A CHARACTERIZATION
OF THE PARALLELEPIPED IN E^n

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A bounded open set in the Euclidean plane E^2 which has a parallelogram as its boundary will be called a two dimensional open parallelepiped. The n-dimensional analogue of such an object is called an open parallelepiped in E^n. In order to motivate the characterization presented here, it is desirable to recall the characterization of starshaped sets given by Krasnosel'skiĭ. In 1946 Krasnosel'skiĭ proved that a bounded closed set S in E^n is starshaped if and only if each set of $s+1$ points of S with $s \le n$ can see at least one point of S via S. As is well known, this result fails if S is unbounded. However, under what circumstances can a set S see infinity via S in the same direction? This and related questions led to the following which is first stated here intuitively. The open parallelepiped in E^n is the only nonempty bounded open convex set S in E^n which has the property that every $2n-1$ of its boundary points can see infinity in a same direction via the complement of S. This result is related to a theorem of Steinitz on convex hulls.

1. **Formal results for parallelepipeds.** The following notations and concepts will be used.

DEFINITION 1. The interior, closure, boundary and relative interior of a set C in E^n are denoted by int C, cl C, bd C, intv C respectively. If R is a ray, a collection of points S in a set C can see infinity in the direction of R via C if for each point $x \in S$ the ray $R(x)$ with endpoint x and having the same direction as R lies in C.

The interior of an n-dimensional parallelepiped is called an open parallelepiped. Its $2n$ (n-1)-dimensional faces will be called faces, and these, of course, consist of n pairs of parallel congruent faces.

The main result of this paper is the following.

THEOREM 1. *Let S be a nonempty bounded open convex set in n-dimensional Euclidean space E^n.*

Then S is an open parallelepiped if and only if for each set of $2n$-1 points of bd S there exists at least one direction R such that all of these $2n$-1 points can see infinity via the complement of S in this direction R.

Proof. To prove this we need to recall the following.

DEFINITION 2.   A smooth point $x \in \text{bd}\, S$ is one at which there exists a unique plane of support to S.   Such a plane is called a tangent plane of support or more briefly a tangent plane.

For each smooth point $x \in \text{bd}\, S$ let $H(x)$ be the unique closed half-space whose bounding plane is the tangent plane of support to S at x, and such that $H(x)$ does not contain S, so that $H(x) \cap S = \varnothing$, $x \in \text{bd}\, H(x)$.   Choose a fixed point $\theta \in E^n$ and let $H_1(x)$ be that translate of $H(x)$ which sends x to θ.   Clearly if $H(x) \neq H(y)$ then $H_1(x) \neq H_1(y)$.   By hypothesis, every s with $s \leqq 2n - 1$ of the family of closed half-spaces $\{H_1(x),\ x = \text{smooth point of } S\}$ have a ray in common.   If S has at least $2n + 1$ distinct tangent planes of support then a theorem of Dines and McCoy [2], Robinson [6], Steinitz [8], Gustin [3], implies that all the members of the set $\{H_1(x),\ x = \text{smooth point of } S\}$ have a ray R in common emanating from θ.   However, since the smooth points of $\text{bd}\, S$ are dense in $\text{bd}\, S$, and since $S = \text{int}\, S$ is a nonempty bounded set, for the direction R there exists a smooth point $z \in \text{bd}\, S$ and a ray $R(z)$ which has the same direction as that of R such that $R(z) \cap S \not\equiv \varnothing$.   The corresponding closed half-space which misses S and whose bounding plane supports S at z cannot contain $R(z)$, which is a contradiction.   Furthermore, if S has fewer than $2n$ distinct tangent planes, then the hypothesis implies that all the boundary of S can see infinity via the complement of S in a same direction.   But just as above this has been shown to be impossible *Hence S has exactly $2n$ distinct tangent planes.   That S is an open polyhedron* follows from the following fact.   If an open bounded convex set in E^n has a finite number m of distinct tangent planes of support, then S is an open polyhedron having m faces.   To prove this let K be the intersection of the m closed half-spaces which contain S and which are bounded by the m tangent planes respectively. If $K \neq \text{cl}\, S$, since the smooth points of $\text{bd}\, S$ are dense in $\text{bd}\, S$, then there exists a smooth point and an associated tangent plane distinct from the m tangent planes given above.   Thus we have proved that *S is an open convex polyhedron whose boundary contains exactly $2n$ faces.*

