EXPANSIVE AUTOMORPHISMS OF
BANACH SPACES

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This paper treats two classes of invertible bounded linear operators on Banach spaces—expansive and uniformly expansive automorphisms—which include the hyperbolic automorphisms. Conditions for an automorphism to be expansive or uniformly expansive are given in terms of the location of its spectrum and approximate point spectrum with respect to the unit circle.

One of the tools used in [3] to determine all expansive automorphisms of compact connected Lie groups was the following result:

THEOREM 0. *Let T be an automorphism of a finite-dimensional real or complex normed linear space. Then a necessary and sufficient condition for T to be expansive is that $|\lambda| \neq 1$ for each complex characteristic root λ of T.*

Theorem 0 was deduced in [2] as a special case of a more general theorem, concerning topological vector spaces over arbitrary nondiscrete scalar fields, whose proof used algebraic methods leaning heavily on the assumption of finite dimensionality. In the present paper we use analytic considerations to treat the infinite-dimensional case.

The results we obtain were suggested by the following observation. In the finite-dimensional case, the condition for an automorphism to be expansive amounts to its being hyperbolic for some norm; in any Banach space, an automorphism is hyperbolic for some norm precisely when its spectrum is disjoint from the unit circle.

1. Preliminaries. If B is a real or complex Banach space, we shall call any bounded linear operator on B having a bounded inverse on B an *automorphism* of B.

The most convenient definition of "expansive" for our purposes is the following.

DEFINITION 1. An automorphism T of a Banach space B is said to be *expansive* provided for each $x \in B$ with $\|x\| = 1$ there exists some nonzero integer i such that $\|T^i x\| \geq 2$.

In this definition any norm equivalent to the given norm on B may be used, and the constant 2 may be replaced by any constant

strictly greater than 1. Moreover, an automorphism T of B is expansive if and only if there exists a constant $c > 0$ such that $0 \neq x \in B$ implies $\| T^i x \| \geq c$ for some integer i. Hence T is expansive precisely when it is "unstable," in the sense of Utz [9], for the norm metric, and precisely when it is "expansive," in the sense of Bryant [1], for the uniformity which B possesses as an abelian topological group.

. The property named next will turn out to be something new only in the infinite-dimensional case.

DEFINITION 2. An automorphism T of a Banach space B is said to be *uniformly expansive* provided there exists some positive integer i such that $x \in B$ with $\| x \| = 1$ implies $\| T^i x \| \geq 2$ or $\| T^{-i} x \| \geq 2$.

Here again, any equivalent norm may be used, and 2 may be replaced by any constant $c > 1$.

We recall the following definition which will play an auxiliary role.

DEFINITION 3. An automorphism T of a Banach space B is said to be *hyperbolic* provided there is a splitting

$$B = B_s \oplus B_u, \qquad T = T_s \oplus T_u,$$

where B_s and B_u are closed T-invariant linear subspaces of B, $T_s = T \,|\, B_s$ is a proper contraction (that is, $\| T_s \| < 1$), and $T_u = T \,|\, B_u$ is a proper dilation (that is, $\| T_u^{-1} \| < 1$).

For an automorphism T, we denote its spectrum by $\varLambda(T)$, its compression spectrum by $\varGamma(T)$, its approximate point spectrum by $\varPi(T)$, its point spectrum by $\varPi_0(T)$, and its spectral radius by $r(T)$. We denote the unit circle $\{\lambda : |\lambda| = 1\}$ in the complex plane by C.

The lemma below is well known (compare [6]), but a proof is included for the sake of completeness.

LEMMA 1. *Let T be an automorphism of a complex Banach space B. Then T is hyperbolic with respect to some norm equivalent to the given norm of B if and only if $\varLambda(T) \cap C = \varnothing$.*

Proof. Assume that for some equivalent norm $\| \cdot \|$ we have a splitting $B = B_s \oplus B_u$, $T = T_s \oplus T_u$ as in Definition 3. Since $\| T_s \| < 1$ and $\| T_u^{-1} \| < 1$, $\varLambda(T_s)$ and $\varLambda(T_u) = (\varLambda(T_u^{-1}))^{-1}$ are both disjoint from C, so the same is true of $\varLambda(T)$.

