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GROUPS WITH MAXIMUM CONDITIONS

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It still seems to be unknown whether there exist Noetherian groups (= groups with maximum condition on subgroups) that are not almost polycyclic, i.e., possess a soluble normal subgroup of finite index. However, the existence of even finitely generated infinite simple groups shows that in general a group whose subnormal subgroups satisfy the maximum condition need not be almost polycyclic. The following theorem gives a number of criteria for a group satisfying a weak form of the maximum condition to be almost polycyclic.

THEOREM. *The following conditions of the group G are equivalent*:

(I) *G is almost polycyclic.*

(II) (a) *If C is a characteristic subgroup of G, then C is finitely generated.*
(b) *Every infinite epimorphic image H of G possesses a locally almost soluble characteristic subgroup $N \neq 1$.*

(III) (a) *If C is a characteristic subgroup of G, then C is finitely generated.*
(b) *Every infinite epimorphic image H of G possesses a locally almost polycyclic accessible subgroup $E \neq 1$.*

(IV) (a) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*
(b) *Every infinite epimorphic image H of G possesses an almost radical accessible subgroup $E \neq 1$.*

(V) (a) *If the normal subgroup N of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of N.*
(b) *Every infinite epimorphic H of G possesses a normal subgroup $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.*

(VI) (a) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*
(b) *Every infinite epimorphic image H of G possesses a characteristic subgroup $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.*

(a1) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*

(VII) (a2) *The maximum condition is satisfied by the normal subgroups of G.*

(b) *Every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ with $\mathfrak{c}_H N \neq 1$.*

(VIII) (a1) *If G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of G.*

(a2) *Abelian normal subgroups of epimorphic images of G are finitely generated.*

(b) *Every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$* with $\mathfrak{c}_H N \neq 1$.

REMARKS. G. Higman [9] has constructed an infinite finitely generated simple group. This group satisfies part (a) of every condition (II) to (VIII) of the theorem without being almost polycyclic. Hence part (b) of the conditions (II) to (VIII) is indispensable. Every group C_{p^∞} of Prüfer's type satisfies part (b) of every condition (II) to (VIII) of the theorem without being almost polycyclic. Hence part (a) of the conditions (II) to (VIII) is likewise indispensable. It is well known that a group G generated by two elements a and b with the relation $b^{-1}ab = a^2$ is metabelian and satisfies the maximum condition for normal subgroups without being almost polycyclic. This group satisfies conditions (VII. a2) and (VII. b) as well as (VIII. a1) and (VIII. b) so that (VII. a1) and (VIII. a2) are indispensable. The existence of infinite locally finite simple groups shows that conditions (II. a) and (III. a) cannot be replaced by (IV. a) or (V. a). We have been unable to decide whether or not conditions (VII. a2) and (VIII. a1) are indispensable. From the proof of the equivalence of (I) and (II) it may easily be seen that one gets a similar criteria if the word 'characteristic subgroup' in (II) is replaced by the word 'normal subgroup'.

NOTATIONS.

$\{\cdots\}$ = subgroup generated by the elements enclosed in braces.

$\mathfrak{z}G$ = center of the group G.

$\mathfrak{c}_G X$ = centralizer of the subset X of G in G.

$G^{(0)} = G$.

$G^{(i+1)} = G^{(i)}$ = commutator subgroup of $G^{(i)}$.

Factor = epimorphic image of a subgroup.

A subgroup U of the group G is Γ-admissible for the automorphism group Γ of G if every element in Γ maps U onto U.

Two subgroups A and B of G are automorphic if there exists an automorphism of G mapping A onto B.

A normal series is a well ordered set of subgroups X_ν of the group G with $0 \leqq \nu \leqq \tau$ such that X_ν is a normal subgroup of $X_{\nu+1}$ for $\nu < \tau$ and $X_\lambda = \bigcup_{\nu<\lambda} X_\nu$ for limit ordinals $\lambda \leqq \tau$; $X_{\nu+1}/X_\nu$ is a factor of the series

A subgroup U is accessible if there exists a normal series from U to G.

