_2$.*

Proof. Fix an element x_2 of E_2, with $0 \leqq x_2 \leqq u_2$, and define the functional f on E_1 by $\langle f, x_1 \rangle = \langle s, x_1 \otimes x_2 \rangle$. Since $0 \leqq f \leqq s_1$ and since s_1 is extreme in S_1, there exists a constant (necessarily equal to $\langle f, u_1 \rangle$) such that $f = \langle f, u_1 \rangle s_1$. Thus, for any x_1 in E_1, we have $\langle s, x_1 \otimes x_2 \rangle = \langle f, x_1 \rangle = \langle f, u_1 \rangle \langle s_1, x_1 \rangle = \langle s_2, x_2 \rangle \langle s_1, x_1 \rangle$. It follows easily from this that $s = s_1 \otimes s_2$.

If K is a nonempty compact convex subset of a locally convex space E, let $M_1^+(K)$ denote the space of all regular Borel probability measures on K and let $Q(K)$ denote the subspace of all maximal such measures on K [9]. We denote by r the affine *resultant mapping* from $M_1^+(K)$ onto K; it is characterized by the equation

$$f(r(\mu)) = \int_K f d\mu , \qquad f \in E, \mu \in M_1^+(K) .$$

The restriction r_m of r to the maximal measures $Q(K)$ is also onto, and K is called a *simplex* if r_m is one-to-one. We will say that K is *simplex-like* if there exists an affine map $\sigma : K \to M_1^+(K)$ such that $r \cdot \sigma$ is the identity map on K. (Note that we are not assuming any continuity properties for the affine cross-section σ.) If K is a simplex, then the inverse of r_m is such an affine cross-section, so every simplex is simplex-like. Eventually (Theorem 1.4), we will prove the converse to this last assertion.

We denote the set of extreme points of a convex set K by ∂K.

THEOREM 1.2. *Let S_1, S_2 and S be the state spaces of (E_1, P_1, u_1), (E_2, P_2, u_2) and $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ respectively. If S_1 is simplex-like, then for each s in ∂S we have $s = s_1 \otimes s_2$ and $s_i \in \partial S_i$.*

Proof. Let σ be an affine cross-section for r. These maps can be extended to linear maps $\bar{r}: M(S_1) \to E_1^\star$ and $\bar{\sigma}: E_1^\star \to M(S_1)$ so that $\bar{r}\cdot\bar{\sigma}$ is the identity on $E_1^\star$, $\bar{r} \geqq 0$ and $\bar{\sigma} \geqq 0$, where $M(S_1)$ is the space of all signed regular Borel measures on S_1. Given any t in S, we can define a linear map $T_t: E_2 \to E_1^\star$ by $\langle T_t(x_2), x_1\rangle = \langle t, x_1 \otimes x_2\rangle$. This map satisfies

$$(a)\ T_t \geqq 0 \text{ and } (b)\ T_t(u_2)(=t_1) \in S_1 .$$

Conversely, given any linear $T: E_2 \to E_1^\star$ which satisfies (a) and (b), there exists a unique t in S such that $T = T_t$. The correspondence $t \leftrightarrow T_t$ is an affine isomorphism, so if $s \in \partial S$, then T_s is extreme in the set of operators from E_2 to $E_1^\star$ which satisfy (a) and (b). Given s in ∂S, then, let $\mu = \sigma(s_1) \in M_1^+(S_1)$, i. e., $\mu = \sigma T_s(u_2)$. Let j be the injection map $L^\infty(S_1, \mu) \to M(S_1)$ defined by $j(f) = f\mu$ ($f \in L^\infty(S_1, \mu)$). Since u_2 is an order unit, given x_2 in E_2 there is a positive number m such that $-mu_2 \leqq x_2 \leqq mu_2$ and hence $-m\mu \leqq \bar{\sigma} T_s(x_2) \leqq m\mu$. By the Radon-Nikodym theorem there is a unique element $L(x_2)$ of $L^\infty(S_1, \mu)$ such that $L(x_2)\mu = jL(x_2) = \bar{\sigma} T_s(x_2)$. The map $L: E_2 \to L^\infty(S_1, \mu)$ is clearly linear, positive and $L(u_2) = 1$.

