PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 2, 1969

APPROXIMATION OF TRANSFORMATIONS WITH CONTINUOUS SPECTRUM

R. V. CHACON

In several recent papers a new approach has been developed in the theory of approximation of automorphisms. Using this approach, Katok and Stepin have developed a new method which is very powerful and which has enabled them to solve several problems which had remained open for some time. Among the results they obtained is a characterization of automorphisms which are not strongly mixing in terms of the speed with which they can be approximated. Counter-examples may be given to show that the speed of approximation cannot be used to characterize those automorphisms which have continuous spectrum. In this paper certain related concepts are developed which do make it possible to deal in general with automorphisms which have continuous spectrum, and to distinguish those which are not strongly mixing from those which are strongly mixing among them. Since it is well-known that automorphisms have continuous spectrum if and only if they are weakly mixing, the result serves to distinguish between strong and weak mixing.

2. Notation and preliminaries. Let $(X, \mathfrak{F}, \mu)$ be the unit interval, the Lebesgue sets and Lebesgue measure. A map T of X onto X is an automorphism of X if it is invertible and measure preserving, i.e., if $A \in \mathfrak{F}$ then $T^{-1}A$, $TA \in \mathfrak{F}$ and $\mu(A) = \mu(T^{-1}A) = \mu(TA)$. If $B \in \mathfrak{F}$ we say that a map T of B onto B is an automorphism of B if it is an automorphism with B regarded as a measure space, that is with respect to $(B, \mathfrak{F} \cap B, \mu_B)$ where μ_B is defined for measurable subsets C of B by setting $\mu_B(C) = \mu(C)/\mu(B)$. As usual all statements are understood to hold almost everywhere, and we'll omit this phrase.

LEMMA 2.1. [3]. *The set $\mathscr{U}$ of automorphisms of X is a topological group with respect to the weak topology, that is, the topology obtained by taking neighborhoods to be finite intersections of sets of the form* $\{S: S \in \mathscr{U}, \mu(SE \triangle TE) < \varepsilon\}$, $E \in \mathfrak{F}$.

In what follows it will be convenient to let $0 \cdot B = \varnothing, 1 \cdot B = B$ for sets $B \in \mathfrak{F}$. In what follows also, the topological space we refer to is $\mathscr{U}$, equipped with the weak topology.

DEFINITION 2.1. We say that ξ is a tower if $\xi = \{C_i, i = 1, \cdots, q\}$, an ordered collection of pairwise disjoint measurable sets of the same measure whose union $A = A_\xi \subset X$. Note that we haven't assumed that $A = X$. If $B \in \mathfrak{F}$ by $B(\xi)$ we mean a set of the form

$$\bigcup_{i=1}^{q} a(i)C_i, \qquad a(i) = 0 \quad \text{or} \quad 1,\ i = 1, \cdots, q$$

for which the measure $\mu(B \triangle \bigcup_{i=1}^{q} a(i)C_i)$ is minimized. If $\xi(n) = \{C_{n,i}, i = 1, \cdots, q(n)\}$ is a sequence of towers, we write $\xi(n) \to \in$ as $n \to \infty$ provided $\mu(B \triangle B(\xi(n)) \to 0$ as $n \to \infty$ for each set $B \in \mathfrak{F}$.

DEFINITION 2.2. Let $\xi = \{C_i, i = 1, \cdots, q\}$ be a tower and let S be an automorphism. We say that S maps the elements of the tower almost cyclically if there exists an integer m, $1 \leqq m \leqq q$ such that the collection of sets $\{S^k C_m, k = 0, \cdots, q\}$ is the same as the collection (without order) of sets which make up the tower ξ.

DEFINITION 2.3. Let $\{\xi(n)\}$ be a sequence of towers. By $\mathscr{M}(\{\xi(n)\})$ we mean the class of automorphisms S with the property that there exists an integer $N(S)$ such that for $n \geqq N(S)$, S maps the elements of $\xi(n)$ almost cyclically.

