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EXTENDING HOMEOMORPHISMS

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Theorem 1 of this paper establishes a necessary and sufficient condition that a locally flat imbedding $f: B^k \to R^n$ of a k-cell in euclidean n-space R^n admits an extension to a homeomorphism $F: R^n \to R^n$ onto R^n such that $F \mid (R^n - B^k)$ is a diffeomorphism which is the identity outside some compact set in R^n. An analogous result for locally flat imbeddings of a euclidean $(n-1)$-sphere into R^n is proved. A lemma which generalizes a theorem of Huebsch and Morse concerning Schoenflies extensions without interior differential singularities is also established.

Let the points of euclidean n-space R^n be written $x = (x^1, \cdots, x^n)$, and provide R^n with the usual euclidean norm $\|x\| = [\Sigma(x^i)^2]^{1/2}$. We set $S_r = \{x \in R^n \mid \|x\| = r\}$, (and $S = S_1$). If M is a topological $(n-1)$-sphere in R^n, we denote the bounded component of $R^n - M$ by $\mathring{J}M$, and the closure of $\mathring{J}M$ in R^n by JM. We refer the reader to § 1 of [2] for the definition of the terms admissible cone K_z, conical point, axis of singular approach, and cone $K_z(\Sigma)$, where Σ is a euclidean $(n-1)$-sphere in R^n.

LEMMA 1. *Let z be an arbitrary point of S and φ a sense-preserving homeomorphism into R^n of an open neighborhood N of S such that φ carries points inside S to points inside $\varphi(S)$, and $\varphi \mid (N - S)$ is a C^m-diffeomorphism. There then exists a homeomorphism Φ of R^n onto R^n and a cone K_z* (resp. $\check{K}_z$) *with axis interiorly normal* (resp. *exteriorly normal*) *to S at z, such that if $X \subset N$ is a sufficiently small open neighborhood of S,*

$$\Phi(x) = \varphi(x) \qquad [x \in X - \{K_z(S) \cup \check{K}_z\}] ,$$

$\Phi \mid (R^n - S)$ is a C^m-diffeomorphism, and Φ is the identity outside some compact set in R^n.

REMARK. We note that a direct application of the proof of Theorem 1.2 of [2] will yield the conclusions of Lemma 1 except for single differential singularities in each component of $R^n - S$.

Proof of Lemma 1. The proof of Lemma 1 will be a variation of the proof of Theorem 1.2 of [2]. We can assume that $0 \in \mathring{J}\varphi(S)$. Let $\delta \in (\frac{1}{2}, 1)$ be a constant so near 1 that $S_\delta \subset N$. Using Theorem 1.1 of [2], there is a homeomorphism $f: JS \to R^n$ into R^n such that

$$f \mid (JS - \mathring{J}S_\delta) = \varphi \mid (JS - \mathring{J}S_\delta)\ ,$$

and $f \mid (\mathring{J}S - 0)$ is a C^m-diffeomorphism. We can also assume that $f(0) = 0$. We now apply Lemma 5.3 of [2] to $f \mid (\mathring{J}S \cap N)$, S_δ, and the fixed point $y = \delta z \in S_\delta$, and conclude that if $\rho > 1$ is a sufficiently large constant, there exists a homeomorphism Θ of R^n onto R^n such that $\Theta(x) = f(x)$ for $x \in Y \cup JS_\delta$, where Y is a suitable neighborhood of S_δ, $\Theta \mid (R^n - \{0 \cup \rho z\})$ is a C^m-diffeomorphism, where the point ρz is a conical point of Θ with cone $K_{\rho z}$ of singular approach to ρz whose axis is interiorly normal to S_ρ at ρz, and if a constant $\nu \in (0, 1)$ is sufficiently near 1, Θ reduces to the identity on $R^n - \{B_{\nu\rho} \cup K_{\rho z}(S_\rho)\}$.