We wish to show that $s_1 \in \partial S_1$. If not, then $s_1 = 1/2(t_1 + t_1')$, where $t_1 \neq t_1'$ and $t_1, t_1' \in S_1$. Define linear maps T, T' from E_2 into $E_1^\star$ by $Tx_2 = \bar{r}[L(x_2)\sigma(t_1)]$, $T'x_2 = \bar{r}[L(x_2)\sigma(t_1')]$. (Since $0 \leqq \sigma(t_1) \leqq 2\mu$, for instance, $L(x_2)\sigma(t_1)$ is well-defined.) Clearly, $T, T' \geqq 0$ and $T(u_2) = t_1$, $T'(u_2) = t_1'$. Thus, T and T' satisfy (a) and (b), and $T(u_2) \neq T'(u_2)$, so $T \neq T'$. Moreover, for every x_2 in E_2, $1/2(T + T')(x_2) = \bar{r}[L(x_2)\sigma(s_1)] = \bar{r}[L(x_2)\mu] = \bar{r}\bar{\sigma}[T_s(x_2)] = T_s(x_2)$, contradicting the fact that T_s is extreme. We now know that $s_1 \in \partial S_1$, and Lemma 1.1 shows that $s = s_1 \otimes s_2$. It is straight-forward to check that $s_2 \in \partial S_2$, so the proof is complete.

The following proposition was proved in [1, 3, 4] under the additional hypothesis that each S_i was a simplex.

PROPOSITION 1.3. *Let S_1, S_2 and S denote state spaces as in Lemma 1.1. If $s_i \in \partial S_i$ ($i = 1, 2$), then $s_1 \otimes s_2 \in \partial S$.*

Proof. Suppose that t is a linear functional on $E_1 \otimes E_2$, that t_1 and t_2 are defined in the obvious way, and that $s_1 \otimes s_2 \pm t \in S$; we want to show that $t = 0$. Since $s_1 \in \partial S_1$ and since $s_1 \pm t_1 \in S_1$, we see that $t_1 = 0$. Suppose that $x_1 \in E_1$, with $0 \leqq x_1 \leqq u_1$. If $x_2 \in P_2$, then $0 \leqq x_1 \otimes x_2 \leqq u_1 \otimes x_2$ and hence

$$\begin{aligned} 0 &\leqq \langle s_1 \otimes s_2, x_1 \otimes x_2\rangle \pm \langle t, x_1 \otimes x_2\rangle \\ &\leqq \langle s_1 \otimes s_2, u_1 \otimes x_2\rangle \pm \langle t, x_1 \otimes x_2\rangle = \langle s_2, x_2\rangle \pm \langle t, x_1 \otimes x_2\rangle . \end{aligned}$$

Since s_2 is extreme (and since $\langle t, x_1 \otimes u_2 \rangle = 0$), this shows that $\langle t, x_1 \otimes x_2 \rangle = 0$ for all x_2 in E_2. Finally, since $E_1 \otimes E_2$ is generated by elements of the form $x_1 \otimes x_2$, where $x_i \in P_i$, we conclude that $t = 0$.

[The special case of [3; Th. 3.1], where all the Γ_α are trivial, follows easily from Theorem 1.2 and Proposition 1.3. By a very small modification of Theorem 1.2, one can prove Theorem 3.1 of [3] in its full generality. We leave such a modification of Theorem 1.2 (and its proof) to the reader.]

Given two partially ordered linear spaces with order unit (E_1, P_1, u_1) and (E_2, P_2, u_2), there is a second ordering one can introduce on $E_1 \otimes E_2$ (cf. [3, 8]) which may be described as follows: Note first that the spaces $E_1 \otimes E_2$ and $E_1^\star \otimes E_2^\star$ are paired by (the linear extension of) the obvious rule $\langle f_1 \otimes f_2, x_1 \otimes x_2 \rangle = \langle f_1, x_1 \rangle \langle f_2, x_2 \rangle$. We define a second cone $P^\wedge$ by

$$P^\wedge = \{x \in E_1 \otimes E_2 \colon \langle f, x \rangle \geqq 0 \text{ for all } f \text{ in } P_1^\star \otimes P_2^\star\} .$$

It is clear that $P_1 \otimes P_2 \subset P^\wedge$ and that $u_1 \otimes u_2$ is an order unit for $P^\wedge$ (since it is one for $P_1 \otimes P_2$). Let S and $S^\wedge$ be the state spaces of $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ and $(E_1 \otimes E_2, P^\wedge, u_1 \otimes u_2)$ respectively. It is evident that $S^\wedge \subset S$, and the following theorem gives conditions under which $S^\wedge = S$. The validity of "(c) implies (a)" was suggested to us by E. Effros.