DEFINITION 2.4. Let $\xi(1)$ and $\xi(2)$ be towers, $\xi(1) = \{C_{1,i}, i = 1, \cdots, q(1)\}$ and $\xi(2) = \{C_{2,i}\, i = 1, \cdots, q(2)\}$. We write $\xi(1) < \xi(2)$ if each set of $\xi(1)$ is the sum of some of the sets of $\xi(2)$. We say that a sequence of towers $\{\xi(n)\}$ is nested if $\xi(n) < \xi(n+1)$, $n = 1, 2, 3, \cdots$.

LEMMA 2.2. *Let $\{\xi(n)\}$ be a nested sequence of towers such that $\xi(n) \to \varepsilon$ as $n \to \infty$. Then the set of automorphisms $\mathscr{M}(\{\xi(n)\})$ is dense in $\mathscr{U}$.*

Proof. Let $T \in \mathscr{U}$. We suppose first that T is antiperiodic, and we need to show that in each neighborhood of T there is an automorphism $S' \in \mathscr{M}$. Let $E_1, \cdots, E_k \in \mathfrak{F}$ and $\varepsilon_1, \cdots, \varepsilon_k$ define the neighborhood, i.e., the neighborhood is given by

$$\{S\colon \mu(SE_1 \triangle TE_1) < \varepsilon_1, \cdots, \mu(SE_k \triangle TE_k) < \varepsilon_k\}\ .$$

It follows from [3 p. 71] that for each $\varepsilon > 0$ there is a set E and an integer m such that the sets $E, TE, \cdots$. $T^m E$ are pairwise disjoint, $\mu(E) < \varepsilon$ and $\mu(\bigcup_{k=0}^{m} T^k E) > 1 - \varepsilon$. We next consider the partition induced on E by the sets

$$\{E_1, T^{-1}E_1, \cdots, T^{-m}E_1, E_2, \cdots, T^{-m}E_2, \cdots, E_k, \cdots, T^{-m}E_k\}$$

and denote its elements $\{A_1, \cdots, A_s\}$. It is clear that

$$\bigcup_{i=1}^{s} T^k A_i = T^k E, \qquad k = 0, \cdots, m$$

and that each set $E_j \cap T^k E$ can be written as the union of some of the sets in $\mathscr{A}_k = \{T^k A_1, \cdots, T^k A_s\}$, $k = 0, \cdots, m$. Since $\xi(n) \to \varepsilon$ for each $\varepsilon' > 0$ there is an N such that each of the sets in the finite collection $\mathscr{A} = \bigcup_{k=0}^{m} \mathscr{A}_k$ can be written as the sum of some of the sets in $\xi(N)$ using these sets at most once and, for each i, $1 \leqq i \leqq s$, using the same number of sets to represent the sets $T^k A_i$, $k = 0, \cdots, m$, with a total error of ε' (in the sense of the measure of the symmetric difference). We let $\xi(N) = \{C_{n,i}, i = 1, \cdots, q(N)\}$ and define S' on $\{C_{n,i}, i = 1, \cdots, q(N)\}$ arbitrarily but so as to map $C_{N,r}$ linearly onto $C_{N,p}$ provided $C_{N,r}$ was used to represent $T^j A_l$ and $C_{N,p}$ was used to represent $T^{j+1} A_l$, $j = 0, \cdots, m-1$, $l = 1, \cdots, s$. On the remaining $C_{N,i}$ S' is defined arbitrarily but so as to have S' map the intervals of $\{C_{N,i}, i = 1, \cdots, q(n)\}$ almost cyclically onto each other. We next extend S' (so that it is defined on X) in stages, so as to have $S' \in \mathscr{M}(\{\xi(n)\})$. It is clear that by taking ε and ε' sufficiently small, S' will be in the required neighborhood.

We next suppose that T is periodic of periodic m so that there exists a set E such that the sets $E, TE, \cdots, T^m E$ are pairwise disjoint and have union equal to X. The automorphism S' is defined as in the anti-periodic case, so as to map $C_{N,r}$ onto $C_{N,p}$ provided $C_{N,r}$ was used to represent $T^j A_l$ and $C_{N,p}$ to represent $T^{j+1} A_l$ for $j = 0, \cdots, m-1$, $l = 1, \cdots, S$. We also, however, define S' so as to map a set of the form $C_{N,r}$ onto a set of the form $C_{N,p}$ provided $C_{N,r}$ was used to represent $T^m A_l$ and $C_{N,p}$ was used to represent A_l, also arbitrarily, but in such a way that $S^m C_{N,r} \neq C_{N,p}$. It is clear that by taking ε sufficiently small (ε' is not present in this case), S' will again be in the required neighborhood.

