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A CONJECTURE AND SOME PROBLEMS ON PERMANENTS

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Let $A = [a_{ij}]$ denote an $n \times n$ matrix and let E be the $n \times n$ identity matrix. We will designate by det A and perm A the determinant and the permanent of A respectively. The polynomial $\varphi(z) = \det(zE - A)$ plays a fundamental role in matrix theory. Similarly we can consider the polynomial $f(z) = \operatorname{perm}(zE - A)$ which has been object of several studies recently, particularly when A is a doubly stochastic matrix. The aim of the present paper is to give some results on the existence of matrices satisfying certain conditions involving the roots of this polynomial.

Let M_n and $\mathscr{M}_n$ be the regions defined as follows: $z \in M_n$ if and only if there exists a stochastic matrix of order n with z as characteristic root; $(z_1, \cdots, z_n) \in \mathscr{M}_n$ if and only if there exists a stochastic matrix of order n whose n characteristic roots are the complex numbers $z_1, \cdots, z_n$.

Similarly we define the regions D_n and $\mathscr{D}_n$ respectively when 'stochastic' is replaced by 'doubly stochastic'. M_n was determined by Karpelevič [3] but the determination of the other three regions seems to be a very difficult problem and has not yet been solved (see [7], [8], [9]).

Replacing in the definitions of M_n, $\mathscr{M}_n$, D_n and $\mathscr{D}_n$ 'characteristic root' by 'root of the polynomial $f(z) = \operatorname{perm}(zE - A)$' we can define four other regions which we shall denote by M_n^*, $\mathscr{M}_n^*$, D_n^* and $\mathscr{D}_n^*$ respectively. To our knowledge no attempt has been made to determine these regions. Their determination is likely to be a much harder problem than the determination of M_n, $\mathscr{M}_n$, D_n and $\mathscr{D}_n$.

Some problems dealing with the characteristic values of a matrix (like some of the problems mentioned in [6]) can be replaced by similar problems dealing with the roots of

$$f(z) = \operatorname{perm}(zE - A) .$$

Examples: (1) find a necessary and sufficient condition for the numbers $a_1, \cdots, a_n$ and $z_1, \cdots, z_n$ to be the principal elements of a symmetric A and the roots of $f(z) = \operatorname{perm}(zE - A)$ respectively; (2) find a necessary and sufficient condition for the numbers $\lambda_1, \cdots, \lambda_n$ and $z_1, \cdots, z_n$ to be the characteristic roots of an $n \times n$ matrix A and the roots of $f(z) = \operatorname{perm}(zE - A)$ respectively. In the sequel we give some results on problems of this nature.

2. Let

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \qquad \text{(of type } s_i \times s_i\text{)},$$

$$X_i = \begin{bmatrix} x_1^i \\ \vdots \\ x_{s_i}^i \end{bmatrix}, \qquad Y_i = [y_1^i, \cdots, y_{s_i}^i]$$

and

$$C = \begin{bmatrix} J_1 & 0 \cdots 0 & X_1 \\ 0 & J_2 \cdots 0 & X_2 \\ \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 \cdots J_m & X_m \\ Y_1 & Y_2 \cdots Y_m & q \end{bmatrix}.$$

LEMMA. *If C is the matrix described above and E denotes the appropriate identity matrix then*

$$\operatorname{perm}(zE - C) = \sum_{i=1}^{m} \left[\sum_{h=0}^{s_i - 1} b_{ih}(z - \lambda_i)^h \prod_{\substack{j=1 \\ j \neq i}}^{m} (z - \lambda_j)^{s_j} \right] + (z - q) \prod_{j=1}^{m} (z - \lambda_i)^{s_j},$$

where

$$b_{ih} = (-1)^{s_i + h + 1} \sum_{j=1}^{h+1} y_j^i x_{j+s_i-1-h}^i \qquad (h = 0, \cdots, s_i - 1).$$

Proof. Let

$$C_i = \begin{bmatrix} J_i & 0 & \cdots & 0 & X_i \\ 0 & J_{i+1} & \cdots & 0 & X_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & J_m & X_m \\ Y_i & Y_{i+1} & \cdots & Y_m & q \end{bmatrix}.$$