THEOREM 1.4. *Let S_1 be the state space of (E_1, P_1, u_1); then the following assertions are equivalent*

(a) *S_1 is a simplex.*

(b) *S_1 is simplex-like.*

(c) *For any partially ordered linear space with order unit (E_2, P_2, u_2), the two state spaces resulting from the two orderings on $E_1 \otimes E_2$ coincide.*

Proof. We have already noted that (a) implies (b). Denoting the two state spaces by S and $S^\wedge$ as above, Theorem 1.2 shows that if S_1 is simplex-like, then every extreme point of S is of the form $s_1 \otimes s_2$. Since such functionals are clearly in $S^\wedge$, we have $\partial S \subset S^\wedge \subset S$. By the Krein-Milman theorem, then, $S = S^\wedge$. It remains to prove that (c) implies (a). Let $E = R^3$, let e_1, e_2, e_3 be the usual unit basis vectors and let P be the cone in E generated by the vectors e_1, e_2, e_3 and $e_1 + e_2 - e_3$. For an order unit u we may choose any point in the interior of P (e.g., $u = e_1 + e_2$). The order dual $(E^\star, P^\star)$ is again isomorphic to R^3, and if d_1, d_2, d_3 are basis vectors dual to e_1, e_2, e_3, then $P^\star$ is generated by $d_1, d_2, d_1 + d_3, d_2 + d_3$. (Equivalently, P is the intersection of the half-spaces through the origin defined by these four functionals.) Let S and $S^\wedge$ be the state spaces of $(E_1 \otimes E, P_1 \otimes P, u_1 \otimes u)$ and $(E_1 \otimes E, P^\wedge, u_1 \otimes u)$ respectively, and suppose that $S = S^\wedge$. We will show that $(E_1^\star, P_1^\star)$ is a vector lattice, which is equivalent to showing that S_1 is a simplex (c.f. [9].).

It is clear that the algebraic dual of $E_1 \otimes E$ can be identified with $E_1' \otimes E^\star$ (where E_1' is the algebraic dual of E_1) by means of the usual pairing between $E_1 \otimes E$ and $E_1' \otimes E^\star$, and each element of $E_1' \otimes E^\star$ can be represented uniquely as $f_1 \otimes d_1 + f_2 \otimes d_2 + f_3 \otimes d_3$, where $f_i \in E_1'$. Such an element is in $(P_1 \otimes P)^\star$ if and only if $f_i \geqq 0$ $(i = 1, 2, 3)$ and $f_1 + f_2 - f_3 \geqq 0$. It is immediate from the definition of $P^\wedge$ and the separation theorem that $(P^\wedge)^\star$ is the $w(E_1' \otimes E^\star, E_1 \otimes E)$-closure of $P_1^\star \otimes P^\star$ in $E_1' \otimes E^\star$. We claim that $P_1^\star \otimes P^\star$ is, in fact, $w(E_1' \otimes E^\star, E_1 \otimes E)$-closed. Indeed, an arbitrary element of $P_1^\star \otimes P^\star$ may be written (not uniquely) in the form $g_1 \otimes d_1 + g_2 \otimes d_2 + g_3 \otimes (d_1 + d_3) + g_4 \otimes (d_2 + d_3)$, where $g_i \in P_1$ $(i = 1, 2, 3, 4)$. Suppose that there is a net $\{g^\alpha\} = \{g_1^\alpha \otimes d_1 + g_2^\alpha \otimes d_2 + g_3^\alpha \otimes (d_1 + d_3) + g_4^\alpha \otimes (d_2 + d_3)\}$ of such elements, converging to $f_1 \otimes d_1 + f_2 \otimes d_2 + f_3 \otimes d_3$ in the $w(E_1' \otimes E^\star, E_1 \otimes E)$ topology. Then $g_1^\alpha + g_3^\alpha$ converges to f_1 in the weak* topology in $E_1^\star$, hence (in particular) $\langle g_1^\alpha, u_1\rangle + \langle g_3^\alpha, u_1\rangle \rightarrow \langle f_1, u_1\rangle$. Since $0 \leqq \langle g_i^\alpha, u_1\rangle$, the net $\{\langle g_1^\alpha, u_1\rangle\}$ is eventually bounded. But the set $\{f : f \in P_1, \langle f, u_1\rangle \leqq M\}$ is weak* compact, and hence $\{g_1^\alpha\}$ has a convergent subnet. Without loss of generality we may assume that $\lim g_1^\alpha = g_1$ exists. It then follows easily that $\lim g_i^\alpha = g_i$ exist for $i = 1, 2, 3, 4$, and since $P_1^\star$ is weak* closed, $g_i \in P_1^\star$ $(i = 1, 2, 3, 4)$. Thus, $\lim g^\alpha = g_1 \otimes d_1 + g_2 \otimes d_2 + g_3 \otimes (d_1 + d_3) + g_4 \otimes (d_2 + d_3)$ is in $P_1^\star \otimes P^\star$, which shows that the latter is $w(E_1' \otimes E^\star, E_1 \otimes E)$-closed. Consequently, $(P^\wedge)^\star = P_1^\star \otimes P^\star$.