Soluble group = group with $G^{(i)} = 1$ for almost all i.

Noetherian group = group with maximum condition on subgroups.

Polycyclic group = Noetherian and soluble group.

Nilpotent group = group G with a finite central series from 1 to G.

Let $\mathfrak{e}$ be any group theoretical property.

A group is an $\mathfrak{e}$-group if it has the property $\mathfrak{e}$.

A group G is almost-$\mathfrak{e}$ if there exists a normal $\mathfrak{e}$-subgroup N of G with finite G/N.

A group G is locally-$\mathfrak{e}$ if every finitely generated subgroup of G is an $\mathfrak{e}$-group.

A group G is radical if every epimorphic image $H \neq 1$ of G possesses a locally nilpotent accessible subgroup $E \neq 1$.

In the proof of the theorem we need several lemmas most of which slightly extend known results. We recall that a group G is finitely presented if there exists a free group F of finite rank and a normal subgroup R of F generated by finitely many classes of elements conjugate in F such that $F/R \cong G$. Every almost polycyclic group is finitely presented; see R. Baer [3], p. 276, Folgerung 3.

LEMMA 1. *If $\mathfrak{e}$ is any class of finitely presented groups, if the finitely generated group G is not an $\mathfrak{e}$-group and if Γ is a group of automorphisms of G, then there exists a Γ-admissible normal subgroup N of G such that G/N is not an $\mathfrak{e}$-group, but G/M is an $\mathfrak{e}$-group, for every Γ-admissible normal subgroup M of G containing N properly.*

Proof. The set $\mathfrak{M}$ of all Γ-admissible normal subgroups X of G such that G/X is not an $\mathfrak{e}$-group is not empty, since it contains 1. Let $\mathfrak{T}$ be any nonempty subset of $\mathfrak{M}$ such that $X \subseteq Y$ or $Y \subset X$ for every pair X, Y of Γ-admissible normal subgroups in $\mathfrak{T}$. If V denotes the union of the elements in $\mathfrak{T}$, then V is likewise a Γ-admissible normal subgroup of G. If V is not contained in $\mathfrak{M}$, then G/V is an $\mathfrak{e}$-group and hence finitely presented. Since G is finitely generated, there exists a finite subset T of V such that $V = \{T^G\}$; see R. Baer [3], p. 270, Satz 1. Since V is the union of the elements of $\mathfrak{T}$, for every x in T there exists a Γ-admissible normal subgroup X^* in $\mathfrak{T}$ containing x. Since the subgroups in $\mathfrak{T}$ are comparable and since T is finite, there exists a Γ-admissible normal subgroup Y in $\mathfrak{T}$ such that $X^* \subseteq Y$ for every x in T. Thus T is a subset of the normal subgroup Y of G such that $V = \{T^G\} \subseteq Y \subseteq V$ so that $V = Y$ belongs to $\mathfrak{M}$. This contradiction shows that V is an element of $\mathfrak{M}$. We have shown that the maximum principle of set theory is applicable and that there exists therefore a maximal Γ-admissible normal subgroup N in $\mathfrak{M}$. Since N is contained in $\mathfrak{M}$, the epimorphic image $H = G/N$ of G is not an $\mathfrak{e}$-group. However, if M is a Γ-admissible normal subgroup of G with $N \subset M$, then the maximality of N implies that G/M is an $\mathfrak{e}$-group.

COROLLARY 2. *If the finitely generated group G is not almost polycyclic, then there exists an epimorphic image H of G which is not almost polycyclic, but every proper epimorphic image of H is almost polycyclic. Furthermore, there exists a characteristic subgroup C of G such that G/C is not almost polycyclic, but G/D is almost polycyclic for every characteristic subgroup D of G containing C properly.*

Proof. The class $\mathfrak{e}$ of almost polycyclic groups is finitely presented. Therefore the two statements follow immediately from Lemma 1 if Γ is the group of all inner automorphisms of the group G or the group of all automorphism of G respectively.