Now we expand $\operatorname{perm}(zE_i - C_i)$ (where E_i is the identity matrix of the same order as C_i) in terms of its first s_i rows. The submatrices contained in these rows with permanent nonnecessarily zero are: $zE^{(i)} - J_i$ ($E^{(i)}$ denotes the identity matrix of the same order as J_i) and the submatrices obtained from $zE^{(i)} - J_i$ by striking out the ρ^{th} column ($\rho = 1, \cdots, s_i$) and bordering on the right hand side with the column $-X_i$. We denote this submatrix by H_ρ. It is not difficult to see that

$$\operatorname{perm} H_\rho = \sum_{\tau=0}^{s_i-\rho} (-1)^{\tau+1} x^i_{\rho+\tau} (z-\lambda_i)^{s_i-\tau-1} \,.$$

Let $\tilde{H}_\rho$ denote the complementary submatrix of H_ρ in $zE_i - C_i$. It can be easily seen that

$$\operatorname{perm} \tilde{H}_\rho = -y^i_\rho \prod_{j=i+1}^{m} (z-\lambda_j)^{s_j} \,.$$

We can now write

$$\begin{aligned}
\operatorname{perm}(zE_i - C_i) &= \sum_{\rho=1}^{s_i} \operatorname{perm} H_\rho \operatorname{perm} \tilde{H}_\rho \\
&\quad + \operatorname{perm}(zE^{(i)} - J_i) \operatorname{perm}(zE_{i+1} - C_{i+1}) \\
&= \sum_{\rho=1}^{s_i} \sum_{\tau=0}^{s_i-\rho} (-1)^\tau y^i_\rho x^i_{\rho+\tau} (z-\lambda_i)^{s_i-\tau-1} \prod_{j=i+1}^{m} (z-\lambda_j)^{s_j} \\
&\quad + (z-\lambda_i)^{s_i} \operatorname{perm}(zE_{i+1} - C_{i+1}) \,.
\end{aligned}$$

Interchanging the order of the first two sums we get

$$\begin{aligned}
\operatorname{perm}(zE_i - C_i) &= \sum_{\tau=0}^{s_i-1} \sum_{\rho=1}^{s_i-\tau} (-1)^\tau y^i_\rho x^i_{\rho+\tau} (z-\lambda_i)^{s_i-\tau-1} \prod_{j=i+1}^{m} (z-\lambda_j)^{s_j} \\
&\quad + (z-\lambda_i)^{s_i} \operatorname{perm}(zE_{i+1} - C_{i+1}) \\
&= \sum_{h=0}^{s_i-1} b_{ih} (z-\lambda_i)^h \prod_{j=i+1}^{m} (z-\lambda_j)^{s_j} \\
&\quad + (z-\lambda_i)^{s_i} \operatorname{perm}(zE_{i+1} - C_{i+1}) \,.
\end{aligned}$$

We now set $i = 1$, use induction, and after some manipulation we obtain the formula stated in the lemma.

We proceed to our main result.

THEOREM 1. *Given any n complex numbers $a_1, \cdots, a_n$ and a polynomial $f(z) = z^n - cz^{n-1} + \cdots$, there exists a square matrix A of order n with $a_1, \cdots, a_n$ as principal elements and such that $f(z) = \operatorname{perm}(zE - A)$ if and only if $a_1 + \cdots + a_n = c$. If this condition is satisfied and both $a_1, \cdots, a_n$ and the coefficients of $f(z)$ are real, A can be chosen real.*

Proof. We prove first the 'if' part. If we perform a permutation on the rows of a square matrix A and then the same permutation on its columns, the roots of $f(z) = \operatorname{perm}(zE - A)$ are not altered. Hence we can, without loss of generality, take the numbers $a_1, \cdots, a_n$ in any order. Thus we will assume that the first s_1 numbers from among $a_1, \cdots, a_{n-1}$ have the common value λ_1, the following s_2 numbers have the common value $\lambda_2, \cdots,$ the last s_m numbers have the common value λ_m and that $\lambda_i \neq \lambda_j$ for $i \neq j$. Consider now the matrix C of the lemma with $q = a_n$ and all the $x_h^k = 1$. We will show that we can choose $Y_1, \cdots, Y_m$ such that $\operatorname{perm}(zE - C) = f(z)$.

