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ON ILYEFF'S CONJECTURE

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An apparently easy problem due to Ilyeff states: If all zeros $z_1, z_2, \cdots, z_n$ of a complex polynomial $P(z)$ lie in $|z| \leqq 1$ then there is always a zero of $P'(z)$ in each of the disks $|z - z_j| \leqq 1, j = 1, \cdots, n$. If true, the conjecture is best possible as one can see from the example $P(z) = z^n - 1$. In full generality the conjectured result was proved only for polynomials of degree $\leqq 4$. In this paper the conjecture is proved for quintics and extensions of earlier results are obtained for zeros of higher derivatives of polynomials having multiple roots.

The above conjecture of Ilyeff was published in Hayman's *Research Problems in Function Theory*. Its validity for polynomials of degree $\leqq 4$ was proved in [1] and [5]. Rubinstein has shown in [5] that the statement holds in general if $|z_j| = 1$. A conjecture stronger than that of Ilyeff was announced in [2] and was proved for those zeros z_j of $P(z)$ for which $|z_j| = 1$.

2. Zeros of multiplicity k on the boundary.

THEOREM 1. *Let $P(z) = (z - z_0)^k Q(z)$, $Q(z) = \prod_{j=1}^{n-k} (z - z_j)$ with $|z_0| = 1$, and $|z_j| \leqq 1, z_j \neq z_0$ $(j = 1, \cdots, n - k)$. Then at least one zero of $P^{(\nu)}(z)$ $(1 \leqq \nu \leqq n - 1)$ lies in the disk*

$$\left| z - \frac{k}{\nu + 1} z_0 \right| \leqq 1 - \frac{k}{\nu + 1} . \tag{2.1}$$

For $\nu > k$, strict inequality will hold in (2.1) *except when $\nu = n - 1$ and $P(z) = (z - z_0)^k (z - z_1)^{n-k}$ with $|z_0| = |z_1| = 1$.*

REMARK. The conjectured result of Goodman, Rahman and Ratti [2] for zeros on the boundary is included in Theorem 1 as a special case when $k = 1, \nu = 1$.

Proof. Without loss of generality, we may assume $z_0 = 1, \nu \geqq k$. Then we easily have

$$\frac{p^{(\nu+1)}(1)}{p^{(\nu)}(1)} = \frac{\nu + 1}{\nu - k + 1} \cdot \frac{Q^{(\nu-k+1)}(1)}{Q^{(\nu-k)}(1)} . \tag{2.2}$$

Denoting the zeros of $P^{(\nu)}(z)$ by $\zeta_1, \cdots, \zeta_{n-\nu}$ and those of $Q^{(\nu-k)}(z)$ by $w_1, \cdots, w_{n-\nu}$, we have from (2.2)

$$\operatorname{Re}\sum_{j=1}^{n-\nu}\frac{1}{1-\zeta_j}=\frac{\nu+1}{\nu-k+1}\operatorname{Re}\sum_{j=1}^{n-\nu}\frac{1}{1-w_j}\,.$$

Since by Gauss-Lucas theorem we have $|w_j|\leqq 1$, it follows that

$$\operatorname{Re}(1-w_j)^{-1}\geqq\frac{1}{2}$$

for all j. Thus

$$\frac{1}{n-\nu}\sum_{j=1}^{n-\nu}\operatorname{Re}\frac{1}{1-\zeta_j}\geqq\frac{1}{2}\cdot\frac{\nu+1}{\nu-k+1}\,, \tag{2.3}$$

so that

$$\max_j\operatorname{Re}(1-\zeta_j)^{-1}\geqq\frac{1}{2}\cdot\frac{\nu+1}{\nu-k+1} \tag{2.4}$$

which is equivalent to (2.1).