To complete the proof, we consider an arbitrary automorphism T and apply the arguments already given to its anti-periodic and periodic components.

3. Principal definitions. We first give an extended version of the type of approximation referred to in [4] as cyclic approximation by periodic transformations with speed $f(n)$. We have preferred to call this type of approximation cyclic with speed $f(n)$ since the definition we give does not require that the T_n be periodic.

DEFINITION 3.1. We say that the automorphism T admits of (cyclic) approximation with speed $f(n)$ if we can find towers $\xi(n)$ with $A_{\xi(n)} = A(n) \subset X$ composed of a finite number $q(n)$ of measurable sets $C_{n,i} \subset A(n)$, $i = 1, \cdots, q(n)$ and automorphisms T_n defined on

$(A(n), \mathfrak{F} \cap A(n), \mu_n)$, for which

(1) $\xi(n) \to \epsilon$ as $n \to \infty$,

(2) T_n maps the elements of $\xi(n)$ cyclically, $T_n C_{n,i} = C_{n,i+1}$, $i = 1, \cdots, q(n-1)$, $T_n C_{n,q(n)} = C_{n,1}$,

(3) $\mu_n(T_n \neq T) < f(q(n))$,

and where μ_n is defined for measurable subsets B of $A(n)$ by setting $\mu_n(B) = \mu(B)/\mu(A(n))$.

REMARK 3.1. The automorphism T_n is not necessarily periodic on $A(n)$, although the transformation it defines on the factor space $A(n)/\xi(n)$ is and has period $q(n)$.

REMARK 3.2. A simple calculation shows that an automorphism admits of cyclic approximation by periodic transformations with speed $f(n)$ in the sense of Katok and Stepin if and only if it admits of cyclic approximation with speed $f(n)/2$ in our sense, with $A(n) \equiv X$.

DEFINITION 3.2. A sequence $\{n(k)\}$ is called an m-pair sequence if

$$n(2k) = 1 + mn(2k-1),\ k = 1, 2, 3, \cdots.$$

We say that an automorphism T admits of approximation in m-pairs with speed $f(n)$ if it admits of cyclic approximation with speed $f(n)$, and if the sequence $\{q(n)\}$, where $q(n)$ is the number of elements in $\xi(n)$, has a subsequence which is an m-pair sequence.

We next state several results obtained by Katok and Stepin in [4]. Since only minor modifications are needed in their proofs to account for our more general definition we refrain from giving the proofs.

THEOREM 3.1 [4]. *The set of automorphisms admitting of approximation with a fixed speed $f(n)$ contains a G_δ set which is everywhere dense in $\mathscr{U}$.*

THEOREM 3.2 [4]. *If an automorphism T admits of approximation with speed θ/n, $\theta < 2$, then T is ergodic.*

THEOREM 3.3 [4]. *If an automorphism T admits of approximation with speed θ/n, $\theta < 1$, then T is not strongly mixing.*

THEOREM 3.4 [4]. *Strongly mixing automorphisms form a set of the first category in the group $\mathscr{U}$.*

REMARK 3.3. Theorem 3.4 was originally obtained by Rokhlin using a different method. It follows directly from Theorems 3.1 and 3.3.

REMARK 3.4. Theorem 3.2 is obtained in [4] as a consequence of a more general result. Since we have no occasion to use it here, we chose not to state the more general result for simplicity, although it is also valid. Theorem 3.3 is obtained in [4] in a somewhat more general form, using a slightly different type of approximation which can also be put into our form. Since we make no use of the added generality we have stated the result in this way for the sake of simplicity also.

4. Main results. In this section we obtain results analogous to Theorems 3.1, 3.3, and 3.4, for automorphisms with continuous spectrum or what is the same thing as we have remarked, for automorphisms which are weakly mixing. Theorem 4.2 is a key result, as it implies that the theory is not vacuous and is used in the proofs of the remaining results.