A set $\mathfrak{M}$ of normal subgroups of the group G is *independent*, if their product is direct.

LEMMA 3. *If 1 is the only finite characteristic subgroup of the group G, if 1 is the only finite Abelian accessible subgroup of G, and if independent sets of finite simple isomorphic normal subgroups of characteristic subgroups of G are finite, then 1 is the only finite accessible subgroup of G.*

Proof. If this statement is false, then there exists a finite accessible subgroup $M \neq 1$ of G, and we can assume that M is minimal. Our hypotheses imply that M is non-Abelian. If β is an automorphism of G, then the image M^β of M is automorphic to M and is likewise a finite simple non-Abelian accessible subgroup of G. Thus the subgroup M^* of G generated by all the subgroups of G which are automorphic to M is a characteristic subgroup of G which possesses a normal series with finite factors leading from 1 to G. It follows that M^* is locally finite; see for instance R. Baer [5], p. 53, bottom. *If* A and B are two different subgroups of G which are automorphic to M, then $V = \{A, B\}$ is finite, since it is a finitely generated subgroup of M^*. A and B are also accessible subgroups of the finite group V, so that A and B are subnormal subgroups of V. Application of H. Wielandt, [12], p. 463 (1. a), shows that A and B normalize each other. Thus A, B and $A \cap B$ are normal subgroups of V. Since $A \neq B$ and A and B are simple, we have $A \cap B = 1$. It follows that A and B centralize each other. Since all subgroups of G automorphic to M are finite and centralize each other pairwise, M^* is a direct product of finite simple groups automorphic to M. The hypotheses of our lemma now imply that M^* is a finite characteristic subgroup of G, which is impossible. Thus the lemma is proved.

COROLLARY 4. *If* 1 *is the only Abelian accesible subgroup of the group* G *and if independent sets of finite normal subgroups of characteristic subgroups are finite, then the product* P *of all finite normal subgroups of* G *is finite and* 1 *is the only almost Abelian accessible subgroup of* $H = G/P$.

Proof. Clearly the product P of all finite normal subgroups of G is a characteristic subgroup of G, so that independent sets of finite normal subgroups of P are finite. Application of R. Baer [7], p. 26, Lemma 5.1, now yields that P is finite.

If E is a finite normal subgroup of $H = G/P$, then there exists a normal subgroup X of G with $P \subseteqq X$ and $E = X/P$. Since P and E are finite, X is also finite, thus, X must be contained in P. This implies $E = 1$, and we have shown:

(1) 1 is the only finite normal subgroup of H.

Now let $h \neq 1$ be an element of H which generates an accessible subgroup $\{h\}$ of H. It follows from K. Grünberg, [8], p. 158, Th. 2, or R. Baer [5], p. 57, Satz 3.3, that the set T of all elements of H which generate accessible subgroups of H is a locally nilpotent characteristic subgroup of H. Since $h \neq 1$, we have $T \neq 1$, so that T is infinite by (1).

Let Q be the uniquely defined characteristic subgroup of G such that $P \subset Q$ and $T = Q/P$. Since 1 is the only Abelian accessible subgroup of G, we have $1 = \mathfrak{z}P = \mathfrak{c}_G P \cap P$. The finiteness of P implies the finiteness of $G/\mathfrak{c}_G P$. If $Q \cap \mathfrak{c}_G P = 1$, then

$$Q = Q/(Q \cap \mathfrak{c}_G P) \cong Q\mathfrak{c}_G P/\mathfrak{c}_G P \subseteqq G/\mathfrak{c}_G P$$