In (2.3), strict inequality will hold unless all the zeros of $Q^{(\nu-k)}(z)$ lie on the unit circle. This can happen only if $z_1=z_2=\cdots=z_{n-k}$ and $|z_1|=1$. For suppose $Q^{(\nu-k)}(z)$ has p distinct zeros $w_1,\cdots,w_p$ with multiplicities $m_1,\cdots,m_p$. Then by the Gauss-Lucas theorem $Q(z)$ must have the same zeros with multiplicities $m_1+\nu-k,\cdots,m_p+\nu-k$ so that the degree of $Q(z)$ will be $n-\nu+p(\nu-k)=n-k$. Hence $p=1$, i.e., $Q^{(\nu-k)}(z)=(z-w_1)^{n-\nu}$ and so $w_1=z_1$ and $Q(z)=(z-z_1)^{n-k}$. Thus $P(z)=(z-1)^k(z-z_1)^{n-k}$ and so all zeros of $P^{(\nu)}(z)$ must lie on the line segment connecting z_1 and 1. Strict inequality will hold in (2.4) unless

$$\left|\zeta_j-\frac{k}{\nu+1}\right|=\frac{\nu-k+1}{\nu+1}\,,\qquad j=1,\cdots,n-\nu\,, \tag{2.5}$$

so that $\zeta_1=\zeta_2=\cdots=\zeta_{n-\nu}=(k/(\nu+1)+(\nu-k+1)/(\nu+1))z_1$. Since the centroid of the zeros of polynomial is invariant under differentiation, we must also have

$$\zeta_1=\frac{k+(n-k)z_1}{n}=\frac{k}{\nu+1}+\frac{\nu+k+1}{\nu+1}z_1\,,$$

so that $\nu=n-1$, which proves the assertion.

Taking $P_a(z)=(z-1)(z^2-2az+1)$ with $-1/2\leqq a\leqq 1$, we see that the zeros of $P_a'(z)$ fill the entire circumference of the circle $|z-1/2|=1/2$, so that for $\nu=k=1$, the result (2.1) cannot be improved.

3. Some lemmas. If the polynomial $P(z)=(z-z_0)^kQ(z)$, $Q(z)=\prod_{j=1}^{n-k}(z-z_j)$, $z_0\neq z_j$, $j=1,\cdots,n-k$, then as in (2.2), we have for

$\nu \geqq k$ (if $P^{(\nu)}(z_0) \neq 0$),

$$P^{(\nu+1)}(z_0) / P^{(\nu)}(z_0) \tag{3.1}$$

$$\frac{P^{(\nu+1)}(z_0)}{P^{(\nu)}(z_0)} = \frac{\nu+1}{\nu-k+1} \cdot \frac{Q^{(\nu-k+1)}(z_0)}{Q^{(\nu-k)}(z_0)} \, . \tag{3.1}$$

Denoting the zeros of $P^{(\nu)}(z)$ by $\zeta_1, \cdots, \zeta_{n-\nu}$ and of $Q^{(\nu-k)}(z)$ by $w_1, \cdots, w_{n-\nu}$ and setting

$$|z_0 - w_j| = r_j, \, |z_0 - \zeta_j| = \rho_j, \, j = 1, \cdots, n-\nu,$$

$(r_1 \leqq r_2 \leqq \cdots \leqq r_{n-\nu}; \rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_{n-\nu})$ we have

$$\sum_{j=1}^{n-\nu} \frac{1}{z_0 - \zeta_j} = \frac{\nu+1}{\nu-k+1} \sum_{j=1}^{n-\nu} \frac{1}{z_0 - w_j} \, . \tag{3.2}$$

$$\prod_{j=1}^{n-\nu} r_j = \frac{\binom{n}{k}}{\binom{\nu}{k}} \prod_{j=1}^{n-\nu} \rho_j \, , \tag{3.3}$$

where the last relation follows from the fact that

$$\begin{aligned} P^{(\nu)}(z_0) &= \binom{n}{\nu} \nu! \prod_{j=1}^{n-\nu} (z_0 - \zeta_j) \\ &= \binom{n-k}{\nu-k} \nu! \prod_{j=1}^{n-\nu} (z_0 - w_j) \, . \end{aligned}$$

In the sequel we shall need the following lemmas.

LEMMA 1. *Let* $f(z) = \sum_{j=0}^{n} \binom{n}{j} a_j z^j$, $g(z) = \sum_{j=0}^{n} \binom{n}{j} b_j z^j$, $h(z) = \sum_{j=0}^{n} \binom{n}{j} a_j b_j z^j$, *and suppose that the zeros of* $f(z)$ *lie in the annulus* $p \leqq |z| \leqq q$, *and those of* $g(z)$ *lie in* $r \leqq |z| \leqq s$, *then the zeros of* $h(z)$ *lie in* $pr \leqq |z| \leqq qs$.