Conversely, assume $\varLambda(T) \cap C = \varnothing$. Define

$$A_s = \{\lambda \in \Lambda(T): |\lambda| < 1\} \,,$$
$$A_u = \{\lambda \in \Lambda(T): |\lambda| > 1\} \,.$$

By the spectral decomposition theorem (see, for example, [8, §148]), there exist closed T-invariant linear subspaces B_s and B_u of B such that $B = B_s \oplus B_u$, $T = T_s \oplus T_u$, $\Lambda(T_s) = A_s$, and $\Lambda(T_u) = A_u$, where T_s and T_u are the restrictions of T to B_s and B_u respectively. Since $r(T_s) < 1$ and $r(T_u^{-1}) < 1$, we may renorm B_s and B_u so that $\|T_s\| < 1$ and $\|T_u^{-1}\| < 1$ (see, for example, [5]). Then these norms on B_s, B_u may be used to renorm B, and T is hyperbolic for this new norm.

2. Main results.

THEOREM 1. *Let T be an automorphism of a complex Banach space B. Then*:
   (1)   *If $\Lambda(T) \cap C = \varnothing$, then T is uniformly expansive.*
   (2)   *If T is uniformly expansive, then $\Pi(T) \cap C = \varnothing$.*
   (3)   *If $\Pi(T) \cap C = \varnothing$, then T is expansive.*
   (4)   *If T is expansive, then $\Pi_0(T) \cap C = \varnothing$.*
   (5)   *The converses of implications* (1), (3), *and* (4) *all fail, even when B is a Hilbert space.*

Proof. (1) In view of Lemma 1, we need only show that T is uniformly expansive provided it is hyperbolic for the given norm of B. Assume we have a T-invariant splitting $B = B_s \oplus B_u$, $T = T_s \oplus T_u$ with $\|T_s\| < 1$, $\|T_u^{-1}\| < 1$. Choose a constant c with

$$\max \{r(T_s), r(T_u^{-1})\} < c < 1 \,.$$

By the spectral radius formula, there exists a positive integer i such that

$$\|T_s^i\| < c^i, \qquad \|T_u^{-i}\| < c^i, \qquad c^i \leqq 1/4 \,.$$

Renorm B by setting

$$|x| = \|x_s\| + \|x_u\|$$

when $x = x_s + x_u$ with $x_s \in B_s$, $x_u \in B_u$.

We show T is uniformly expansive using this new norm. Let $x \in B$ with $|x| = 1$. Write $x = x_s + x_u$, where $x_s \in B_s$, $x_u \in B_u$. Then $\|x_s\| \geqq 1/2$ or $\|x_u\| \geqq 1/2$. If $\|x_s\| \geqq 1/2$, then $\|x_s\| \leqq \|T_s^i\| \cdot \|T_s^{-i}x_s\|$, so

$$|T^{-i}x| \geqq \|T_s^{-i}x_s\| \geqq \|T_s^i\|^{-1} \|x_s\| \geqq 4(1/2) = 2 \,.$$

Similarly, if $\|x_u\| \geqq 1/2$, then $|T^i x| \geqq 2$.
   (2) Suppose there exists some $\lambda \in \Pi(T) \cap C$. Since the properties

that T be uniformly expansive and that $\Pi(T)$ be disjoint from C are unaffected when T is multiplied by a scalar of modulus 1, we may assume $\lambda = 1$. There exists a sequence $(x_n \mid n = 1, 2, \cdots)$ in B with $\|x_n\| = 1$ for each n and $\|Tx_n - x_n\| \to 0$. Then $\|x_n - T^i x_n\| \to 0$ for every integer i. For small $\varepsilon > 0$ and any integer i, there is some n for which $\|T^i x_n\| < 1 + \varepsilon$ and $\|T^{-i} x_n\| < 1 + \varepsilon$. Hence T is not uniformly expansive.

(3) Suppose T is not expansive; in other words, suppose there exists $x \in B$ with $\|x\| = 1$ and $\|T^n x\| \leq 2$ for every integer n. Define E to be the least closed linear subspace of B which is invariant under T and T^{-1} and which contains x, and set $S = T \mid E$. Clearly S is not expansive. Since $\Pi(S) \subset \Pi(T)$, it suffices to show $\Pi(S) \cap C \neq \varnothing$.