Now, since $S = S^\wedge$, we have $(P_1 \otimes P)^\star = (P^\wedge)^\star = P_1^\star \otimes P^\star$. Thus, given f_1, f_2, f_3 in $P_1^\star$ such that $f_3 \leqq f_1 + f_2$, there exist elements g_1, g_2, g_3, g_4 in $P_1^\star$ such that

$$
\begin{aligned}
f_1 \otimes d_1 + f_2 \otimes d_2 + f_3 \otimes d_3 &= g_1 \otimes d_1 + g_2 \otimes d_2 + g_3 \otimes (d_1 + d_3) + g_4 \otimes (d_2 + d_3) \\
&= (g_1 + g_3) \otimes d_1 + (g_2 + g_4) \otimes d_2 + (g_3 + g_4) \otimes d_3 .
\end{aligned}
$$

This shows that $f_1 = g_1 + g_3, f_2 = g_2 + g_4$ and $f_3 = g_3 + g_4$, i.e., the space $(E_1^\star, P_1^\star)$ has the Riesz decomposition property (cf. [7, p. 27]). The following lemma then completes the proof.

LEMMA 1.5. *Let (E, P) be a partially ordered linear space such that $E = P - P$. Then $(E^\star, P^\star)$ is a vector lattice if and only if it has the Riesz decomposition property.*

Proof. Every vector lattice has the decomposition property, so we want to prove the converse. By a standard argument, the decomposition property implies that if $g_1, g_2 \leqq f_1, f_2$ in $E^\star$, then there exists h in $E^\star$ such that $g_1, g_2 \leqq h \leqq f_1, f_2$. Thus, if $f_1, f_2 \in P^\star$, then the set $A = \{g: 0 \leqq g \leqq f_1, f_2\}$ is directed by $\leqq$. For each x in P, the net $\{\langle g, x\rangle: g \in A\}$ is monotone and bounded, hence $\lim \{\langle g, x\rangle: g \in A\}$ exists. Since $P - P = E$, the net $\{g: g \in A\}$ converges pointwise to a positive functional on E, which is readily verified to be $f_1 \wedge f_2$. This suffices (cf. e.g., [9, p. 60]) to show that $E^\star$ is a lattice.

2. Tensor products of compact convex sets. Suppose that K is a compact convex set (always assumed to be a nonempty subset of some locally convex Hausdorff space) and let $A(K)$ (or simply A) denote the space of all real-valued continuous affine functions on K. If A^+ denotes the cone of nonnegative functions in A and 1 denotes the function identically equal to 1, then $(A, A^+, 1)$ is a partially ordered linear space with order unit. Now A is a Banach space under the supremum norm and the order dual of (A, A^+) is precisely the space A^* of all continuous linear functionals on A, ordered in the usual way (cf. [7, p. 45]). Under the evaluation mapping, K is affinely homeomorphic to the state space of $(A, A^+, 1)$.

Suppose that K_1 and K_2 are compact convex subsets of locally convex Hausdorff spaces, let $A_i = A(K_i)$ and consider the partial orderings on $A_1 \otimes A_2$ induced by the cones $A_1^+ \otimes A_2^+$ and $P^\wedge$ defined in § 1. Let $K_1 \square K_2$ denote the state space S of $(A_1 \otimes A_2, A_1^+ \otimes A_2^+, 1 \otimes 1)$ and let $K_1 \triangle K_2$ denote the state space $S^\wedge$ of $(A_1 \otimes A_2, P^\wedge, 1 \otimes 1)$. As noted before, the inclusion $K_1 \triangle K_2 \subset K_1 \square K_2$ is always valid.