This lemma is a special case of a theorem due to Szego [4; p. 65, Th. 16.1]. In particular if $R(t)$ is a polynomial of degree $n-k$, and $f(t) = d^\nu/dt^\nu\{t^k R(t)\}$ and $h(t) = R^{(\nu-k)}(t) (\nu \geqq k)$, then an easy computation shows that the polynomial $g(t)$ of the above lemma may be chosen, except for a constant factor, as follows:

$$g(t) = \sum_{j=0}^{n-\nu} \frac{\binom{n-\nu}{j}\binom{n}{k}}{\binom{\nu+j}{k}} t^j \, . \tag{3.3a}$$

LEMMA 2. *Let* $r_1, \cdots, r_m$ *and* a, b, c $(a^m \leqq c \leqq b^m)$ *be positive numbers satisfying*

$$a \leqq r_j \leqq b \tag{3.4}$$

$$\prod_{j=1}^{m} r_j \geqq c\ . \tag{3.5}$$

Then

$$\sum_{j=1}^{m} \frac{1}{r_j^2} \leqq \frac{m-\mu}{a^2} + \frac{\mu-1}{b^2} + \left(\frac{a^{m-\mu}b^{\mu-1}}{c}\right)^2 \tag{3.6}$$

where

$$\mu = \min\{\nu \mid b^\nu a^{m-\nu} \geqq c,\ \nu\ \textit{integer}\}\ . \tag{3.7}$$

Proof. We first observed that the maximum of $\sum_1^m r_j^{-2}$ is not attained unless equality holds in (3.5) for if $\prod_{j=1}^m r_j > c$, then at least one of the r_j's say r_1 is strictly greater than a and so replacing it by $(1-\varepsilon)\cdot r_1$ with a suitable ε, we can increase the sum $\sum r_j^{-2}$.

Also at most one of the r_j's can lie in the open interval (a, b). For if we had for some i and j, $a < r_i \leqq r_j < b$, then replacing r_i by $r_i/1+\varepsilon$, and r_j by $r_j(1+\varepsilon)$ with suitable ε, such that (3.4) and (3.5) remain valid, the sum $\sum r_j^{-2}$ would be increased by

$$\frac{(1+\varepsilon)^2-1}{r_i^2} + \frac{(1+\varepsilon)^{-2}-1}{r_j^2}$$

which is strictly positive.

So to maximize $\sum r_j^{-2}$, we must have

$$r_1 = r_2 = \cdots = r_{m-\nu} = a \leqq r_{m-\nu+1} \leqq r_{m-\nu+2} = \cdots = r_m = b$$

so that from $a^{m-\nu} r_{m-\nu+1} b^{\nu-1} = c$ we obtain

$$a^{m-\nu+1}\cdot b^{\nu-1} < c \leqq a^{m-\nu}b^\nu$$

which gives (3.7).

LEMMA 3. *Let $0 < \alpha \leqq 1$ and suppose w is a point in the closed unit disk. Then*

$$\operatorname{Re}\frac{1}{\alpha-w} \geqq \frac{1}{2\alpha} - \frac{1-\alpha^2}{2\alpha}\cdot\frac{1}{r^2}\,, \quad r = |\alpha - w|\ . \tag{3.8}$$

The proof follows from elementary geometric considerations.

4. Zeros inside the disk.

We shall prove the theorems:

THEOREM 2. *If $P(z) = (z-z_0)^k Q(z)$, $(k \geqq 1, n \geqq 2+k)$, $|z_0| \leqq 1$, $Q(z) = \prod_1^{n-k}(z-z_j)$, $z_j \neq z_0$, $|z_j| \leqq 1$ $(j = 1, \cdots, n-k)$, then at least one zero of $P^{(n-2)}(z)$ lies in the closed disk*

$$(4.1) \qquad |z - z_0| \leqq \frac{2(n-k-1)}{n-1}\sqrt{\frac{n-1+|z_0|}{n}}\,.$$

REMARKS. (i) When $n = 3$, $k = 1$, the theorem asserts the existence of a zero of $P'(z)$ in $|z - z_0| \leqq \sqrt{2 + |z_0|/3}$ which implies the Ilyeff's conjecture in this case. A comparison of (4.1) with (2.1) in the special case $n = 4$, $k = 1$, $\nu = 2$ and $z_0 = 1$ shows that Theorem 1 asserts the existence of a zero of $P''(z)$ in $|z - 1/3| \leqq 2/3$, while (4.1) does so in the disk $|z - 1| \leqq 4/3$. However, Theorem 2 holds even when $|z_0| < 1$.