We show first that $|\lambda| \leq 1$ for each $\lambda \in \Gamma(S)$. Let $\lambda \in \Gamma(S)$. Then $\lambda \in \Pi_0(S^*)$, where S^* is the automorphism induced by S on the dual E^* of E. Choose $y^* \in E^*$ with $\|y^*\| = 1$ and $S^* y^* = \lambda y^*$. By definition of E, $\langle S^i x, y^* \rangle \neq 0$ for some integer i; set $c = \langle S^i x, y^* \rangle$. For each positive integer n,

$$\langle S^{i+n} x, y^* \rangle = \langle S^i x, (S^*)^n y^* \rangle = \langle S^i x, \lambda^n y^* \rangle = \lambda^n c ,$$

so that

$$2 \geq \|S^{i+n} x\| \geq |\langle S^{i+n} x, y^* \rangle| = |\lambda|^n |c| .$$

It follows that $|\lambda| \leq 1$. (Actually, $\Gamma(S) \subset C$. In fact, the same argument as above but with negative n shows that $|\lambda| \geq 1$ for each $\lambda \in \Gamma(S)$.)

We have $\Lambda(S) \cap C \neq \varnothing$, for otherwise S would be (uniformly) expansive by (1). Choose $\lambda \in \Lambda(S) \cap C$. Either $\lambda \in \text{bdy } \Lambda(S)$ or $\lambda \in \text{int } \Lambda(S)$. If $\lambda \in \text{bdy } \Lambda(S)$ we are done, since bdy $\Lambda(S) \subset \Pi(S)$. Suppose $\lambda \in \text{int } \Lambda(S)$. Choose a sequence $(\lambda_n \mid n = 1, 2, \cdots)$ in $\Lambda(S)$ such that $|\lambda_n| > 1$ for each n and $\lambda_n \to \lambda$. By what was proved above, no $\lambda_n \in \Gamma(S)$, hence each $\lambda_n \in \Pi(S)$. Since $\Pi(S)$ is closed, $\lambda \in \Pi(S)$ also.

The proof of (4) is trivial and valid for any normed space. The examples in the next section establish (5).

COROLLARY 1. *Each hyperbolic automorphism of B is uniformly expansive.*

COROLLARY 2. *When B is finite-dimensional, an automorphism of B is expansive if and only if it is uniformly expansive.*

Although we are unable to determine whether, in general, an automorphism is uniformly expansive if its approximate point spectrum is disjoint from C, we do have the following special result.

THEOREM 2. *Let T be an automorphism of a complex Hilbert space H. Then a necessary and sufficient condition for T to be uniformly expansive is that $\Pi(T) \cap C = \varnothing$.*

For the proof we require an algebraic lemma.

LEMMA 2. *Given any n complex numbers $c_1, \cdots, c_n$, there exists $\lambda \in C$ such that $\operatorname{Re}(\sum_{j=1}^{n} \lambda^j c_j) \geqq 0$.*

Proof. Define f by $f(z) = \sum_j c_j z^j$, let $\Gamma = f(C)$, and suppose $0 \notin \Gamma$. It will suffice to show that the winding number $\operatorname{Ind}[\Gamma, 0]$ of Γ with respect to 0 is positive, for then $\arg f(\lambda) = 0$ for some $\lambda \in C$. We have

$$\operatorname{Ind}[\Gamma, 0] = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i} \int_C \frac{f'(\lambda)d\lambda}{f(\lambda)}.$$

Now f is analytic inside C and $f(0) = 0$, so by the argument principle $\operatorname{Ind}[\Gamma, 0] \geqq 1$.

Proof of Theorem 2. Necessity follows from Theorem 1. To prove sufficiency, suppose T is not uniformly expansive. Then for each positive integer n there exists $x_n \in H$ with $\|x_n\| = 1$ and

$$\max\{\|T^n x_n\|, \|T^{-n}x_n\|\} < 2.$$

We are going to use Lemma 2 to construct for each n a vector $y_n \in H$ and a number $\lambda_n \in C$ such that $\|(T - \lambda_n I)y_n\| \leqq A(\log n)^{-1/2}\|y_n\|$, where A is a fixed constant independent of n. The desired conclusion will then follow. In fact, let $\lambda \in C$ be a limit point of $\{\lambda_n : n = 1, 2, \cdots\}$. Given $\varepsilon > 0$,

$$\|(T - \lambda I)y_n\| \leqq \|(T - \lambda_n I)y_n\| + \|(\lambda_n - \lambda)y_n\|$$
$$\leqq [A(\log n)^{-1/2} + |\lambda_n - \lambda|]\|y_n\|$$
$$\leqq \varepsilon\|y_n\|$$

for large enough n. Hence $\lambda \in \Pi(T)$.

Fix a positive integer n; we construct the desired $y_n \in H$ and $\lambda_n \in C$. Set

$$y_n = \sum_{j=k}^{m-1} \lambda^j T^j x_n,$$

where $m \geq 0$ and $k \leq 0$, $k < m$, are integers still to be determined, depending only on n, and where $\lambda \in C$ is still to be determined, depending on m and k. An easy computation gives

$$\|y_n\|^2 = \sum_{j=k}^{m-1} \|T^j x_n\|^2 + 2\operatorname{Re}\sum_{j=1}^{m-1-k} \lambda^j c_j,$$

where each c_j is a sum of terms of the form $(T^i x_n, T^{m-i} x_n)$. By Lemma 2 we may choose $\lambda \in C$ such that the second term above is nonnegative. Then

$$(*) \qquad\qquad \|y_n\|^2 \geqq \sum_{j=k}^{m-1} \|T^j x_n\|^2 \;.$$

Set $\lambda_n = \lambda^{-1}$. Then

$$(**) \qquad\qquad \|(T - \lambda_n I)y_n\| \leqq \|T^m x_n\| + \|T^k x_n\| \;.$$

To determine m and k, upon which the choice of λ depends, we consider three cases.

   Case (i). $\|T^j x_n\|^2 \geqq (\log n)/n$ for $0 \leqq j \leqq n$. Take $m = n$ and $k = 0$. From $(*)$, $\|y_n\|^2 \geqq \log n$; from $(**)$ and the choice of x_n,

$$\|(T - \lambda_n I)y_n\| \leqq \|T^n x_n\| + \|x_n\| \leqq 3 \;.$$

Hence

$$\|(T - \lambda_n I)y_n\| \leqq 3 (\log n)^{-1/2} \|y_n\| \;.$$

   Case (ii). $\|T^j x_n\|^2 \geqq (\log n)/n$ for $-n \leqq j \leqq 0$. Take $m = 0$ and $k = -n$. Exactly as in case (i) we find

$$\|(T - \lambda_n I)y_n\| \leqq 3 (\log n)^{-1/2} \|y_n\| \;.$$

   Case (iii).   Cases (i) and (ii) both fail to apply. Let

$$c = \max \{\|T\|, \|T^{-1}\|\} \;;$$

$c \geqq 1$ since T is not uniformly expansive. Then $\|T^j x_n\| \geqq c^{-j}$ and $\|T^{-j} x_n\| \geqq c^{-j}$ for every j. Since case (i) fails to hold there is an integer m, $0 < m \leqq n$, for which $\|T^m x_n\|^2 < \log n/n$, and we take the least such m. Then m satisfies $c^{-2m} \leqq \|T^m x_n\|^2 < \log n/n$, so that $m > B \log n$ with $B = (4 \log c)^{-1}$. Similarly, since case (ii) fails there is an integer k, $-n \leqq k < 0$, with $\|T^k x_n\|^2 < \log n/n$. We take the greatest such k, which must satisfy $k < -B \log n$.

   By the choice of m and k, $\|T^j x_n\| \geqq \|T^k x_n\|$ for all $k \leqq j < m$. From $(*)$ we obtain

$$\|y_n\|^2 \geqq (m - k) \|T^k x_n\|^2 \geqq (2B \log n) \|T^k x_n\|^2 \;.$$

Now

$$\|T^m x_n\|^2 < \log n/n \leqq \|T^{k+1} x_n\| \leqq c \|T^k x_n\| \;.$$

From $(**)$ we obtain $\|(T - \lambda_n I)y_n\| \leqq (1 + c) \|T^k x_n\|^2$. Hence in this case

$$\|(T - \lambda_n I)y_n\| \leqq (1 + c)(2B)^{-1/2} (\log n)^{-1/2} \|y_n\| \;.$$

To conclude the proof, set $A = \max \{3, (1 + c)(2 \log c)^{1/2}\}$.

3. **Examples.** The examples below establish assertion (5) of Theorem 1.

EXAMPLE 1. The bilateral shift on separable Hilbert space is unitary and, a fortiori, not expansive, but it has no eigenvalues.

EXAMPLE 2. Let H be a separable Hilbert space with orthonormal base $(e_n \mid n = 1, 2, \cdots)$ and let T be the diagonal operator on H given by $T(e_n) = \lambda_n e_n$, where $0 \neq \lambda_n \notin C$ and $\lambda_n \to 1$. Then T is expansive, for $0 \neq x \in H$ implies $(x, e_n) \neq 0$ for some n, so that

$$\|T^i x\|^2 \geq |\lambda_n|^{2i} |(x, e_n)|^2 \geq 4$$

for some integer i sufficiently large in absolute value. However, $1 \in \Pi(T)$.

Notice that if we take $\lambda_2 = \lambda_1$, the new automorphism S of H given by $S(e_1) = e_1$ and $S(e_n) = \lambda_n e_n$ for $n > 1$ is not expansive, yet $\Lambda(S) = \Lambda(T)$.

The preceding is a counterexample to the converse of Theorem 1 (3) when B is a Hilbert space. Next we construct a counterexample for an arbitrary separable Banach space.

EXAMPLE 3. Let B be any (infinite-dimensional) separable Banach space. Choose a maximal biorthogonal family $(e_n, e_n^*)_n$ with $\|e_n^*\| = 1$ for every index n. We may take the index class of this family to be $\{1, 2, \cdots\}$, for it is countable since the unit sphere in the dual B^* of B is weak-$*$ separable. The Hahn-Banach theorem says that $(e_n^* \mid n = 1, 2, \cdots)$ must be total. Let K be the compact operator given by $K = \sum_n \lambda_n \langle \cdot, e_n^* \rangle e_n$, where $\lambda_n = 3^{-n} \|e_n\|^{-1} \leq 3^{-n}$, $n = 1, 2, \cdots$. Let $T = I + K$, where I is the identity operator. Since $\|Kx\| \leq \|x\|/2$, $\|Tx\| \geq \|x\|/2$ for all $x \in B$. Hence T is an automorphism of B. Clearly $1 \in \Pi(T)$.

To see that T is expansive, let $x \in B$ with $\|x\| = 1$. For some n, $\langle x, e_n^* \rangle \neq 0$. For each positive integer i, $\langle T^i x, e_n^* \rangle = (1 + \lambda_n)^i \langle x, e_n^* \rangle$, so $\|T^i x\| \geq (1 + \lambda_n)^i |\langle x, e_n^* \rangle|$. Hence $\|T^i x\| \geq 2$ for large enough i.

EXAMPLE 4. Let H be a separable Hilbert space. We construct a uniformly expansive automorphism T of H with

$$\Lambda(T) = A = \{\lambda : 1/(2\sqrt{2}) \leq |\lambda| \leq 2\sqrt{2}\}.$$

More specifically, $\Pi_0(T) = \varnothing$, $\Pi(T) = \text{bdy } A$, and $C \subset \text{int } A = \Gamma(T) = \Pi_0(T^*)$, so that the adjoint T^* of T is not even expansive.

Let $(e_n)_{-\infty}^{\infty}$ be an orthonormal base of H. Define the weighted bilateral shift T of H by

$$Te_n = \begin{cases} 2\sqrt{2}\, e_{n+1}, & n \geq 0 \,, \\ \dfrac{1}{2\sqrt{2}} e_{n+1}, & n \leq -1 \,. \end{cases}$$

To see that T is uniformly expansive, let $x \in H$ with $\|x\| = 1$. Write $x = y + ce_0 + z$, where y is orthogonal to e_n for all $n \leq 0$ and z is orthogonal to e_n for all $n \geq 0$. Now $\|y\|^2 + |c|^2 + \|z\|^2 = \|x\|^2 = 1$, If $\|y\|^2 + |c|^2 \geq 1/2$, then

$$\|Tx\|^2 \geq (2\sqrt{2})^2(\|y\|^2 + |c|^2) \geq 4 \,;$$

if $|c|^2 + \|z\|^2 \geq 1/2$, then $\|T^{-1}x\|^2 \geq (2\sqrt{2})^2(|c|^2 + \|z\|^2) \geq 4$.

To determine $\Lambda(T)$, note initially that $\|T^n\| = \|T^{-n}\| = (2\sqrt{2})^n$ for every $n \geq 1$. Hence $\Lambda(T) \subset A$. Now T^* is the weighted backwards shift

$$T^*e_n = \begin{cases} 2\sqrt{2}\, e_{n-1}, & n \geq 1 \,, \\ \dfrac{1}{2\sqrt{2}} e_{n-1}, & n \leq 0 \,. \end{cases}$$

To find $\Pi_0(T^*)$, fix a complex number $\lambda \neq 0$. Then $T^*x = \lambda x$ for some vector $0 \neq x = \sum_{-\infty}^{\infty} c_n e_n$ if and only if

$$\lambda c_n = \begin{cases} \dfrac{1}{2\sqrt{2}} c_{n+1}, & n \leq -1 \,, \\ 2\sqrt{2}\, c_{n+1}, & n \geq 0 \,, \end{cases}$$

that is, $c_n = (\lambda/2\sqrt{2})^n c_0$ and $c_{-n} = (1/2\sqrt{2}\,\lambda)^n c_0$ for all $n \geq 1$. Since $x \neq 0$, the series $\sum c_n e_n$ converges if and only if $|\lambda/2\sqrt{2}| < 1$ and $|1/2\sqrt{2}\,\lambda| < 1$, that is, $\lambda \in \text{int } A$.

It follows that $\Lambda(T) \supset \text{cls } \Gamma(T) = A$. A computation similar to the above shows that $\Pi_0(T) = \varnothing$, and elementary estimates give that $\Pi(T) = \text{bdy } A$.

EXAMPLE 5. Let A be the annular region $\{z : 1/2 < |z| < 2\}$ in the complex plane and let H be the space of all analytic functions belonging to the complex Hilbert space $L^2(A, m)$, where m is planar Lebesgue measure. Then [4] H is a separable Hilbert space. The analytic position operator T on H given by $Tf(z) = zf(z)$ is an auto-

morphism of H. It is proved in [4] that $\Lambda(T)$ is the closure of A and $\Pi(T)$ is the boundary of A. Thus $C \subset \Lambda(T)$, yet T is uniformly expansive by Theorem 2.

We sketch a direct proof that T is uniformly expansive which avoids the computation of $\Lambda(T)$ given in [4]. Take $f \in H$ with $\|f\| = 1$ and define

$$\varphi(r) = r \int_0^{2\pi} |f(re^{it})|^2 dt \qquad (1/2 < r < 2) .$$

Thus $1 = \|f\|^2 = \int_{1/2}^2 \varphi(r)dr$. Since f is analytic, $\varphi(r) > 0$. It follows from Hölder's inequality that $\log \varphi(r)$ is a convex function of $\log r$: if $r = r_1^t r_2^{1-t}$ with $0 \leq t \leq 1$, then $\varphi(r) \leq \varphi(r_1)^t \varphi(r_2)^{1-t}$. One concludes that φ is a continuous function having no local maximum in $(1/2, 2)$. Since $\int_{1/2}^2 \varphi(r)dr = 1$, either $\int_1^2 \varphi(r)dr \geq 2/3$ or $\int_{1/2}^1 \varphi(r)dr \geq 1/3$. In the former case the geometrical properties of φ imply that $\int_{3/2}^2 \varphi(r)dr \geq 1/3$, so that

$$\|T^n f\|^2 = \int_{1/2}^2 r^{2n} \varphi(r)dr$$

$$\geq (3/2)^{2n} \int_{3/2}^2 \varphi(r)dr \geq (3/2)^{2n}(1/3)$$

for all $n \geq 1$. Similarly, in the latter case $\int_{1/2}^{3/4} \varphi(r)dr \geq 1/6$, so that $\|T^{-n}f\|^2 \geq (4/3)^{2n}/6$. Hence $\max\{\|T^n f\|, \|T^{-n}f\|\} \geq (4/3)^n/\sqrt{6}$ for all $n \geq 1$.

4. **Remarks.** Implications (1) and (3) of Theorem 1 no longer hold if we take real instead of complex scalars. In fact, let B^* be any nonzero complex Banach space and let T^* be the multiplication in B^* by the complex number i. Then the automorphism T of the real Banach space subordinate to B^* is an isometry, but $\Lambda(T) = \varnothing$.

Now let T be any automorphism of a real Banach space B. Denote by T^* the induced automorphism of the complexification B^* of B (see [7, Th. 1.3.1], for example). There is a norm-preserving real-linear isomorphism f of B into B^* with $fT = T^*f$; also $B^* = B \oplus B$ as a real Banach space, and $T^* = T \oplus T$. Hence T is hyperbolic (uniformly expansive, expansive, respectively) precisely when T^* is. It follows that Theorem 1 (1)-(4) remain valid in the real case if $\Lambda(T)$, $\Pi(T)$, $\Pi_0(T)$ are replaced by $\Lambda(T^*)$, $\Pi(T^*)$, $\Pi_0(T^*)$.

We return to the complex case. The set of all uniformly expansive automorphisms of a Banach space B is easily seen to be open in the

set of all automorphisms of B. It would be interesting to know whether the former set is dense in the latter.

Added in proof. Techniques similar to those used here show that Theorem 2 remains valid for an arbitrary Banach space. Also, the uniformly expansive automorphisms are not even dense in the expansive automorphisms. Proofs will appear in a paper by the second-named author.

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