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A MULTIPLIER THEOREM

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Let G be a locally compact abelian group and φ a complex-valued function defined on the dual Γ. In this paper we prove that φ is a multiplier of type $(L^1 \cap L^\infty, L^1 \cap C)$ if and only if $\varphi = \hat{f}$ for some $f \in L^1(G)$.

Throughout the paper $M(G)$ denotes the measure algebra of the locally compact group G, $L^p(G)$ $(1 \leqq p \leqq \infty)$ the usual Lebesgue space of index p formed with respect to left Haar measure on G, $C(G)$ the set of all bounded continuous complex-valued functions on G and $C_0(G)$, the set of all $f \in C(G)$ which vanish at infinity.

For a locally compact abelian group G with dual Γ the Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dx \qquad (\gamma \in \Gamma)\ .$$

The Fourier-Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \qquad (\gamma \in \Gamma)\ .$$

For $y \in G$, the translate f_y of the function f is defined by

$$f_y(t) = f(t - y) \qquad (t \in G)\ .$$

The translate μ_y of the measure $\mu \in M(G)$ is defined by

$$\mu_y(E) = \mu(E - y)$$

where E is any Borel set in G.

A complex-valued function φ defined on Γ is said to be a multiplier of type $(L^1 \cap L^\infty, L^1 \cap C)$ if given $f \in L^1(G) \cap L^\infty(G)$ there corresponds a $g \in L^1(G) \cap C(G)$ such that $\varphi\hat{f} = \hat{g}$. The set of all multipliers of type $(L^1 \cap L^\infty, L^1 \cap C)$ will be denoted by $(L^1 \cap L^\infty, L^1 \cap C)$. The multiplier problem $(L^1 \cap L^\infty, L^1 \cap C)$ is then the determination of necessary and sufficient conditions which insure that $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$. The multiplier problems, $(L^1 \cap L^\infty, L^1 \cap C_o)$, $(L^1 \cap C_o, L^1)$, etc., are defined similarly.

For the classical groups T and $\mathbf{R}$, the multiplier problem $(L^1 \cap L^\infty, L^1 \cap C_o)$ has been solved by Zygmund [9] and Doss [1, p. 191], respectively. The solution for $G = \mathrm{T}$ has also been given by Verblunsky [8, p. 303]. Edwards [3, pp. 376–378] has solved the problem for compact groups satisfying the first axiom of countability. Hewitt and Ross have recently solved the problem (to appear in [5]) for all compact groups. We prove for arbitrary *LCA* groups the following theorem:

THEOREM 1. $(L^1 \cap L^\infty,\ L^1 \cap C) = (L^1 \cap L^\infty,\ L^1 \cap C_o) = L^1(G)^\wedge$.

Proof. By $L^1(G)^\wedge$ we mean the set of $\hat{f}$ on Γ which are Fourier transforms of functions $f \in L^1(G)$. Suppose $\varphi = \hat{f}$ for some $f \in L^1(G)$. If $g \in L^1(G) \cap L^\infty(G)$ then, by [6, p. 4], the convolution $f * g \in L^1(G) \cap C_0(G)$. Thus $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$ and $(L^1 \cap L^\infty, L^1 \cap C_o)$.

Next suppose $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$. We first show that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$. Assume temporarily that G is compact. Since $\varphi \in (L^\infty, C)$ we have $\varphi \in (L^\infty, L^\infty)$. By a result of Edwards [3, p. 374] $\varphi = \hat{\mu}$ for some $\mu \in M(G)$.

If G is a noncompact *LCA* group we proceed as follows. $\varphi \in (L^1 \cap L^\infty, L^1 \cap C)$ implies $\varphi \in (L^1 \cap C_o, L^1)$. Doss [1, p. 189] has proved that, for $G = \mathbf{R}$, $\varphi \in (L^1 \cap C_o,\ L^1)$ if and only if $\varphi = \hat{\mu}$ for some $\mu \in M(\boldsymbol{R})$. We have been able to generalize his proof to noncompact *LCA* groups, but the proof is rather lengthy. Frank Forelli has recently given a simple proof that $(L^1 \cap C_o, L^1) = M(G)^\wedge$. (See Theorem 3.2 of [4].)

So for $f \in L^1(G) \cap L^\infty(G)$

$$g(x) = \int_G f(x-t)d\mu(t) \text{ a.e.}$$

where $\hat{g} = \hat{\mu}\hat{f}$ and $g \in L^1(G) \cap C(G)$. We now show that μ is absolutely continuous with respect to Haar measure. Let A be any relatively compact Borel subset of G and ψ the characteristic function of A. Then the convolution $\psi * \mu$ is equal a.e. to a continuous function. Thus for each relatively compact Borel subset A of G the function

$$x \longrightarrow \mu\,(x + A)$$

is equal a.e. to a continuous function. This implies by the following theorem that $d\mu(x) = f\,(x)\,dx$ for some $f \in L^1(G)$ and hence concludes the proof.

THEOREM 2. *Let G be a locally compact group and $\mu \in M(G)$ such that for each relatively compact Borel subset A of G, the function $x \rightarrow \mu(x + A)$ is equal locally a.e. to a continuous function on G. Then $d\mu(x) = f(x)\,dx$ for some $f \in L^1(G)$.*

Compare this with Theorem 2 of [2, p. 407], where μ can be any Radon measure but where G is assumed to be a first countable *LCA* group. In this connection see also 1.6 of [7, p. 230]. The proof of the present theorem may be obtained by simple modifications of the proof of Theorem (35.13) of [5], which we omit.

Remark. Let G be a noncompact LCA group. Since

$$(L^1 \cap C_o, L^1) = M(G)^\wedge$$

we have that

$$\begin{aligned} M(G)^\wedge &= (L^1 \cap C, L^1) \\ &= (L^1 \cap L^P, L^1) \\ &= (L^1 \cap L^P, L^1 \cap L^P) \\ &= (L^1 \cap C, L^1 \cap L^P) \\ &= (L^1 \cap C_o, L^1 \cap L^P) \\ &= (L^1 \cap C_o, L^1 \cap C_o) \\ &= (L^1 \cap C_o, L^1 \cap C) \\ &= (L^1 \cap C, L^1 \cap C) \end{aligned}$$

where $M(G)^\wedge$ is the set of $\hat{\mu}$ on Γ which are the Fourier-Stieltjes transforms of measures $\mu \in M(G)$. For infinite compact groups it is false that $(C, L^1) = M(G)^\wedge$ since $(L^2, L^2) = L^\infty(\Gamma)$.

The author wishes to thank the referee for bringing to his attention theorem (35.13) of [5] and for the reference to [4]. Theorem (35.13) enabled the removal of the (unnecessary) hypothesis that G be first countable in Theorem 1.

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