THEOREM 4.1. *If the automorphism T admits of approximation in m-pairs with speed θ/n, $\theta < 1$, then T is weakly mixing.*

Proof. For any integer k we have that $\mu_n(T^k \neq T_n^k) \leqq k\mu_n(T \neq T_n)$ so that $\mu_n(T^{q(n)} \neq T_n^{q(n)}) \leqq \theta$. To see this for $k = 2$, note that

$$\begin{aligned}\mu_n(T^2 \neq T_n^2) &= \mu_n(T = T_n,\ T^2 \neq T_n^2) + \mu_n(T \neq T_n,\ T^2 \neq T_n^2)\\ &= \mu_n(T = T_n,\ TT_n \neq T_nT_n) + \mu(T \neq T_n,\ T^2 \neq T_n^2)\\ &\leqq \mu_n(TT_n \neq T_nT_n) + \mu_n(T \neq T_n)\\ &= \mu_n(T \neq T_n) + \mu_n(T \neq T_n)\ .\end{aligned}$$

This implies that

$$\mu(T^{q(n)} = T_n^{q(n)}) \geqq (1-\theta)(1-\delta(n)) \tag{1}$$

since

$$\mu_n(T^{q(n)} = T_n^{q(n)}) = 1 - \mu_n(T^{q(n)} \neq T_n^{q(n)}) \geqq 1 - \theta,$$

and since $\mu_n(C) = \mu(C)/\mu(A(n))$ where $A(n) = \bigcup_{i=1}^{q(n)} C_{n,i}$, so that we obtain equation (1), where $\delta(n) = \mu(D(n))$, and $D(n) = cA(n)$. The sets of the tower $\xi(n)$, $C_{n,i}$, $i = 1, \cdots, q(n)$ and the complement of their union $D(n)$ form a partition of X we therefore have, for $f(x)$ a fixed eigenfunction with $|f| = |\lambda| = 1$, where λ is the eigenvalue of $f(x)$, that

$$f(x) = \sum_{j=1}^{q(n)} f(x)\chi_{C_{n,j}}(x) + f(x)\chi_{D(n)}(x)$$

where χ_C is the characteristic function of C. Further, since $\xi(n) \to \epsilon$, given $\varepsilon > 0$ there exists a set $E = E(n) \subset A(n)$ of measure less than ε, $q(n)$ constants $f_1, \cdots, f_{q(n)}$ and a function $\varepsilon(x)$ such that

$$(2) \qquad f(x) = \sum_{j=1}^{q(n)} (f_j + \varepsilon(x))\chi_{C_{n,j}}(x) + f(x)\chi_{D(n)}(x)$$

where $|\varepsilon(x)| < \varepsilon$, $x \in cE$.

Since $f(x)$ is an eigenfunction with eigenvalue λ we have that

$$(3) \qquad \lambda^{q(n)} f(x) = \sum_{j=1}^{q(n)} (f_j + \varepsilon'(x))\chi_{C'_{n,j}}(x) + f'(x)\chi_{D'(n)}(x) ,$$

where

$$\varepsilon'(x) = \varepsilon(T^{-q(n)}x),\ C'_{n,j} = T^{q(n)}C_{n,j},\ f'(x) = f(T^{-q(n)}x) ,$$

and $D'(n) = T^{q(n)}D(n)$. If we let $B(n) = \bigcup_{j=1}^{q(n)} T^{q(n)}C_{n,j} \cap C_{n,j}$, equation (1) implies that $\mu(B(n)) \geqq (1 - \theta)(1 - \delta(n))$ and it follows from equation (2) that for $x \in cE \cap B(n)$ we have

$$(4) \qquad f(x) = \sum_{j=1}^{q(n)} (f_j + \varepsilon(x))\chi_{C_{n,j} \cap C'_{n,j}}(x) .$$

Further, equation (3) implies that for $x \in cT^{q(n)}E \cap B(n)$ we have

$$(5) \qquad \lambda^{q(n)} f(x) = \sum_{j=1}^{q(n)} (f_j + \varepsilon'(x))\chi_{C_{n,j} \cap C'_{n,j}}(x) .$$