is also finite. But $T = Q/P$ infinite implies that Q is infinite. Hence $Q \cap \mathfrak{c}_G P \neq 1$. If $a \neq 1$ is an element in $Q \cap \mathfrak{c}_G P$, then Pa is an element in T and therefore $\{Pa\}$ is an accessible subgroup of T and H; see R. Baer [5], p. 59, Zusatz 3.6. It follows that $\{P, a\}/P$ is an accessible subgroup of the locally nilpotent group T; see K. Grünberg, [8], p. 158, Lemma 7, or R. Baer [5], p. 48, Lemma 1.4. Hence $\{P, a\}$ is an accessible subgroup of Q and G, and this implies that $\{P, a\} \cap \mathfrak{c}_G P$ is an accessible subgroup of $\mathfrak{c}_G P$ and G. Since $\{a\} \subseteqq \mathfrak{c}_G P$ the application of Dedekind's Modular Law yields

$$\{P, a\} \cap \mathfrak{c}_G P = P\{a\} \cap \mathfrak{c}_G P = \{a\}(P \cap \mathfrak{c}_G P) = \{a\} \neq 1 .$$

Thus there exists a cyclic accessible subgroup of G, which contradicts our hypotheses, and we have shown:

(2) 1 is the only Abelian accessible subgroup of H.

If U is any almost Abelian accessible subgroup of H, then (2) implies that U is finite. The statements (1) and (2) show that the hypotheses of Lemma 3 are satisfied by H. Thus $U = 1$, and our assertion is proved.

PROPOSITION 5. *Let $N \neq 1$ be a normal subgroup of the group G such that $G/\mathfrak{c}_G N$ is almost polycyclic. Then there exists an almost Abelian normal subgroup $A \neq 1$ of G. If N is a characteristic subgroup of G, then A is a characteristic subgroup of G.*

Proof. If $N \cap \mathfrak{c}_H N \neq 1$, then $\mathfrak{z}N$ is an Abelian normal subgroup of G, and clearly $\mathfrak{z}N$ is even characteristic in G whenever N is characteristic in G. If $N \cap \mathfrak{c}_H N \neq 1$, then

$$N = N/(N \cap \mathfrak{c}_G N) \cong N\mathfrak{c}_G N/\mathfrak{c}_G N \subseteqq G/\mathfrak{c}_G N\,,$$

so that N is isomorphic to a subgroup of the almost polycyclic group $G/\mathfrak{c}_G N$. It follows that N is likewise almost polycyclic, and there exists a soluble characteristic subgroup S of N with finite N/S; see R. Baer [3], p. 276, Satz 3. If $S = 1$, then N is a nontrivial finite normal subgroup of G. If $S \neq 1$, there exists an Abelian characteristic subgroup $A \neq 1$ of N, which is a nontrivial Abelian normal subgroup of G. Clearly, A is also characteristic in S, N and G whenever N is a characteristic subgroup of G.

REMARK. The above proposition may be generalized easily.

LEMMA 6. *If 1 is the only almost Abelian normal subgroup of the group G, and if every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ such that $\mathfrak{c}_H N \neq 1$, then every nontrivial normal subgroup of G possesses an infinite independent set of normal subgroups of G.*

Proof. If $X \neq 1$ is a normal subgroup of G, then our hypotheses imply that X is infinite. Since 1 is the only Abelian normal subgroup of G, we have $X \cap \mathfrak{c}_G X = \mathfrak{z}X = 1$. This implies that $X\mathfrak{c}_G X/\mathfrak{c}_G X \cong X/(X \cap \mathfrak{c}_G X) = X$ and therefore $G/\mathfrak{c}_G X$ are infinite. As in the proof of R. Baer [6], p. 177, Folgerung 5.2, one shows by using Lemma 5.1 of this paper that every nontrivial normal subgroup of G possesses an infinite set of independent normal subgroups of G.

COROLLARY 7. *If every independent set of infinite normal subgroups of any epimorphic image of the group G is finite, then the following two properties of G are equivalent*:

(I) *Every infinite epimorphic image H of G possesses an almost Abelian normal subgroup $N \neq 1$.*

(II) *Every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ such that $\mathfrak{c}_H N \neq 1$.*

Proof. Let G be a group satisfying (I) and let H be an infinite epimorphic image of G. Then there exists an almost Abelian normal subgroup $N \neq 1$ of H. If N is finite, then $H/\mathfrak{c}_H N$ is finite, so that $\mathfrak{c}_H N$ is infinite. If N is infinite, then there exists an Abelian characteristic subgroup A of N with finite N/A; see R. Baer [2], p. 152, Lemma 2. Clearly A is an infinite normal subgroup of H with $\mathfrak{c}_H A \neq 1$. Thus (I) implies (II).