Finally, we prove that S is an open parallelepiped.   Let F_i $(i = 1, \cdots, 2n)$ denote the $2n$ faces of S, and choose points $P_i \in \text{intv}\, F_i$, where $\text{intv}\, F_i$ is the relative interior of F_i.   Also let N_i be the normal ray to F_i through p_i which is exterior to S.   Translate N_i so that p_i goes to the fixed point $\theta \in E^n$ and denote this translate by T_i.   Since $\text{bd}\, S$ bounds a compact convex set in E^n with $S = \text{int}\, S$, we have

$$\text{conv} \left(\bigcup_{i=1}^{2n} T_i \right) = E^n$$

where conv = convex hull. Since $\theta \in$ int conv $\bigcup_{i=1}^{2n} T_i$, a theorem of Steinitz [8] implies there exist points x_i $(i = 1, \cdots, j)$, $x_i \neq \theta$, $j \leqq 2n$, with $x_i \in \bigcup_{k=1}^{2n} T_k$ such that $\theta \in$ int conv $(x_1 \cup x_2 \cup \cdots \cup x_j)$. Also this theorem of Steinitz implies that if for each and every such set of points $(x_1, x_2, \cdots, x_j)$ it is true that $j = 2n$, then every such collection can be broken down into pairs, each of which is collinear with θ. Suppose $j < 2n$. Without loss of generality, rearrange subscripts so that $x_i \in \bigcup_{k=1}^{j} T_k$. Hence, as stated above, the rays T_i $(i = 1, \cdots, j)$ encompass E^n (i. e. conv $(\bigcup_{i=1}^{j} T_i) = E^n$). Now let R be any ray through θ. Since conv $(\bigcup_{i=1}^{j} T_i) = E^n$, $j < 2n$, the ray R makes an angle with some ray, say T_k, $k \leqq j$, which exceeds $\pi/2$. This implies that the translate $R(p_k)$ of R which has endpoint $p_k \in$ intv F_k must pierce S, since N_k is the outward normal to S at p_k, and since R makes an angle with N_k which exceeds $\pi/2$. Thus, we have shown, that if $j < 2n$, the j points $(p_1, \cdots, p_j)$ cannot all see infinity in a same direction via the complement of S. Since this violates the hypothesis, we have $j = 2n$. As mentioned above, the Steinitz theorem [8] implies that $(x_1, \cdots, x_{2n})$ are collinear in pairs with θ. This, in turn, implies that the $2n$ faces of S consists of n pairs of parallel faces. Since this implies, by induction, that S is an open parallelepiped, Theorem 1 has been established.

2. **Starshaped sets.** Theorem 1 is clearly related to the extended Euclidean space in which points at infinity are involved. As a consequence we will use the following notion. For related ideas see Allen [1].

DEFINITION 3. A set S is starshaped in the extended sense if all of S can see some finite point of S via S or if all of S can see infinity in the same direction via S (see Definition 1).

If S is compact, Krasnosel'skii's theorem as stated in our introduction states that the *Helly number* is $n + 1$. This means that if every $n + 1$ points of S can see a point of S via S, then S is starshaped. Theorem 1 shows that the Helly number is not $n + 1$ for unbounded closed sets. Robkin [7] showed that if every $2n$ boundary points of a closed set in E^n can see infinity in the same direction via S then all of S can see infinity in the same direction via S. Robkin generalized this result so that the hypothesis applied only to points of spherical support. We will obtain another theorem of this sort. To do this we need the following definition.

DEFINITION 4. A point x in a closed set $S \subset E^n$ is a point of spherical support if there exists an open sphere B such that $x \in \operatorname{bd} B$ and such that $B \cap S = \varnothing$.

We obtain the following result.

THEOREM 2. *Let S be a nonempty closed set in E^n. Suppose each set of $s + 1$ points of spherical support of S with $s \leq n$ can see some point of S via S.*
Then S is starshaped in the extended sense (see Definition 3).

Notice that the hypothesis in Theorem 2 does not involve infinity, for then the Helly number could not be $n + 1$, by virtue of Theorem 1. However, the *conclusion* does involve infinity.