Equations (4) and (5) in turn imply that

$$(6) \qquad \lambda^{q(n)} = \sum_{j=1}^{q(n)} \left(\frac{f_j + \varepsilon(x)}{f_j + \varepsilon'(x)}\right)\chi_{C_{n,j} \cap C'_{n,j}}(x) ,$$

on $F(n) = B(n) \cap cE \cap cT^{q(n)}E$. By choosing $\varepsilon > 0$ sufficiently small we can make the measure of $F(n)$ as close to $1 - \theta$ as we like, and we also have that $|\varepsilon(x)| \leqq \varepsilon$, $|\varepsilon'(x)| \leqq \varepsilon$. Since $|f(x)| = 1$ we may assume that $|f_j| = 1$ (T is ergodic by Theorem 3.2) and (6) implies that on $F(n)$

$$(7) \qquad |\lambda^{q(n)} - 1 \leqq | \frac{2\varepsilon}{1 - \varepsilon} ,$$

where $\mu(F(n)) > 0$.

Since we have assumed that $\{q(n)\}$ contains an m-pair sequence we may apply (7) with $q(n) = k$ and with $q(n) = 1 + mk$ to obtain in the first case that ($|\lambda| = 1$)

$$(8)\quad \begin{aligned} |\lambda^{1+mk} - 1| &= |\lambda\lambda^{mk} - \lambda^{mk} + \lambda^{mk} - 1| \\ &= |\lambda^{mk}(\lambda - 1) + (\lambda^{mk} - \lambda^{(m-1)k}) \\ &\quad + \cdots + (\lambda - 1)| \geqq |\lambda - 1| - \frac{2m\varepsilon}{1-\varepsilon} \end{aligned}$$

and in the second that

$$(9)\quad |\lambda^{1+mk} - 1| \leqq \frac{2\varepsilon}{1-\varepsilon}\,.$$

There two inequalities imply that

$$|\lambda - 1| \leqq \frac{2(m+1)\varepsilon}{1-\varepsilon}\,,$$

from which it follows that $\lambda = 1$

DEFINITION 4.1. A 1-tower is a tower ξ (see Definition 2.1) composed of subintervals of the unit interval, so that ξ is a 1-tower if $\xi = \{I_j, j = 1, \cdots, q\}$ where the $I_j, j = 1, \cdots, q$ are pairwise disjoint intervals of the same length whose union is contained in the unit interval. The transformation $T(\xi)$ induced by the 1-tower ξ is defined as the linear map of I_j onto $I_{j+1}, j = 1, \cdots, q-1$ so that the domain of $T(\xi)$ is $\bigcup_{j=1}^{q-1} I_j$ and the range is $\bigcup_{j=2}^{q} I_j$. The geometric figure which we associate with $T(\xi)$ is an ordered stack of the intervals I_j, $j = 1, \cdots, q$ with I_1 on the bottom and I_q on the top, so that each point lies below its image. This enables us to regard the action of $T(\xi)$ as an upward flow through the stack, the flow stopping at I_q where $T(\xi)$ is undefined.

A k-tower ξ is an ordered collection of k 1-towers $\xi_1, \cdots, \xi_k$ such that all intervals are pairwise disjoint. The transformation $T(\xi)$ induced by the k-tower ξ is defined as $T(\xi_j)$ on the domain of $T(\xi_j)$, $j = 1, \cdots, k$. Since all intervals are pairwise disjoint, these domains are pairwise disjoint.

DEFINITION 4.2. Let ξ_1 and ξ_2 be 1-towers having all intervals pairwise disjoint and of equal length, and let $\xi_1 = \{I_{1,1}, \cdots, I_{q(1),1}\}$, $\xi_2 = \{I_{1,2}, \cdots, I_{q(2),2}\}$. By $\xi_1^*\xi_2$ we mean the 1-tower defined by $\xi_1^*\xi_2 = \{I_{1,1}, \cdots, I_{q(1),1}, I_{1,2}, \cdots, I_{q(2),2}\}$ so that the stack associated with $\xi_1^*\xi_2$ is obtained by placing the stack associated with ξ_2 on the top of the stack associated with ξ_1. Note also that $\xi_1^*\xi_2 \neq \xi_2^*\xi_1$.