(ii) Under the hypothesis of Theorem 2, it is possible to replace the right side of (4.1) by

$$\frac{n-k-1}{n-1}\theta(z_0)$$

where $\theta(z_0) = |z_0| + \sqrt{2 - |z_0|^2}$, which for large values of n yields a disk smaller than the one given by (4.1).

Proof. Without loss of generality, we may take $z_0 = \alpha$, $0 \leqq \alpha \leqq 1$. Setting in Lemma 1, $f(t) = P^{(n-2)}(\alpha + t) = (d^{n-2}/dt^{n-2})(t^k Q(\alpha + t))$ and $\mathrm{h}(t) = Q^{(n-2-k)}(\alpha + t)$, we have by (3.3a)

$$g(t) = t^2 + \frac{2n}{n-k}t + \frac{n(n-1)}{(n-k-1)(n-k)}\,.$$

For the zeros β_1 and β_2 of $g(t)$, we have

$$|\beta_1|^2 = |\beta_2|^2 = \frac{n(n-1)}{(n-k)(n-k-1)}\,.$$

Assuming that $\rho_1 \leqq \rho_2$ and $r_1 \leqq r_2$ (see notation proceeding (3.2)) we have by Lemma 1,

$$(4.2) \qquad \rho_1\sqrt{\frac{n(n-1)}{(n-k)(n-k-1)}} \leqq r_1 \leqq r_2\,,$$

whence

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} \leqq \frac{(n-k)(n-k-1)}{n(n-1)}\frac{2}{\rho_1^2}\,.$$

Suppose now the theorem is false. Then

$$(4.3) \qquad \rho_2 \geqq \rho_1 > \frac{2(n-k-1)}{n-1}\sqrt{\frac{n-1+\alpha}{n}}\,,$$

and thus

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} < \frac{1}{2}\,\frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)}\,. \tag{4.4}$$

Also from (4.3) and (4.2) and from $n - k \geqq 2$, we have $r_1 > 1$, which for $\alpha = 0$ yields the desired contradiction.

If $\alpha \neq 0$ then from (3.2), (4.4) and Lemma 3 we get

$$\begin{aligned}
&\frac{1}{2}\operatorname{Re}\sum_{j=1}^{2}\frac{1}{\alpha-\zeta_j}\\
&\quad > \frac{n-1}{2(n-k-1)}\left\{\frac{1}{\alpha} - \frac{1-\alpha^2}{4\alpha}\cdot\frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)}\right\}\\
&\quad \geqq \frac{n-1}{2(n-k-1)}\cdot\frac{1}{2\alpha}\left\{2 - \frac{(1-\alpha^2)(n-1)}{n-1+\alpha}\right\} \qquad (4.5)\\
&\quad > \frac{n-1}{4\alpha(n-k-1)}\left\{1 + \alpha^2\cdot\frac{n}{n-1+\alpha}\right\}\\
&\quad \geqq \frac{n-1}{2(n-k-1)}\sqrt{\frac{n}{n-1+\alpha}}\,,
\end{aligned}$$

observing that $n - k \leqq 2(n-k-1)$. Since $1/\rho_j \geqq \operatorname{Re} 1/\alpha - \zeta_j$, $j = 1, 2$, (4.5) yields a contradiction to (4.3) which completes the proof of the theorem.

THEOREM 3. *Suppose $P(z) = (z-z_0)^k Q(z)$, $(k \geqq 1, 2k \leqq n-2)$, $|z_0| \leqq 1$, $Q(z) = \prod_{j=1}^{n-k}(z - z_j)$, $z_j \neq z_0$, $|z_0| \leqq 1$ $(j = 1, \cdots, n-k)$. Then at least one zero of $P^{(n-3)}(z)$ lies in the disk*

$$|z - z_0| \leqq \frac{(n-k-2)}{n-2}\theta(z_0) \tag{4.6}$$

where $\theta(z_0) = |z_0| + \sqrt{2 - |z_0|^2}$.