Conversely, let condition (II) be satisfied by G, and let H be an infinite epimorphic image of G. Then every independent set of infinite normal subgroups of H is finite, and Lemma 6 shows the existence of an almost Abelian normal subgroup $N \neq 1$ of H. Thus (II) implies (I), and our assertion is proved.

LEMMA 8. *Let G be a group satisfying the following condition*:

($\mathfrak{M}$) *If the characteristic subgroup C of G is not finitely generated, then the maximum condition is satisfied by the normal subgroups of C.*

Then the following conditions hold:

(a) *If A and B are characteristic subgroups of G with $A \subseteq B$, then B/A likewise satisfies* ($\mathfrak{M}$).

(b) *Products of independent finite normal subgroups of characteristic subgroups of G are finite.*

(c) *The product $\mathfrak{R}G$ of all almost polycyclic characteristic subgroups of G is an almost polycyclic characteristic subgroup of G.*

(d) *1 is the only almost radical accessible subgroup of $G/\mathfrak{R}G$.*

(e) *$\mathfrak{R}G$ contains every almost radical accessible subgroup of G.*

(f) *If G is not almost polycyclic, then there exists an epimorphic image H of G such that H/C is almost polycyclic for every characteristic subgroup $C \neq 1$ and 1 is the only almost radical accessible subgroup of H; H satisfies* ($\mathfrak{M}$).

Proof. It is easy to see that every characteristic subgroup and every factor group modulo a characteristic subgroup of a group with property ($\mathfrak{M}$) likewise satisfies ($\mathfrak{M}$). This implies (a).

Let C be a characteristic subgroup of G and let $\mathfrak{S}$ be an independent set of nontrivial finite normal subgroups of C. Then the product P of all finite normal subgroups of C is a locally finite characteristic subgroup of C different from 1. Thus P is finite, if it is finitely generated. If P is not finitely generated, then by ($\mathfrak{M}$) the normal subgroups of P satisfy the maximal condition. Then P is the product of finitely many finite groups and hence finite. The finiteness of P implies the finiteness of $\mathfrak{S}$, since every element of $\mathfrak{S}$ is contained in P. This proves (b).

Clearly the product $\mathfrak{N}G$ of all almost polycyclic characteristic subgroups of G is a characteristic subgroup of G which satisfies ($\mathfrak{M}$). Assume $\mathfrak{N}G$ is not almost polycyclic. If $\mathfrak{N}G$ is not finitely generated, then by ($\mathfrak{M}$) the normal subgroups of $\mathfrak{N}G$ satisfy the maximum condition. It follows that $\mathfrak{N}G$ is the product of finitely many almost polycyclic characteristic subgroups of G. This implies that $\mathfrak{N}G$ is likewise almost polycyclic, since every extension of an almost polycyclic group by an almost polycyclic group is almost polycyclic; see for instance W.R. Scott, [11], p. 150, 7. 1. 2. Hence $\mathfrak{N}G$ is finitely generated. Since $\mathfrak{N}G$ is not almost polycyclic, Corollary 2 shows the existence of an epimorphic image K of $\mathfrak{N}G$ with the following properties:

(1) K is not almost polycyclic, but every proper epimorphic image of K is almost polycyclic.

Clearly K is infinite. Since $\mathfrak{N}G$ is the product of almost polycyclic normal subgroups, K is likewise the product of almost polycyclic normal subgroups. Hence there exists an almost polycyclic normal subgroup $N \neq 1$ of K. By (1) K/N is almost polycyclic, and this implies that K is almost polycyclic, since every extension of an almost polycyclic group by an almost polycyclic group is almost polycyclic. Since this contradicts (1), we have proved (c).