Proof. Consider the collection K of points of spherical support of S. If $K = \varnothing$, then $S = E^n$, and the theorem is obviously true. Hence, suppose $S \neq E^n$. For $x \in K$, let $\{B_\alpha(x), \alpha \in A\}$ be the collection of all open spheres of support to S at x, and let $H_\alpha(x)$ denote the closed half-space whose bounding plane supports $B_\alpha(x)$ at x, and for which $B_\alpha(x) \cap H_\alpha(x) = \varnothing$. Define

$$(1) \qquad\qquad H(x) \equiv \bigcap_{\alpha \in A} H_\alpha(x)$$

and let

$$M \equiv \{H(x),\ x \in K\}\ .$$

Since $S \neq E^n$, K contains at least one point. Since every $s + 1$ ($s \leq n$) members of M have a point in common, and since $H(x), x \in K$ is convex, Helly's theorem [4] implies that each finite subcollection of M has a point in common. If some finite subcollection of M has a bounded intersection, then compactness implies that

$$(2) \qquad\qquad \bigcap_{x \in K} H(x) \neq \varnothing\ .$$

Hence, suppose every finite subcollection has unbounded intersection. If $H_1(x)$ is the translate of $H(x)$ which sends x to a fixed point $\theta \in E^n$, then each finite subset of $\{H_1(x),\ x \in K\}$ has a ray in common through θ. The theorem of Dines and McCoy [2] implies that all the members of $\{H_1(x), x \in K\}$ have a ray in common through θ. This, in turn, implies that there exists a direction R such that each point $x \in K$ is the endpoint of a ray $R(x) \subset H(x)$ having x as endpoint.

To complete the proof, we need to show that either S is starshaped with respect to a point in (2) or that S can see infinity via

S in the direction of R, defined above. In case 1, if (2) holds, let $p \in \bigcap_{x \in K} H(x)$, and suppose a point $x \in S$ exists such that $px \not\subset S$. In case 2, if (2) fails, suppose a point $x \in S$ exists such that $R(x) \not\subset S$. In either case, a point y exists with $y \in px$ in case 1, and with $y \in R(x)$ in case 2 such that $y \notin S$. Set up a linear order on xy from x to y, so that $x < y$. Since S is closed, there exists a real value $t > 0$ and an open sphere $B(y, t)$ with center y and radius t such that $S \cap B(y, t) = \varnothing$. Also let $B(u, t/2)$ be a sphere with center u and radius $t/2$ where $x < u \leqq y$ on xy. Now, define the convex set

$$N(u) = \mathrm{cl\ conv}\ [B(y, t) \cup B(u, t/2)]$$

where cl conv = closed convex hull. Since S is closed and since $N(y) \cap S = \varnothing$, there exists a point $w \in xy$ such that $x \leqq w < y$ and such that

(3) $$N(w) \cap \mathrm{bd}\ S \neq \varnothing,\ S \cap \mathrm{int}\ N(w) = \varnothing\ .$$

Choose $z \in S \cap \mathrm{bd}\ N(w)$. Clearly z is a point of spherical support of S since each point of bd $N(w)$ lies on the boundary of an open sphere lying in int $N(w)$, and since (3) holds. Since the set $H(z)$ in (1) at z contains no ray in the direction of $\overrightarrow{xy}$, in case 1, $H(z)$ cannot contain p which violates (2), and in case (2), $H(z)$ cannot contain a ray $R(z)$ with endpoint z and parallel to R. Since this is a contradiction, we have proved that S is starshaped in the extended sense.

BIBLIOGRAPHY

1. J. E. Allen, *A Generalization of convexity*, Abstr. 582-29 Amer. Math. Soc. Notices, **8** (1962), 344.

2. L. L. Dines, and N. H. McCoy, *On linear inequalities*, Trans. Roy. Soc. Canada, **27** (1933), 37-70.

3. William Gustin, *On the interior of the convex hull of a Euclidean Set*, Bull. Amer. Math. Soc. **53** (1947),209-301.

4. E. Helly, *Über Mengen Konvexer Körper mit gemeinshaftlichen Punkten*, Jahr. Deut. Math. Verein. **32** (1923), 175-176.

5. M. A. Krasnosel'skiï, *Sur un Critère pour qu'on domain soit étoilé*, Mat. Sb. N. S. **19** (1946), 309-310.

6. C. V. Robinson, *Spherical theorems of Helly type and congruence indices of spherical caps*, Amer. J. Math. **64** (1942), 260-272.

7. E. E. Robkin, *Characterization of starshaped sets*, Dissertation, University of California, Los Angeles, 1965.

8. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, I, II, III, J. Reine Angew. Math. **143**, (1913), 128-175; **144** (1914), 1-40; **146** (1916), 1-52.

9. F. A. Valentine, *Convex sets*, McGraw-Hill Book Company, 1964.

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