Let ω be a radial C^∞-diffeomorphism of R^n onto R^n such that $\omega(x) = x$ for $\|x\| \leqq \varepsilon$, and $\omega(x) = \rho x$ for $\|x\| \geqq 1 - \varepsilon$, where $0 < \varepsilon < \frac{1}{2}$. We then set $\Phi(x) = f\omega^{-1}\Theta^{-1}\omega(x)$ for $x \in JS$. If $\zeta: R^n \to R^n$ is the C^∞-diffeomorphism $\zeta(x) = \rho x$, we set $K_z = \zeta^{-1}(K_{\rho z})$ and $\hat{X} = \zeta^{-1}(JS_\rho - B_{\nu\rho})$. We can assume that ν is so near 1 that $1 - \varepsilon < \nu$ and $\delta < \nu$, so that $\zeta^{-1}(JS_\rho - B_{\nu\rho}) = \omega^{-1}(JS_\rho - B_{\nu\rho})$ and $\hat{X} \subset JS - JS_\delta$. Then we see that $\Phi \mid (\hat{X} - K_z(S)) = \varphi \mid (\hat{X} - K_z(S))$ and $\Phi \mid \mathring{J}S$ is a C^m-diffeomorphism (which reduces to the identity on a neighborhood of 0). We have therefore defined $\Phi \mid JS$ with the desired properties. We then define Φ on $R^n - \mathring{J}S$ in an analogous manner to satisfy the conclusions of Lemma 1 by regarding R^n, with the "point at infinity" added, as an n-sphere, and using the geometry of inversion. This completes the proof of Lemma 1.

REMARK. As the proof of Lemma 1 shows, we also could state corresponding "one-sided" lemmas in which the differentability is only assumed either outside or inside of S. For example, if only $\varphi \mid (N - JS)$ is a C^m-diffeomorphism, then $\check{K}_z$ and Φ exist where $\Phi \mid (R^n - JS)$ is a C^m-diffeomorphism which is the identity outside some compact set in R^n.

We now fix the integer n, and for any integer $k \leqq n$, we regard $R^k \subset R^n$ as consisting of those points $x = (x^1, \cdots, x^n)$ with $x^{k+1} = \cdots = x^n = 0$. We denote the unit k-cell in $R^k \subset R^n$ by B^k. For convenience, we assume in what follows that "diffeomorphism" means "C^∞-diffeomorphism."

THEOREM 1. *Let $f: B^k \to R^n$ be a homeomorphism into R^n such that each point $x \in B^k$ has an open neighborhood V_x in R^n and a sense-preserving homeomorphism f_x: $V_x \to R^n$ into R^n satisfying*

$$f_x \mid (V_x \cap B^k) = f \mid (V_x \cap B^k)$$

and $f_x \mid (V_x - B^k)$ is a diffeomorphism. Then there exists a homeomorphism F of R^n onto R^n such that

(i) $F \mid B^k = f$,
(ii) $F \mid (R^n - B^k)$ *is a diffeomorphism,*
(iii) F *is the identity outside some compact set in* R^n.

Proof. An examination and easy modification of the proof of Proposition C in [1] shows that there exists an open neighborhood N of B^k in R^n and a sense-preserving homeomorphism $\varphi\colon N \to R^n$ into R^n such that $\varphi \mid B^k = f$ and $\varphi \mid (N - B^k)$ is a diffeomorphism. Let $J\mathscr{S} \subset N$ be a smooth convex n-cell in R^n, where $\mathscr{S}$ is a smooth $(n-1)$-sphere in R^n such that $B^k \subset \mathscr{S}$, and let z be an arbitrary point in B^k. Using the remark following Lemma 1, there exists an open neighborhood Y of $\mathscr{S}$ in R^n, a cone K_z with axis exteriorly normal to $\mathscr{S}$ at z, and a homeomorphism Φ of R^n onto itself such that $\Phi(x) = \varphi(x)$ for $x \in (Y - K_z)$, and $\Phi \mid (R^n - J\mathscr{S})$ is a diffeomorphism which is the identity outside some compact subset of R^n. We then define $F\colon R^n \to R^n$ by

$$F(x) = \varphi(x) \qquad [x \in (Y - K_z) \cup J\mathscr{S}]$$
$$F(x) = \Phi(x) \qquad [x \in Y \cup (R^n - J\mathscr{S})]\,.$$

It is clear that F satisfies the conclusions of Theorem 1.