Let $g(z) = \prod_{j=1}^{m}(z - \lambda_j)^{s_j}$. Using the formula of the lemma we can write

$$\frac{\operatorname{perm}(zE - C)}{g(z)} = \sum_{i=1}^{m}\sum_{h=0}^{s_i-1}\frac{b_{ih}}{(z - \lambda_i)^{s_i-h}} + z - q \,.$$

Let us now resolve $f(z)/g(z)$ into partial fractions. Bearing in mind that $f(z) = z^n - (\sum_{i=1}^{n} a_i)z^{n-1} + \cdots$ we get

$$\text{(I)} \qquad \frac{f(z)}{g(z)} = \sum_{i=1}^{m}\sum_{h=0}^{s_i-1}\frac{d_{ih}}{(z - \lambda_i)^{s_i-h}} + z - q \,.$$

Let us take $b_{ih} = d_{ih}$. With this choice of the b_{ih} we have $f(z) = \operatorname{perm}(zE - C)$ as required. Now we compute the y_h^k by $b_{ih} = (-1)^{s_i+h+1}\sum_{j=1}^{h+1} y_j^i$ ($h = 0, \cdots, s_i - 1$; $i = 1, \cdots, m$) which is a system of linear equations, always compatible.

If we suppose the numbers $a_1, \cdots, a_n$ as well as the coefficients of $f(z)$ real it follows from (I) that the d_{ih} and therefore the b_{ih} are also real. In this case C can, clearly, be chosen real.

The "only if" part of the theorem is an immediate consequence of the formula

$$\operatorname{perm}(zE - A) = z^n + \sum_{p=1}^{n}\sum_{1\leq i_1<\cdots<i_p\leq n}(-1)^p \operatorname{perm} A\begin{pmatrix} i_1, \cdots, i_p \\ i_1, \cdots, i_p \end{pmatrix} z^{n-p}$$

where $A\begin{pmatrix} i_1, \cdots, i_p \\ i_1, \cdots, i_p \end{pmatrix}$ denotes the principal submatrix of A contained in the rows $i_1, \cdots, i_p$.

Concerning the problem (1) mentioned in § 1 of the present paper, we have been able to prove the following partial result.

THEOREM 2. *Let $a_1, \cdots, a_n$ be real numbers and suppose that there exists an index i_0 such that $i \neq j$; $i, j \neq i_0$ implies $a_i \neq a_j$. Let $f(z) = z^n - cz^{n-1} + \cdots$ be a given polynomial with real coefficients such that $c = \sum_{i=1}^{n} a_i$.*

$$\textit{If } f(a_j).\ \prod_{\substack{i=1\\ i\neq j, i_0}}^{n}(a_j - a_i) \geqq 0 \qquad (j = 1, \cdots, n; j \neq i_0)\,,$$

there exists an $n \times n$ real symmetric matrix A with $a_1, \cdots, a_n$ as principal elements and such that $f(z) = \operatorname{perm}(zE - A)$.

We omit the proof which follows closely the technique used in the proof of the Theorem 1.

3. We denote by Ω_n the set of all doubly stochastic matrices of order n. When $A \in \Omega_n$, $f(z) = \operatorname{perm}(zE - A)$ enjoys some interesting

properties as for instance: the roots of $f(z)$ lie in or on the boundary of the unit disc $|z| \leqq 1$ (see [1] and [4]). For the real roots of $f(z)$ it is known that they lie in the interval $0 < x \leqq 1$. We have been led to the following

CONJECTURE. Let A be an $n \times n$ doubly stochastic irreducible matrix. If n is even, then $f(z) = \operatorname{perm}(zE - A)$ has no real roots; if n is odd, then $f(z) = \operatorname{perm}(zE - A)$ has one and only one real root.

It can be seen by direct computation that the conjecture is true in the following cases:

(a) A is a 2×2 real (not necessarily nonnegative) irreducible matrix all of whose row and column sums are 1.

(b) A is a 3×3 real (not necessarily nonnegative) irreducible symmetric matrix all of whose row and column sums are 1.

(c) A is the $n \times n$ matrix all of whose entries are equal to $1/n$.

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