If $x_i \in K_i$, then (as before) $x_1 \otimes x_2$ is defined to be the element in $K_1 \triangle K_2 \subset K_1 \square K_2$ which satisfies $\langle x_1 \otimes x_2, f_1 \otimes f_2\rangle = f_1(x_1)f_2(x_2)$, $(f_i \in A_i)$. The following theorem is a consequence of Theorem 1.2 and Proposition 1.3. A complete description of the extreme points of $K_1 \square K_2$ remains an open problem.

THEOREM 2.1. *If $x_i \in \partial K_i$ $(i = 1, 2)$, then $x_1 \otimes x_2$ is extreme in $K_1 \square K_2$. If K_1 is a simplex, then any x in $\partial(K_1 \square K_2)$ is of the form $x = x_1 \otimes x_2$, where $x_i \in \partial K_i$.*

Theorem 1.4 yields the following result. (The implication "(d) implies (a)" is contained in the *proof* of Theorem 1.4)

THEOREM 2.2. *The following assertions about a compact convex set K_1 are equivalent:*

(a) *K_1 is a simplex.*

(b) *K_1 is simplex-like.*

(c) *$K_1 \square K_2 = K_1 \triangle K_2$ for every compact convex set K_2.*

(d) *$K_1 \square K_2 = K_1 \triangle K_2$ if K_2 is a two-dimensional square.*

Denote by $BA(K_1 \times K_2)$ (or simply by BA) the space of all continuous real-valued biaffine functions on $K_1 \times K_2$, i.e., those which are continuous and affine in each variable. The space BA is, in a natural way, a partially ordered linear space with order unit, and the corresponding state space in its order dual is denoted by $K_1 \otimes K_2$. (This is the projective tensor product of K_1 and K_2 defined by Semadeni [11].)

If $f_1 \otimes f_2 \in A(K_1) \otimes A(K_2) = A_1 \otimes A_2$, then we can regard $f_1 \otimes f_2$ as an element of $BA(K_1 \times K_2)$ by

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2) .$$

This embedding of the generating elements of $A_1 \otimes A_2$ extends to a linear embedding of $A_1 \otimes A_2$ into $BA(K_1 \times K_2)$; we will henceforth regard $A_1 \otimes A_2$ as a subspace of BA. Note that this subspace contains the constant functions and separates points of $K_1 \times K_2$. The partial ordering on $A_1 \otimes A_2$ defined by the cone $P^\wedge$ is easily seen to be the same as that induced by BA, i.e., $\sum_{i=1}^n f_1^i \otimes f_2^i \in P^\wedge$ if and only if $\sum f_1^i(x_1)f_2^i(x_2) \geqq 0$ for all $(x_1, x_2) \in K_1 \times K_2$. Note, too, that the order duals of each of these spaces coincide with their duals as normed linear spaces, using the supremum norm. Let $\rho: BA(K_1 \times K_2)^* \rightarrow [A(K_1) \otimes A(K_2)]^*$ denote the restriction mapping. If $s \in K_1 \otimes K_2$, then, $\rho(s) \in K_1 \triangle K_2$. It is readily verified (using the extension theorem for positive functionals, cf. [7, p. 8] that ρ is an affine map of $K_1 \otimes K_2$ onto $K_1 \triangle K_2$ and is continuous in the weak* topologies. Let $\omega: K_1 \times K_2 \rightarrow K_1 \otimes K_2$ denote the evaluation mapping: If $f \in BA$, then $< \omega(x_1, x_2), f > = f(x_1, x_2)$ for all $(x_1, x_2) \in K_1 \times K_2$. Clearly, $\rho\omega(x_1, x_2) = x_1 \otimes x_2 \in K_1 \triangle K_2$. The two tensor products $K_1 \otimes K_2$ and $K_1 \triangle K_2$ will be isomorphic if ρ is a bijection; the conditions under which this can occur are considered below. We first investigate further the extreme points of all three tensor products. This has been done before [1, 3, 4], but except for [3] it was assumed that both sets were simplexes.

THEOREM 2.3. *Every extreme point of* $K_1 \triangle K_2$ [*or of* $K_1 \otimes K_2$] *is of the form* $x_1 \otimes x_2$ [*of the form* $\omega(x_1, x_2)$], *where* $x_i \in \partial K_i$, $i = 1, 2$. *If* $(x_1, x_2) \in \partial K_1 \times \partial K_2$, *then* $x_1 \otimes x_2$ *is extreme in* $K_1 \square K_2$ *and in* $K_1 \triangle K_2$, *and* $\omega(x_1, x_2)$ *is extreme in* $K_1 \otimes K_2$.

Proof. Suppose that M is a subspace of $C(K_1 \times K_2)$ (the continuous real valued functions on $K_1 \times K_2$) which contains the constants, so that M is (in the natural ordering) a partially ordered space with order unit. A standard argument shows that every extreme state on M is the restriction of an extreme state on $C(K_1 \times K_2)$, hence is evaluation at a point (x_1, x_2) in $K_1 \times K_2$. If the functions in M are biaffine and separate points, then $x_i \in \partial K_i$, $i = 1, 2$. By applying this observation to $M = A_1 \otimes A_2$ and to $M = BA$, we obtain the first part of the theorem.

If $(x_1, x_2) \in \partial K_1 \times \partial K_2$, then $x_1 \otimes x_2$ is extreme in $K_1 \square K_2$ by Theorem 2.1. Since $x_1 \otimes x_2 \in K_1 \triangle K_2 \subset K_1 \square K_2$, we also have $x_1 \otimes x_2$ extreme in $K_1 \triangle K_2$. Since the map $\rho: K_1 \otimes K_2 \to K_1 \triangle K_2$ is a weak* continuous affine surjection, $\rho^{-1}(x_1 \otimes x_2)$ contains an extreme point of $K_1 \otimes K_2$. *By the first part of the theorem, this functional is evaluation* at some point of $K_1 \times K_2$ which, since $A_1 \otimes A_2$ separates points of $K_1 \times K_2$, is precisely the point (x_1, x_2).

DEFINITION. A Banach space E is said to have the *approximation property* [2] if for each compact convex subset C of E and each $\varepsilon > 0$, *there exists a continuous linear transformation* (or equivalently, affine transformation) $T: E \to E$ such that $T(E)$ is finite dimensional and $\| Tx - x \| < \varepsilon$ if $x \in C$. It remains open whether every Banach space has the approximation property.

THEOREM 2.4. *The restriction mapping $\rho: K_1 \otimes K_2 \to K_1 \triangle K_2$ is one-to-one for every pair of compact convex sets K_1 and K_2 if and only if every Banach space has the approximation property.*

The proof of this proceeds by a series of simple lemmas. Note that the restriction mapping ρ is a linear operator from $BA(K_1 \otimes K_2)^*$ onto $(A_1 \otimes A_2)^*$ which is the adjoint of the embedding of $A_1 \otimes A_2$ into $BA(K_1 \times K_2)$. *Consequently,* ρ will be one-to-one on $BA(K_1 \times K_2)^*$ if and only if $A_1 \otimes A_2$ is dense (in the supremum norm topology) in $BA(K_1 \times K_2)$. Since $BA(K_1 \times K_2)^*$ is generated by $K_1 \otimes K_2$, ρ is one-to-one on BA^* if and only if it is one-to-one on $K_1 \otimes K_2$.

LEMMA 2.5. *Let K_1 be a compact convex set. If the Banach space $A(K_1)$ has the approximation property, then $A(K_1) \otimes A(K_2)$ is dense in $BA(K_1 \times K_2)$, for every compact convex set K_2.*

Proof. Suppose that K_1 and K_2 are compact convex sets and let $A[K_2, A(K_1)]$ denote the space of all affine continuous mappings $F: K_2 \to A(K_1)$ with norm $\|F\| = \sup\{\|F(x_2)\|: x_2 \in K_2\}$. If $f \in BA(K_1 \times K_2)$, then $(Fx_2)(x_1) = f(x_1, x_2)(x_i \in K_i)$ defines an element of $A[K_2, A(K_1)]$, and it is readily verified that the correspondence $f \to F$ is a linear isometry between these two spaces. Furthermore, under this isometry the subspace $A(K_1) \otimes A(K_2)$ corresponds to the set of those elements in $A[K_2, A(K_1)]$ having finite dimensional range. Suppose, now, that $A(K_1)$ has the approximation property. Given f in $BA(K_1 \times K_2)$, with corresponding function F, let $C = F(K_2)$. This is a compact convex subset of $A(K_1)$, so for any $\varepsilon > 0$ there exists a continuous linear operator $T: A(K_1) \to A(K_1)$ such that the range of T is finite dimensional and $\| Tg - g \| < \varepsilon$ for each g in C. It follows that $T \circ F$ is in $A[K_2, A(K_1)]$, has finite dimensional range, and satisfies $\| T \circ F - F \| < \varepsilon$, so $A_1 \otimes A_2$ is dense in BA.

COROLLARY 2.6. *If K_1 is a simplex and K_2 is any compact convex set, then $K_1 \otimes K_2$ is affinely homeomorphic to $K_1 \triangle K_2$.*

Proof. It is known [10] that if K_1 is a simplex, then $A(K_1)^*$ is an abstract (L)-space. From a theorem of Grothendieck [2], it follows that $A(K_1)$ has the approximation property, so the above lemma and preceding remarks show that ρ is a homeomorphism.

This corollary is also an immediate consequence of a result of Lazar [5, Lemma 3.1; 6].

LEMMA 2.7. *If $A(K_1) \otimes A(K_2)$ is dense in $BA(K_1 \times K_2)$ for each compact convex K_2, then for each compact convex $C \subset A(K_1)$ and each $\varepsilon > 0$, there exists a finite dimensional subspace $M \subset A(K_1)$ and an affine map $\varphi: C \to M$ such that $\| \varphi(c) - c \| < \varepsilon$ for each c in C.*

Proof. Let $g \in BA(K_1 \times C)$ be defined by $g(x_1, c) = c(x_1)$. By hypothesis there exists f in $A(K_1) \otimes A(C)$ such that $\| f - g \| < \varepsilon$. If φ and ψ denote the elements of $A[C, A(K_1)]$ corresponding to f and g respectively, then φ has range contained in a finite-dimensional subspace M of $A(K_1)$ and ψ is the inclusion map $C \to A(K_1)$. Hence for each c in C, $\| \varphi(c) - c \| = \| \varphi(c) - \psi(c) \| < \varepsilon$.

LEMMA 2.8. *Suppose that E is a normed linear space and that C is a nonempty compact convex subset of E. Suppose that $\varepsilon > 0$ and that there exists a finite dimensional subspace $M \subset E$ and a continuous affine map $\varphi: C \to M$ such that $\| x - \varphi x \| < \varepsilon$ for each $x \in C$. Then there exists a continuous affine map $\psi: E \to M$ such that $\| x - \psi x \| < 2\varepsilon$ for each x in C.*

Proof. Let $x_1, \cdots, x_n$ be a basis for M; then there are continuous real valued affine functions $\varphi_1, \varphi_2, \cdots, \varphi_n$ on C such that $\varphi(x) = \sum \varphi_i(x) x_i$ for $x \in C$. By [9, p. 31] we can choose continuous affine functionals $\psi_1, \cdots, \psi_n$ on E such that $| \varphi_i(x) - \psi_i(x) | < \varepsilon (\sum \| x_i \|)^{-1}$ for x in C. It follows that if $\psi(x) = \sum \psi_i(x) x_i$ for x in E, then $\| \varphi x - \psi x \| < \varepsilon$ for $x \in C$, hence $\| \psi x - x \| < 2\varepsilon$ if $x \in C$.

This result, together with Lemma 2.7, shows that under the density hypothesis of Lemma 2.7, $A(K_1)$ has the approximation property.

LEMMA 2.9. *Suppose that $A(K)$ has the approximation property, for each compact convex set K. Then every Banach space B has the approximation property.*

Proof. Let K denote the unit ball of B^*, in its weak* topology. It is easily verified that if $B \times R$ is normed by $\|(x, r)\| = \|x\| + |r|$, then the correspondence between (x, r) and the functional on K defined by $k \to \langle x, k\rangle + r$ is an isometric isomorphism between $B \times R$ and $A(K)$. Thus, if C is a compact convex subset of $B \subset B \times R$ and $\varepsilon > 0$, then there exist a finite dimensional subspace M of $B \times R$ and a continuous linear map $T: B \times R \to M$ with $\|Tx - x\| < \varepsilon$ for $x \in C$. If P denotes the natural projection $P(x, r) = x$ of $B \times R$ onto B, then $\|P\| = 1$ and the range of $P \circ T$ is finite dimensional in B. Furthermore, if $x \in C$ then $\|P(Tx) - x\| = \|P(Tx - x)\| \leqq \|Tx - x\| < \varepsilon$.

This completes the proof of Theorem 2.4.