LEMMA 2. *Let* $f\colon R^{n-1} \to R^n$, $n \geqq 4$ *be an imbedding of* R^{n-1} *as a closed subset of* R^n. *Suppose for each* $x \in R^{n-1}$ *there is a neighborhood* V_x *of* x *in* R^n *and a homeomorphism* $f_x\colon V_x \to R^n$ *into* R^n *such that* $f_x \mid (V_x \cap R^{n-1}) = f \mid (V_x \cap R^{n-1})$, *and* $f_x \mid (V_x - R^{n-1})$ *is a diffeomorphism. Then there is a homeomorphism* F *of* R^n *onto* R^n *such that* $F \mid R^{n-1} = f$ *and* $F \mid (R^n - R^{n-1})$ *is a diffeomorphism.*

PROOF. As in the proof of Lemma 2, (cf. Proposition C_1 of [1]), there is an open neighborhood U of R^{n-1} in R^n and a homeomorphism $\Phi\colon U \to R^n$ into R^n such that $\Phi \mid R^{n-1} = f$ and $\Phi \mid (U - R^{n-1})$ is a diffeomorphism. Let $\mathscr{R}_1^{n-1}$, $\mathscr{R}_2^{n-1}$ be diffeomorphs (under good C^1-approximations to the inclusion) of R^{n-1} as closed subsets of R^n such that $\mathscr{R}_1^{n-1}$ and $\mathscr{R}_2^{n-1}$ are contained in opposite components of $R^n - R^{n-1}$, and if V denotes the component of $R^n - \{\mathscr{R}_1^{n-1} \cup \mathscr{R}_2^{n-1}\}$ which contains R^{n-1}, then $\bar{V} = V \cup \mathscr{R}_1^{n-1} \cup \mathscr{R}_2^{n-1} \subset U$. Let V_1 (resp. V_2) denote that component of $R^n - \mathscr{R}_1^{n-1}$ (resp. $R^n - \mathscr{R}_2^{n-1}$) which does not contain $\mathscr{R}_2^{n-1}$ (resp. $\mathscr{R}_1^{n-1}$). Applying the corollary to Theorem 1 of [3], there are diffeomorphisms Θ_1, Θ_2 of R^n onto R^n such that $\Theta_i \mid \mathscr{R}_i^{n-1} = \Phi \mid \mathscr{R}_i^{n-1}$, $i = 1, 2$. Since any orientation-preserving diffeomorphism of R^{n-1} on itself is diffeotopic to the identity, we may assume that $\Theta \mid 0_i = \Phi \mid 0_i$, where 0_i is an open neighborhood of $\mathscr{R}_i^{n-1}$ in $R^n - R^{n-1}$, $i = 1, 2$. Then $F\colon R^n \to R^n$ defined by

$$F(x) = \Theta_1(x) \qquad [x \in 0_1 \cup V_1]\,,$$
$$F(x) = \Theta_2(x) \qquad [x \in 0_2 \cup V_2]\,,$$
$$F(x) = \Phi(x) \qquad [x \in 0_1 \cup 0_2 \cup V]$$

satisfies the conclusions of Lemma 2.

Using one point compactification and stereographic projection, Theorem 2 below is obtained readily from Lemmas 2 and 1.

THEOREM 2. *Let* $f\colon S \to R^n$ *be a homeomorphism into* R^n, $n \geqq 4$, *and let* p *be an arbitrary point in* S. *Suppose each point* $x \in S - p$ *has an open neighborhood* V_x *in* R^n *and a sensepreserving homeomorphism* $f_x\colon V_x \to R^n$ *into* R^n *such that* $f_x \mid (V_x \cap S) = f \mid (V_x \cap S)$, $f \mid (V_x - S)$ *is a diffeomorphism, and* f_x *carries points inside* S *to points inside* $f(S)$. *Then there is a homeomorphism* F *of* R^n *onto itself such that* $F \mid S = f$, *and* $F \mid (R^n - S)$ *is a diffeomorphism which is the identity outside some compact subset in* R^n.

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