If $C \neq 1$ is an almost polycyclic characteristic subgroup of $G/\mathfrak{N}G$, then there exists a characteristic subgroup D of G such that $\mathfrak{N}G \subset D$ and $C = D/\mathfrak{N}G$ is almost polycyclic. Since $\mathfrak{N}G$ and D are almost polycyclic, D is an almost polycyclic characteristic subgroup of G and thus contained in $\mathfrak{N}G$. This contradiction shows:

(2) 1 is the only almost polycyclic characteristic subgroup of $G/\mathfrak{N}G$.

Assume there exists a nontrivial radical accessible subgroup of $G/\mathfrak{N}G$. Then there exists also a nontrivial locally nilpotent accessible subgroup of $G/\mathfrak{N}G$, and the subgroup S generated by all locally nilpotent accessible subgroups of $G/\mathfrak{N}G$ is a nontrivial locally nilpotent characteristic subgroup of $G/\mathfrak{N}G$; see R. Baer [5], p. 57, Lemma 3. If S is finitely generated, then S is a finitely generated nilpotent group and therefore Noetherian and polycyclic; see R. Baer [1], p. 299, Satz B. This contradicts (2) so that S is not finitely generated. Since

$\mathfrak{R}G$ is a characteristic subgroup of G, by (a) $G/\mathfrak{R}G$ satisfies condition ($\mathfrak{M}$), and the normal subgroups of S fulfill the maximum condition. This implies that S is Noetherian and polycyclic, since S is locally nilpotent; see D.H. McLain, [10], Theorem 3.2, p. 10. This contradicts (1), and we have shown:

(3) 1 is the only radical accessible subgroup of $G/\mathfrak{R}G$.

By (b) independent sets of finite normal subgroups of characteristic subgroups of $G/\mathfrak{R}G$ are finite. Application of Corollary 4 yields that the product P of all finite normal subgroups of $G/\mathfrak{R}G$ is finite and that 1 is the only almost Abelian accessible subgroup of $(G/\mathfrak{R}G)/P$. It is a consequence of (2) that $P = 1$. This together with (3) implies that 1 is the only almost radical accessible subgroup of $G/\mathfrak{R}G$. We have proved (d).

If the almost radical accessible subgroup E of G is not contained in $\mathfrak{R}G$, then $E\mathfrak{R}G/\mathfrak{R}G \cong E/(E \cap \mathfrak{R}G)$ is a nontrivial almost radical accessible subgroup of $G/\mathfrak{R}G$. This contradicts (d), and thus (e) is proved.

Let G be not almost polycyclic. By condition ($\mathfrak{M}$)G is finitely generated or the normal subgroups of G satisfy the maximum condition. By Corollary 2 there exists a characteristic subgroup C of G such that G/C is not almost polycyclic, but G/D is almost polycyclic for every characteristic subgroup D of G containing C properly. By (a) $H = G/C$ satisfies ($\mathfrak{M}$). By (c) the product $\mathfrak{R}H$ of all almost polycyclic characteristic subgroups of H is an almost polycyclic characteristic subgroup of H. If $\mathfrak{R}H \neq 1$ then $H/\mathfrak{R}H$ is almost polycyclic, and this implies that H is almost polycyclic. Thus $\mathfrak{R}H = 1$, and by (d) 1 is the only almost radical accessible subgroup of H.

Proof of the theorem. If G is almost polycyclic, then G is especially Noetherian and every infinite epimorphic image of G possesses a finitely generated Abelian normal subgroup, not 1. These properties imply that the conditions (II) to (VIII) are consequences of (I).

Assume now that the group G is not almost polycyclic, but that at least one of the conditions (II) to (VIII) is satisfied. Then especially G is finitely generated or the maximum condition is satisfied by the normal subgroups of G. By Corollary 2 this implies the existence of a characteristic subgroup C of G with the following properties:

(1) $H = G/C$ is not almost polycyclic, but H/D is almost polycyclic for every characteristic subgroup $D \neq 1$ of H.

If (II) is satisfied, then H possesses a locally almost soluble characteristic subgoup $N \neq 1$ of H. Clearly H likewise satisfies condition (II. a), so that N is finitely generated. Since N is a finitely generated almost soluble group, there exists a soluble characteristic subgroup S of N and H; see for instance W.R. Scott, [11], p. 152, 7.7. If $S \neq 1$, then there exists an Abelian characteristic subgroup $A \neq 1$ of S, N and H. As a characteristic subgroup of H the group A is finitely generated and therefore Noetherian. This implies that there exists an almost polycyclic characteristic subgroup $D \neq 1$ of H. By (1) H/D is almost polycyclic, so that H is almost polycyclic. This contradicts (1), and G does not satisfy condition (II).

If (III) is satisfied, then H possesses a locally almost polycyclic accessible subgroup $E \neq 1$. Hence the subgroup R generated by all locally almost polycyclic accessible subgroup of H is a locally almost polycyclic characteristic subgroup, not 1, of H, since the product of two normal almost polycyclic subgroups is almost polycyclic; see R. Baer [4], p. 360, Folgerung 1. Since R is a characteristic subgroup of H, it is finitely generated by (III. a). Thus H is an extension of the almost polycyclic group R by H/R which is almost polycyclic by (1). But then H must be almost polycyclic, which contradicts (1). Hence G does not satisfy (III).

If one of the conditions (IV) to (VII) is satisfied, then by Lemma 8 (f) we may assume that the epimorphic image H of G satisfies, in addition to (1), the following condition:

(2) 1 is the only almost radical accessible subgroup of H.

Clearly (2) implies that G does not satify (IV).

If (V) is satisfied, (V. b) and (2) imply the existence of an infinite independent set $\mathfrak{S}$ of normal subgroups of H; see Lemma 6. Then the product P of all normal subgroups in $\mathfrak{S}$ is a normal subgroup of H, and (V. a) implies that P is finitely generated or the maximum condition is satisfied by the normal subgroups of P. In both cases $\mathfrak{S}$ must be a finite set. This contradiction shows that G does not satisfy (V).

If (VI) is satisfied, then there exists a characteristic subgroup $N \neq 1$ of H such that $\mathfrak{c}_H N \neq 1$. Since $\mathfrak{c}_H N$ is likewise a characteristic subgroup of H, $H/\mathfrak{c}_H N$ is almost polycyclic by (1). Now Proposition 5 yields the existence of an almost Abelian characteristic subgroup $A \neq 1$ of H. This contradicts (2), and G does not satisfy (VI).

If (VII) is satisfied, (VII. b) and (2) imply the existence of an infinite independent set $\mathfrak{S}$ of normal subgroups of H; see Lemma 6. But by (VII. a2) the normal subgroups of H satisfy the maximum condition. Hence $\mathfrak{S}$ must be finite, and G does not fulfill (VII).

Thus (VIII) must be satisfied. By (VIII. a1) and Lemma 2 there exists an epimorphic image H of G with the following properties:

(3) H is not almost polycyclic, but every proper epimorphic image of H is almost polycyclic.

By (VIII. b) there exists a normal subgroup $N \neq 1$ of H such that $\mathfrak{c}_H N \neq 1$. Condition (3) yields that $H/\mathfrak{c}_H N$ is almost polycyclic. Application of Proposition 5 shows the existence of an almost Abelian normal subgroup $B \neq 1$ of H. If B is infinite, then there exists an Abelian characteristic subgroup $C \neq 1$ of B which is an Abelian normal subgroup of H; see R. Baer [2], p. 152, Lemma 2. By (VIII. a2) C is finitely generated and therefore Noetherian. Thus there exists a Noetherian almost Abelian normal subgroup $A \neq 1$ of H. Since H/A is almost polycyclic by (3), H must be almost polycyclic also. This contradiction finally proves our theorem.

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