It has been shown by Lazar [4] and by Davies and Vincent-Smith [1] that if K_1 and K_2 are simplexes, then $K_1 \otimes K_2$ is a simplex. (It follows from Corollary 2.6 and Theorem 2.2, of course, that in this case all three tensor products are the same.) The next result shows that the converse is valid. We first require a definition.

A subset F of a compact convex set K is called a *face* if F is compact convex and if $x, y \in F$ whenever $x, y \in K$ and $\alpha x + (1 - \alpha)y \in F$ for some $0 < \alpha < 1$. It is readily seen that the cone in $A(K)^*$ generated by F is a "hereditary" subcone of the cone generated by K, hence (cf. [9, p. 64]) if K is a simplex, then so is the face F.

PROPOSITION 2.10. *If K_1 and K_2 are compact convex sets and if any of the sets $K_1 \otimes K_2$, $K_1 \square K_2$ or $K_1 \triangle K_2$ is a simplex, then K_1 and K_2 are simplexes (and the three tensor products are the same).*

Proof. Since a face of a simplex is a simplex, it suffices to show that K_1 is affinely homeomorphic to a face of each of the tensor products. Choose x_2 in ∂K_2 and let $F = \{x \otimes x_2 \in K_1 \square K_2 : x \in K_1\}$. It is clear that F is affinely homeomorphic to K_1; we will show that it is a face of $K_1 \square K_2$. Suppose, then, that $x \otimes x_2 \in F$ and that $x \otimes x_2 = \alpha s + (1 - \alpha)t$, where $0 < \alpha < 1$ and $s, t \in K_1 \square K_2$. We have $x = \alpha s_1 + (1 - \alpha)t_1$ and $x_2 = \alpha s_2 + (1 - \alpha)t_2$. Since x_2 is extreme, $s_2 = x_2 = t_2$. By Lemma 1.1, this implies that $s = s_1 \otimes x_2$ and $t = t_1 \otimes x_2$, which shows that F is a face of $K_1 \square K_2$. It is clear that $F \subset K_1 \triangle K_2 \subset K_1 \square K_2$, so F is also a face of $K_1 \triangle K_2$. Finally, the inverse image of F under the restriction map $\rho: K_1 \otimes K_2 \to K_1 \triangle K_2$ is a face of $K_1 \otimes K_2$ which is easily seen to be affinely homeomorphic to K_1; it equals $\{\omega(x, x_2): x \in K_1\}$. The last assertion of the proposition was noted abave. (By "the same," we mean that $K_1 \triangle K_2 = K_1 \square K_2$ and that the map ρ is a homeomorphism between $K_1 \otimes K_2$ and $K_1 \triangle K_2$.)

3. Problems and remarks. There remain a few open questions. Although every point of the form $x_1 \otimes x_2$ (with $(x_1, x_2) \in \partial K_1 \times \partial K_2$) is extreme in $K_1 \square K_2$, there will be additional extreme points if neither of the sets is a simplex and (for instance) one of them is a square. (These additional points are easily seen to be outside of the set $\{x_1 \otimes x_2: (x_1, x_2) \in K_1 \times K_2\}$.) Is there a simple description of these points?

Another question is related to the characterization of simplexes in terms of affine cross-sections for the resultant map. Suppose that such a cross-section σ exists for the compact convex set K (so that K is a simplex); must σ coincide with the inverse of r_m? Equivalently, does σ map K into the maximal measures $Q(K)$? [Since $\sigma(x)$ is necessarily the maximal measure ε_x whenever $x \in \partial K$, we have $\sigma(\operatorname{conv} \partial K) \subset Q(K)$.][1]

It should also be noted that it is possible that $K_1 \triangle K_2$ is always a face of $K_1 \square K_2$. This is suggested by the fact (Theorem 2.3) that $\partial(K_1 \triangle K_2) \subset \partial(K_1 \square K_2)$.

We have dealt solely with tensor products of two spaces (or of two convex sets). Those results which can be formulated for finite tensor products can, however, be readily proved by appropriate induction arguments and can then be extended by standard methods to infinite tensor products. The techniques for doing this are well known [1, 3, 4] so we have restricted ourselves to the simplest case in order to exhibit the essential ideas of the proofs. Similarly, these results can be extended fairly easily to sets of states which are invariant under appropriate actions of semigroups, as was done in [3]. (The extension of Theorem 1.4 to more than two spaces requires that all but one of the state spaces be simplexes.)

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