REMARK. (i) In the special case $n = 4$, $k = 1$, the above theorem gives an improvement on Theorem 2 of [5], since it guarantees the existence of a zero of $P'(z)$ in $|z - z_0| \leqq 1/2(|z_0| + \sqrt{2 - |z_0|^2}) < 1$ if $|z_0| \neq 1$.

(ii) In case $2k > n - 2$, $n \geqq k + 3$ we can prove that under the conditions of Theorem 3, the disk $|z - z_0| \leqq (n - k - 1/n - 1)\theta(z_0)$ will contain at least one zero of $P^{(n-3)}(z)$. In particular the disk

$$|z - z_0| \leqq \frac{1}{2}\theta(z_0) \leqq 1$$

will include at least one zero of $P^{(n-3)}(z)$ when

$$k > \frac{n-2}{2}\,.$$

Proof. As in Theorem 2, we set $z_0 = \alpha$, $0 \leqq \alpha \leqq 1$ and identify the polynomials $f(t)$, $g(t)$ and $h(t)$ of Lemma 1, as follows:

$$f(t) \equiv P^{(n-3)}(\alpha + t),\, h(t) = Q^{(n-3-k)}(\alpha + t)\ ,$$

and except for a constant factor

$$g(t) = \sum_{j=0}^{3} \binom{n}{j}\binom{3}{j} t^{3-j} \Big/ \binom{n-k}{j}\ .$$

Since $g'(t) > 0$ for real t, it follows that $g(t)$ has exactly one real zero. A straightforward substitution yields

$$g\left(-\frac{n}{n-k-1}\right) \leqq 0 \leqq g\left(\frac{n-2}{n-k-2}\right)$$

on using the assumption $2k \leqq n-2$. So denoting the zeros of $g(t)$ by t_1, t_2, t_3 then for the real zero, say t_3, we have

$$\frac{n-2}{n-k-2} \leqq |t_3| \leqq \frac{n}{n-k-1}\ .$$

Since $\bar{t}_2 = t_1$, and $|t_1 t_2 t_3| = |t_1|^2 |t_3| = \binom{n}{3} \Big/ \binom{n-k}{3}$, we obtain

$$\begin{aligned}\frac{(n-1)(n-2)}{(n-k)(n-k-2)} &\leqq |t_1|^2 \\ &\leqq \frac{n(n-1)}{(n-k)(n-k-1)} \leqq \left(\frac{n-2}{n-k-2}\right)^2 \leqq |t_3|^2\ .\end{aligned}$$

Now by Lemma 1 (using the notation of § 3)

$$\rho_1^2 \frac{(n-1)(n-2)}{(n-k)(n-k-2)} \leqq r_1^2 \leqq r_2^2 \leqq r_3^2\ . \tag{4.7}$$

Suppose the theorem were not true, i.e.,

$$\rho_1 > \frac{n-k-2}{n-\alpha}(\alpha + \sqrt{2-\alpha^2})\ . \tag{4.8}$$

Then for all α, $\rho_1 > (n-k-2/n-2)\sqrt{2}$ which would imply that

$$r_1^2 > 2 \cdot \frac{(n-1)(n-k-2)}{(n-2)(n-k)} \geqq 1\ . \tag{4.9}$$

For $\alpha = 0$, this already gives a contradiction. If $0 < \alpha \leqq 1$, then from (3.2) with $\nu = n-3$, from (4.7), and (4.8) and Lemma 3 we have

$$\frac{1}{3} \operatorname{Re} \sum_{j=1}^{3} \frac{1}{\alpha - \zeta_j} = \frac{1}{3} \frac{n-2}{(n-k-2)} \operatorname{Re} \sum_{j=1}^{3} \frac{1}{\alpha - w_j}$$

$$> \frac{n-2}{n-k-2} \cdot \left[\frac{1}{2\alpha} - \frac{1-\alpha^2}{2\alpha} \cdot \frac{(n-k)(n-2)}{(n-1)(n-k-2)} \frac{1}{\theta^2(\alpha)} \right].$$

Since $(n-k)(n-2) \leqq 2(n-1)(n-k-2)$ and $|\mathrm{Re}\,(\alpha - \zeta_j)^{-1}| \leqq 1/\rho_j$, we have

$$\frac{1}{3}\sum_{j=1}^{3} \frac{1}{\rho_j} > \frac{n-2}{2(n-k-2)} \left\{ \frac{1}{\alpha}\left(1 - \frac{2}{\theta^2}\right) + \frac{2\alpha}{\theta^2} \right\}$$
$$\geqq \frac{n-2}{(n-k-2)} \cdot \frac{1}{\theta}.$$

Therefore

$$\rho_1 < \frac{n-k-2}{n-2}\theta$$

which contradicts (4.8). This completes the proof of Theorem 3.

5. Quintic polynomials.

We shall prove the

THEOREM 4. *If $P(z) = (z - z_0)Q(z)$, $Q(z) = \prod_1^4 (z - z_j)$, $|z_j| \leqq 1$ $(j = 0, 1, \cdots, 4)$, then at least one zero of $P'(z)$ lies in the disk*

$$|z - z_0| \leqq \frac{1}{2}\sqrt{2 - |z_0|^2)}\,. \tag{5.1}$$

REMARK. This in particular proves Ilyeff's conjecture for quintics since the right side of (5.1) is < 1 if $|z_0| < 1$.

Proof. Without loss of generality we may assume $z_0 \neq z_j$ $(j = 1, \cdots, 4)$ and $0 \leqq z_0 \leqq 1$.

From (3.3) with $n = 5$, $\nu = 1$, we have

$$r_1 r_2 r_3 r_4 = 5\rho_1\rho_2\rho_3\rho_4\,. \tag{5.2}$$

Now identifying in Lemma 1, $f(t)$ with $P'(z_0 + t)$, $h(t)$ with $Q(z_0 + t)$, $g(t)$ becomes, except for a constant factor, the polynomial

$$t^{-1}[(1 + t)^5 - 1]$$

whose zeros t_1, t_2, t_3, t_4 satisfy

$$|t_1|^2 = |t_2|^2 = 4 \sin^2 \frac{\pi}{5},\ |t_3|^2 = |t^4|^2 = 4 \sin^2 \frac{2\pi}{5}$$

and $t_1 t_2 t_3 t_4 = 5$. It follows then from Lemma 1 that

$$\rho_1 \cdot |t_1| \leqq r_j \leqq \rho_4 \cdot |t_4|\,, \qquad (j = 1, \cdots, 4)\,. \tag{5.3}$$

From Lemma 2, (5.2) and (5.3) we conclude that $\sum_{j=1}^{4} r_j^{-2}$ cannot be larger than the corresponding expression for

$$r'_1 = r'_2 = |t_1|\rho_1,\ r'_3 = \frac{\rho_2\rho_3}{\rho_1}|t_4|,\ r'_4 = |t_4|\rho_4 .$$

Thus on using $\rho_1 \leqq \rho_2 \leqq \rho_3 \leqq \rho_4$ and $|t_1|^{-2} + |t_4|^{-2} = 1$, we have

$$\sum_{j=1}^{4} r_j^{-2} \leqq \sum_{j=1}^{4} r'^{-2} \leqq 2\rho_1^{-2} . \tag{5.4}$$

If $z_0 \neq 0$, then on using Lemma 3 and (3.2) with $k = 1, \nu = 1, n = 5$, we have from (5.4)

$$\frac{4}{\rho_1} \geqq \operatorname{Re} \sum_{j=1}^{4} \frac{1}{z_0 - \zeta_j} \geqq 2\left\{\frac{2}{z_0} - \frac{1 - z_0^2}{2z_0}\cdot\frac{2}{\rho_1^2}\right\} ,$$

from which the result follows by elementary calculation. If $z_0 = 0$, then $r_j \leqq 1$ $(j = 1, 2, 3, 4)$ and so by (5.2) $\rho_1 \leqq 5^{-(1/4)} < 2^{-(1/2)}$. This completes the proof.

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