DEFINITION 4.3. Let ξ be a 1-tower, $\xi = \{I_1, \cdots, I_q\}$ and write each interval I_j, $j = 1, \cdots, q$ as the sum of k consecutive intervals of the same length (and therefore of length $1/k$, the common length of the I_j, $j = 1, \cdots, q$), $I_j = I_j^1 + \cdots + I_j^k$, $j = 1, \cdots, q$. By $\gamma(k, \xi)$ we mean the 1-tower obtained by setting $\gamma(k, \xi) = \xi_1^* \xi_2^* \cdots {}^* \xi_k$ where $\xi_l = \{I_1^l, \cdots, I_q^l\}$, $l = 1, \cdots, k$. The stack associated with $\gamma(k, \xi)$ is obtained by splitting the stack associated with ξ into k consecutive and equal substacks, and placing the second over the first and then the third over the second until finally we place the kth substack over the first $k - 1$ substacks arranged one over the other.

DEFINITION 4.4. Fix the positive integer m and define $R_1 = R_1(m)$ as the interval $(0, 1/2m - 1)$, $D_{1,1} = D_{1,1}(m)$ as the interval $(1/2m - 1, m/2m - 1)$ and $D_{1,2} = D_{1,2}(m)$ as the interval $(m/2m - 1, 1)$ so that $D_{1,1}$ and $D_{1,2}$ have the same length, and consequently $\eta(1) = \eta(1, m) = \{D_{1,i}(m), i = 1,2\}$ is a 1-tower. We next suppose that R_n and $\eta(n)$ are given and define R_{n+1} and $\eta(n + 1)$ inductively as follows. We first form $\gamma(m, \eta(n))$ (see Definition 4.3) and then remove from the right side of R_n an interval $I(n)$ of length equal to the length of the intervals in $\gamma(m, \eta(n))$, and then define

$$R_{n+1} = R_n - I(n), \qquad \eta(n + 1) = \gamma(m, \eta(n))^* \rho(n) ,$$

where $\rho(n)$ is the 1-tower consisting of the single interval $I(n)$ (so that $\rho(n) = \{I(n)\}$).

REMARK 4.1. Definition 4.4 gives us a sequence of intervals $\{R_n(m)\}$ such that for each fixed m, $\mu(R_n(m)) \to 0$ as $n \to \infty$ and a sequence of 1-towers $\{\eta(n, m)\}$ such that for each fixed m, $\{\eta(n, m)\}$ is a nested sequence of towers with $\eta(n, m) \to \in$ as $n \to \infty$.

REMARK 4.2. The geometric interpretation of Definition 4.4 is the following. We start with three consecutive intervals, the last two of which have the same length. The lengths of the three intervals are chosen so that they add to one and so that the length of the first is just enough to permit the following sequence of operations. At the outset we suppose that the last two intervals are stacked, with the third over the second. This stack is split into m equal substacks which are stacked in order, and an extra interval is extracted from the right hand side of the first interval. This yields an interval and a stack (composed of $2m + 1$ intervals). We then repeat, obtaining an interval and a stack (composed of $m(2m + 1) + 1$ intervals), and so on.

REMARK 4.3. If we let $q(n)$ be the number of elements in $\eta(n)$, then we have that $q(1) = 2$, $q(2) = m \cdot q(1) + 1$, and in general that $q(n+1) = mq(n) + 1$.

THEOREM 4.2. *Let m be a positive integer and let $\theta > 1/m$. Then there exists an automorphism $T(m)$ admitting of approximation in m-pairs with speed θ/n.*

Proof. We define $T = T(m)$ by setting

$$T = \lim_{n\to\infty} T(\eta(n, m))$$

where $\{\eta(n, m)\}$ is the sequence of towers of Definition 4.4. It follows easily from Remark 4.1 that T is an automorphism which is well defined. We next define T_n on $\bigcup_{i=1}^{q(n)} D_{n,i}$, where

$$\eta(n) = \{D_{n,i}, i = 1, \cdots, q(n)\}$$

as $T(\eta(n+1))$ on the domain of $T(\eta(n+1))$ (which is contained in $\bigcup_{i=1}^{q(n)} D_{n,i}$), and as the linear map of $D_{n+1,q(n+1)}$ onto $D_{n+1,1}$. If we define T_n in this way, then T_n maps the sets $D_{n,i}$, $i = 1, \cdots, q(n)$ cyclically in order and we also have that

$$\mu_n(T \neq T_n) = \frac{1}{mq(n)}\,.$$

Since the other conditions are satisfied by Remarks 4.1 and 4.3, we see that T admits of approximation in m-pairs with speed θ/n, $\theta = 1/m$.

THEOREM 4.3. *The set of automorphisms admitting of approximation in m-pairs with speed θ/n, $\theta > 1/m$ contains an everywhere dense G_δ set (in $\mathscr{U}$ with respect to the weak topology).*

Proof. Let $\eta(n, m) = \eta(n) = \{D_{n,i}, i = 1, \cdots, q(n)\}$ be as in Definition 4.4. Let $\mathscr{U}_n$ be the set of automorphisms which permute the $D_{n,i}$, $i = 1, \cdots, q(n)$, cyclically in any order. Define $\mathscr{S}_n$ by setting

$$\mathscr{S}_n = \bigcup_{\substack{U_n \in \mathscr{U}_n \\ U_{n+1} \in \mathscr{U}_{n+1}}} \left\{T \colon \mu_n(T \neq U_n) < \frac{\theta}{q(n)}\,,\ \mu_{n+1}(T \neq U_{n+1}) < \frac{\theta}{q(n+1)}\right\},$$

where μ_n and μ_{n+1} are the measures obtained when μ is restricted to $\bigcup_{i=1}^{q(n)} D_{n,i}$ and to $\bigcup_{i=1}^{q(n+1)} D_{n+1,i}$ and normalized. A simple calculation shows that $\mathscr{S}_n$ is open in the weak topology (note that the $\mathscr{U}_n$ are not assumed linear), and so $\mathscr{S} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \mathscr{S}_n$ is a G_δ set. The automorphism $T(m)$ of Theorem 4.2 is in $\mathscr{S}_n$ for all n. We know by Lemma 2.2 that $\mathscr{M}(\{\eta(n)\})$ is dense in $\mathscr{U}$ and since $S \in \mathscr{M}\{\eta(n)\}$ implies that there exists an $N = N(S)$ such that for $n \geqq N$, S maps the intervals $D_{n,i,i=1,\ldots,q(n)-1}$, almost cyclically onto each other, it follows that $S \in \mathscr{S}_n$, $n > N(S)$, and therefore $\mathscr{M}\{\eta(n)\} \subset \mathscr{S}$.

THEOREM 4.4. *The automorphisms admitting of approximation in m-pairs with speed θ/n, $\theta = 1/m$ are weakly mixing, but not strongly mixing and they contain an everywhere dense G_δ set (in $\mathscr{U}$ with respect to the weak topology).*

Proof. Follows immediately from Theorems 4.1, 3.3, and 4.3.

REMARK 4.4. Theorem 4.4 implies a result due to Halmos [2] that the weakly mixing automorphisms contain an everywhere dense G_δ set.

BIBLIOGRAHPY

1. R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, to appear.
2. P. R. Halmos, *In general a measure-preserving transformation is mixing*, Annals of Math., (2) 45 (1944), pp. 786–792.
3. ———, *Lectures on ergodic theory*, Math. Soc. of Japan, 1956, (Chelsea).
4. A. B. Katok, and A. M., Stepin, *Approximations in ergodic theory*, Uspekhi Math. Nauk, (5) 22 (1967), pp. 81–106. Translation: Russian Math. Surveys (5) 22, 1967, pp. 77–102.
5. V. A. Rokhlin, *In general a measure-preserving transformation is not mixing*, Dokl. Akad. Nauk SSSR (N. S.) 60 (1948), pp. 349–351.

Received January 17, 1969. Research supported in part by NSF Grant GP-7401.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINNESOTA