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SELF-ADJOINT DIFFERENTIAL OPERATORS

ARNOLD L. VILLONE

Let $\mathscr{H}$ denote the Hilbert space of square summable analytic functions on the unit disk, and consider the formal differential operator

$$L = \sum_{i=0}^{n} p_i D^i$$

where the p_i are in $\mathscr{H}$. This paper is devoted to a study of symmetric operators in $\mathscr{H}$ arising from L. A characterization of those L which give rise to symmetric operators S is obtained, and the question of when such an S is self-adjoint or admits of a self-adjoint extension is considered. If A is a self adjoint extension of S and $E(\lambda)$ the associated resolution of the identity, the projection E_Δ corresponding to the interval $\Delta = (a, b]$ is shown to be an integral operator whose kernel can be expressed in terms of a basis of solutions for the equation $(L - \ell)u = 0$ and a spectral matrix.

Let $\mathscr{A}$ denote the space of functions analytic on the unit disk and $\mathscr{H}$ the subspace of square summable functions in $\mathscr{A}$ with inner product

$$(f, g) = \iint_{|z|<1} f(z)\overline{g(z)}dxdy .$$

Then $\mathscr{H}$ is a Hilbert space with the reproducing property, i.e., for each z there exists a unique element K_z of $\mathscr{H}$ such that

$$f(z) = (f, K_z) .$$

Moreover, if the sequence $\{f_n\}$ converges to f in norm, $f_n(z)$ converges to $f(z)$ uniformly on compact subsets of the disk. A complete orthonormal set for $\mathscr{H}$ is provided by the normalized powers of z,

$$e_n(z) = [(n + 1)/\pi]^{1/2}z^n, \ n = 0, 1, \cdots .$$

From this it follows that $\mathscr{H}$ is identical with the space of power series $\sum_{n=0}^{\infty} a_n z^n$ which satisfy

$$\sum_{n=0}^{\infty} | a_n |^2/(n + 1) < \infty . \tag{1.1}$$

Consider the formal differential operator

$$L = p_n D^n + \cdots + p_1 D + p_0 ,$$

where $D = d/dz$ and the p_i are in $\mathscr{A}$. For f in $\mathscr{H}$ the element Lf is in $\mathscr{A}$, but not necessarily in $\mathscr{H}$. To see this we take $L = d/dz$ and $f(z) = \sum_{n=1}^{\infty} n^{-1/2}z^n$, from (1.1) it follows that f is in $\mathscr{H}$ but Lf is not. In order to consider L as an operator in $\mathscr{H}$ we must restrict the class of functions on which L acts in some suitable manner. Since our concern is with densely defined operators it is only natural to demand that powers of z be mapped into $\mathscr{H}$. This requires some restrictions on the coefficients of L. As an example consider the operator $L = pD$ where $p(z) = \sum_{n=0}^{\infty}(n+1)z^n$.

We have $Le_k(z) = k(k+1)^{1/2}\pi^{-1/2}\sum_{n=k-1}^{\infty}(n-k)^{1/2}z^n$, and hence $Le_k \notin \mathscr{H}$. A sufficient condition for the Le_k to be in $\mathscr{H}$ is that the coefficients p_i be in $\mathscr{H}$.

Let $L = \sum_{i=0}^{n} p_i D^i$, where the p_i are in $\mathscr{H}$, and let $\mathscr{D}_0$ denote the span of the e_k and $\mathscr{D}$ the set of all f in $\mathscr{H}$ for which Lf is in $\mathscr{H}$. We now define the operators T_0 and T as follows.

$$T_0 f = Lf \quad f \in \mathscr{D}_0 \,,$$
$$Tf = Lf \quad f \in \mathscr{D} \,.$$

THEOREM 1.1. *T_0 and T are densely defined operators with range in $\mathscr{H}$, $T_0 \subseteqq T$, and T is closed.*

Proof. We first show that T is closed. Let $\{f_n\}$ be a sequence of functions in $\mathscr{D}$ such that $f_n \to f$ and $Tf_n \to g$, hence $f_n(z)$ and $Lf_n(z)$ converge uniformly on compact subsets to $f(z)$ and $g(z)$ respectively. But $Lf_n(z)$ also converges to $Lf(z)$. Hence $Lf(z) = g(z)$, $|z| < 1$, so $Tf \in \mathscr{H}$ and $Tf = g$.

Since $\mathscr{D}_0$ is dense in $\mathscr{H}$ and $T_0 f = Tf$ for $f \in \mathscr{D}_0 \cap \mathscr{D}$ it suffices to show that the e_j are in $\mathscr{D}$. Since $Le_j = \sum_{i=0}^{n} p_i D^i e_j$ and $p_i D^i e_j$ is either zero or of the form $p_i e_k$ for some nonnegative integer k, it sufficies to show that $p_i e_k \in \mathscr{H}$. Let $p_i = \sum_{j=0}^{\infty} a_j e_j$, a simple computation yields

$$e_k e_j = [(k+1)\pi]^{1/2}[(j+1)/(j+k+1)]^{1/2} e_{j+k} \,,$$

and consequently,

$$\| e_k p_i \|^2 \leqq [(k+1)\pi] \, \| p_i \|^2 < \infty \,.$$

T_0 and T are respectively the minimal and maximal operators in $\mathscr{H}$ associated with the formal operator L. We now proceed to study the class of formal differential operators for which T_0 is symmetric.

It is clear that the operator T_0 associated with the formal differential operator L is symmetric if and only if

$$(Le_n, e_m) = (e_n, Le_m) , \quad n, m = 0, 1, \cdots . \tag{1.2}$$

We shall refer to those formal operators satisfying (1.2) as formally symmetric. As an example we have the real Euler operator

$$L = \sum_{i=0}^{n} a_i z^i D^i ,$$

a_i real. Then $Le_j = p(j)e_j$ where p is the characteristic polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x(x-1) \cdots (x-n+1) .$$

Since $p(j) = \overline{p(j)}$, L is formally symmetric. A characterization of formally symmetric L in terms of the coefficients p_i is given in the next section. We now proceed to the consideration of the adjoint operators T_0^* and T^*. In what follows we shall make use of the result that if L is formally symmetric of order n, then the coefficients p_i are polynomials of degree at most $n+i$, $i = 0, 1, \cdots, n$. A proof of this is given in Theorem 2.2.

THEOREM 1.2. *If T_0 is symmetric, $T_0^* = T$ and $T^* \subseteqq T$. The closure of T_0, S, is self adjoint if and only if $S = T$.*

Proof. By Theorem 2.2 the coefficients p_i are polynomials of degree at most $n+i$. This implies that T_0 maps $\mathscr{D}_0$ into itself. In particular,

$$\begin{aligned} Le_m &= \sum_{i=0}^{n+m} \alpha_i e_i , \quad 0 \leqq m \leqq n , \\ Le_{n+j} &= \sum_{i=j}^{2n+j} \alpha_i e_i , \quad j = 1, 2, \cdots . \end{aligned} \tag{1.3}$$

Using this we show that $T_0^* \subseteqq T$. Let $g = \sum_{j=0}^{\infty} a_j e_j$ and $g^* = \sum_{j=0}^{\infty} b_j e_j$ be in the graph of T_0^* and consider the sequence $\{g_p\}$ in $\mathscr{D}_0$ defined as $g_p = \sum_{j=0}^{p} a_j e_j$. Since $g_p \to g$ we have $(T_0 e_k, g_p) \to (T_0 e_k, g) = (e_k, g^*)$. Hence $(e_k, T_0 g_p) \to (e_k, g^*)$. Now Lg is in $\mathscr{A}$ and $T_0 g_p$ converges to Lg uniformly on compact subsets. Since the e_j are just the normalized powers of z, the power series expansion of Lg can be written as $\sum_{j=0}^{\infty} c_j e_j(z)$. Since $Lg_p(z) = \sum_{j=0}^{p} a_j Le_j(z)$ converges uniformly to $\sum_{j=0}^{\infty} c_j e_j(z)$, it follows from (1.3) that Lg_p has the same coefficient of e_m as does Lg for $p > n+m+1$. Hence $(e_m, T_0 g_p) = \bar{c}_m$ for $p > n+m+1$ and since $(e_m, T_0 g_p) \to (e_m, g^*)$ we have $c_m = b_m$. Therefore $g^* = Lg$, so that $g \in \mathscr{D}$ and $g^* = Tg$.

To show that $T \subseteqq T_0^*$ it will suffice to show that $(T_0 e_m, g) = (e_m, Tg)$ for all g in $\mathscr{D}$ and $m = 0, 1, \cdots$. Let $g = \sum_{j=0}^{\infty} a_j e_j$ be in $\mathscr{D}$ and g_p as before. Since T_0 is symmetric and $g_p \to g$ we have $(e_m, T_0 g_p) = (T_0 e_m, g_p) \to (T_0 e_m, g)$. By precisely the same argument as before $(e_m, T_0 g_p) = (e_m, Tg)$ for $p > n + m + 1$, from which it follows that $(e_m, Tg) = (T_0 e_m, g)$ and $T_0^* = T$. Since $T_0 \subseteqq T$, $T^* \subseteqq T_0^* = T$.

The closure S of the symmetric operator T_0 is given by $T_0^{**} = T^* \subseteqq T$. Since T is closed $T^{**} = T$, from which it follows that $S^* = T$. Hence $S = T$ implies $S = S^*$. Conversely if S is self-adjoint we have $S = T^* = S^* = T$.

A sufficient condition for T to be self-adjoint is given by the following theorem.

THEOREM 1.3. *For* $f = \sum_{j=0}^{\infty} a_j e_j$ *set* $f_m = \sum_{j=0}^{m} a_j e_j$. *If* $\sup_m \| Tf_m \| < \infty$ *for each* f *in* $\mathscr{D}$, *then* S *is self-adjoint.*

Proof. Since $T^* \subseteqq T$, T symmetric implies $T = T^*$ and hence $S = S^*$. We show that $(Tf, g) - (f, Tg)$ vanishes for all f, g in $\mathscr{D}$. If L is of order n we have $(Tf_m, g_p) = (Tf, g_p)$ for $m > n + p + 1$. Using this fact and the symmetry of T_0 we obtain

$$\begin{aligned}(Tf, g_{kn}) &= (Tf_{kn+n+1}, g_{kn}) = (f_{kn+n+1}, Tg_{kn}) \\ &= (f_{kn-n-1}, Tg_{kn}) + (f_{kn+n+1} - f_{kn-n-1}, Tg_{kn}) \\ &= (f_{kn-n-1}, Tg) + (f_{kn+n+1} - f_{kn-n-1}, Tg_{kn}) \\ &\qquad\qquad\qquad\qquad\qquad k = 1, 2, \cdots .\end{aligned}$$

Therefore,

$$(Tf, g) - (f, Tg) = \lim_{k\to\infty} (f_{kn+n+1} - f_{kn-n-1}, Tg_{kn}) .$$

Since the Tg_{kn} are bounded in norm this implies $(Tf, g) - (f, Tg) = 0$.

COROLLARY. *If* L *is a formally symmetric Euler operator, then* S *is self-adjoint.*

Proof. For $f = \sum_{j=0}^{\infty} b_j e_j$ in $\mathscr{D}$, Tf and Tf_m are given by $\sum_{j=0}^{\infty} p(j) b_j e_j$ and $\sum_{j=0}^{m} p(j) b_j e^j$ respectively, where $p(x)$ is the characteristic polynomial for L. Hence

$$\| Tf_m \|^2 = \sum_{j=0}^{m} p(j)^2 | b_j |^2 \leqq \| Tf \|^2 ,$$

and the result follows.

2. Formal considerations. The formal operator $L = \sum_{i=0}^{n} p_i D^i$ is formally symmetric if

$$(Le_n, e_m) = (e_n, Le_m),\ n, m = 0, 1, \cdots .$$

To obtain a characterization of the formally symmetric operators in terms of their coefficients we first determine the action of L on e_k.

LEMMA 2.1. *Let* $L = \sum_{i=0}^{n} p_i D^i$ *where* $p_i(z) = \sum_{k=0}^{\infty} a_k(i)\, z^k$. *Then* $Le_i = \sum_{j=0}^{\infty} c_{ij} e_j$ *where*

$$
\begin{aligned}
c_{ij} &= A(i, j) \sum_{k=0}^{n} B(i, k) a_{j-i+k}(k)\ , \quad i, j = 0, 1 \cdots\ , \\
A(i, j) &= [(i+1)/(j+1)]^{1/2}\ , \\
B(i, k) &= i!/(i-k)! \quad i \geqq k \\
&= 0 \qquad\qquad\quad i < k\ .
\end{aligned}
\tag{2.1}
$$

Proof. Consider the elementary operators $L_{pq} = z^p D^q$, $p, q = 0, 1, \cdots$. A simple calculation yields

$$L_{pq} e_m = B(m, q) A(m, m + p - q) e_{m+p-q}\ .$$

Now consider Le_m (as an element of $\mathscr{A}$),

$$
\begin{aligned}
Le_m(z) &= \sum_{i=0}^{n} \sum_{k=0}^{\infty} a_k(i) L_{ki} e_m(z) \\
&= \sum_{i=0}^{n} \sum_{k=0}^{\infty} a_{k-m+i}(i) B(m, i) A(m, k) e_k(z) \\
&= \sum_{k=0}^{\infty} c_{mk} e_k(z) \quad |z| < 1\ .
\end{aligned}
$$

But $e_k(z)$ is just a multiple of z^k, therefore it follows from the uniqueness of power series representation of elements of $\mathscr{A}$, that $\sum_{k=0}^{\infty} c_{mk} e_k$ converges to Te_m in $\mathscr{H}$.

It follows that L is formally symmetric if and only if the coefficients $a_k(i)$, i, $k = 0, 1, \cdots$, satisfy the linear system

$$c_{ij} = \overline{c_{ji}}\ , \quad i, j = 0, 1, \cdots. \tag{2.2}$$

The following provides a simplification of the system (2.2).

THEOREM 2.2. *If* $L = \sum_{i=0}^{n} p_i D^i$ *is formally symmetric the* p_i *are polynomials of degree at most* $n + i$.

Proof. Consider $c_{n+p,0}$ for $p \geqq 1$. Since $j - n - p < 0$ for $p \geqq 1$ and $j = 0, \cdots, n$, $a_{j-n-p}(j) = 0$. Consequently $c_{n+p,0} = \bar{c}_{0,n+p}$ reduces to $A(0, n+p) a_{n+p}(0) = 0$, $p \geqq 1$, and p_0 is of degree at most n. We now proceed inductively. Consider

$$c_{n+p,k+1} = \bar{c}_{k+1,n+p}\ , \quad p \geqq k + 2\ . \tag{2.3}$$

Since $k+1+j-n-p<0$ for $p \geqq k+2$ and $j=0, \cdots, n$, (2.3) reduces to

$$A(k+1, n+p) \sum_{j=0}^{k-1} B(k+1, j) a_{n+p+j-k-1}(j)=0, \quad p \geqq k+2 .$$

Since $n+p+j-k-1 \geqq n+j+1$, it follows from the inductive hypothesis that $a_{n+p+j-k-1}(j)=0$ for $j=0, \cdots, k$, and hence

$$A(k+1, n+p)(k+1)!\, a_{n+p}(k+1)=0, \quad p \geqq k+2 .$$

Therefore degree $p_{k+1} \leqq n+k+1$.

This result allows a considerable simplification of the system (2.2). For each nonnegative integer p consider the subsystem S_p of (2.2)

$$c_{i, i+p}=\bar{c}_{i+p, i}, \quad i=0,1, \cdots .$$

Since the equation $c_{ij}=\bar{c}_{ji}$ appears only in $S_{|i-j|}$ we have a partition of (2.2). Since the p_i are polynomial of degree at most $n+i$,

$$a_{\ell+p}(\ell)=0 \quad p>n, \quad \ell=0, \cdots, n,$$

from which it follows that S_p is trivial for $p>n$. From (2.1) we see that $a_{\ell}(i)$ appears only in $S_{|\ell-i|}$. Hence (2.2) is equivalent to the $n+1$ systems,

$$S_p: c_{i, i+p}=\bar{c}_{i+p, i}, \quad i=0,1, \cdots,$$

where the $a_{j+p}(j)$ appear only in S_p. Using (2.1) this becomes

$$S_p: \sum_{k=0}^{n} a_{p+k}(k) B(i, k)=\sum_{k=p}^{n} \bar{a}_{k-p}(k) B(i+p, k) A^2(i+p, i) . \tag{2.4}$$

THEOREM 2.3. *The system S_p is satisfied if and only if*

$$j!\, a_{j+p}(j)=R_0^j \quad j=0,1, \cdots, n, \tag{2.5}$$

where $R_i^0=\sum_{k=p}^n \bar{a}_{k-p}(k) B(i+p, k) A^2(i+p, i)$, and the R_i^j are obtained recursively by

$$R_i^j=R_{i+1}^{j-1}-R_i^{j-1} . \tag{2.6}$$

Proof. For fixed p denote the left and right hand sides of the ith member of S_p by L_i^0 and R_i^0 respectively. We now employ a reduction scheme. Form the sequence of systems $\{L_i^1=R_i^1\}, \{L_i^2=R_i^2\}, \cdots$, where

$$\begin{aligned} L_i^{j+1} &= L_{i+1}^j-L_i^j \\ R_i^{j+1} &= R_{i+1}^j-R_i^j \qquad i, j=0,1, \cdots . \end{aligned}$$

By induction on j it can be shown that

$$L_i^j = \sum_{k=0}^{n} a_{k+p}(k) B(i, k-j) P_j(k)$$

where $P_j(k) = k(k-1) \cdots (k-j+1)$. Consequently, $L_0^j = j!\, a_{j+p}(j)$ and the necessity follows.

For the sufficiency we use the fact that for a given system of linear equations, $L^j = R^j$, $j = 0 \cdots, n$, there exists a unique set of linear systems $\{\hat{L}_i^0 = \hat{R}_i^0\}, \cdots, \{\hat{L}_i^n = \hat{R}_i^n\}$ which have the properties $P1$ thru $P3$.

$$\begin{aligned}
&P1\ \hat{L}_i^j = \hat{L}_{i+1}^{j-1} - \hat{L}_i^{j-1} \\
&\hat{R}_i^j = \hat{R}_{i+1}^{j-1} - \hat{R}_i^{j-1} && j = 1, \cdots, n \\
& && i = 0, 1, \cdots \\
&P2\ \hat{L}_0^j = L^j,\ \hat{R}_0^j = R^j && j = 0, \cdots, n \\
&P3\ \hat{L}_i^n = L^n,\ \hat{R}_i^n = R^n && i = 0, 1, \cdots .
\end{aligned}$$

This set is constructed in the following manner.

The system $\{\hat{L}_i^n = \hat{R}_i^n\}$ is defined by $P3$. To satisfy $P1$ and $P2$ we define the system $\{\hat{L}_i^{n-1} = \hat{R}_i^{n-1}\}$ inductively by $\hat{L}_0^{n-1} = L^{n-1}$, $\hat{R}_0^{n-1} = R^{n-1}$, $\hat{L}_{i+1}^{n-1} = \hat{L}_i^{n-1} + L^n$, and $\hat{R}_{i+1}^{n-1} = \hat{R}_i^{n-1} + R^n$. Similarly we define the system $\{\hat{L}_i^{n-2} = \hat{R}_i^{n-2}\}$ through $\{\hat{L}_i^0 = \hat{R}_i^0\}$ by means of the equations

$$\begin{aligned}
\hat{L}_0^{n-2} &= L^{n-2},\ \hat{R}_0^{n-2} = R^{n-2} \\
\hat{L}_{i+1}^{n-2} &= \hat{L}_i^{n-2} + \hat{L}_i^{n-1},\ \hat{R}_{i+1}^{n-2} = \hat{R}_i^{n-2} + \hat{R}_i^{n-1} \\
\hat{L}_0^0 &= L^0,\ \hat{R}_0^0 = R^0 \\
\hat{L}_{i+1}^0 &= \hat{L}_i^0 + \hat{L}_i^1,\ \hat{R}_{i+1}^0 = \hat{R}_i^0 + \hat{R}_i^1 .
\end{aligned}$$

From the method of construction the systems $\{\hat{L}_i^0 = \hat{R}_i^0\}$ thru $\{\hat{L}_i^n = \hat{R}_i^n\}$ are the unique systems satisfying $P1$ thru $P3$.

Since $P_j(k)$ vanishes for $0 \leqq k \leqq j-1$ it follows that $L_i^j = 0$ for $j > n$ and all i. Moreover, for $j = n$ we have $L_i^n = n!\, a_{n+p}(n)$, a constant independent of i. From (2.4) we see that $R_i^0 = \sum_{k=p}^{n} \bar{a}_{k-p}(k) C_k(i)$, where the $C_k(i)$ are polynomials in i of degree k. Hence $R_i^1 = R_{i+1}^0 - R_i^0$ can be written in the form $\sum_{k=p}^{n} \bar{a}_{k-p}(k) C_k^1(i)$, where the $C_k^1(i)$ are of degree $k-1$. Continuing in this manner we obtain

$$\begin{aligned}
&R_i^j = 0 \quad j > n && i = 0, 1, \cdots , \\
&R_i^n = R_0^n && i = 0, 1, \cdots .
\end{aligned}$$

Hence the systems $\{L_i^j = R_i^j\}$ $j = 0, \cdots, n$ satisfy $P1$ thru $P3$ where $L_0^j = R_0^j$ corresponds to the $L^j = R^j$ and the system $\{\hat{L}_i^0 = \hat{R}_i^0\}$ corresponds to the system S_p. This yields the sufficiency.

This theorem provides an algorithm for determining all formally symmetric operators of a given order. As an application we give the general formally symmetric first order operator. Use of 2.5 for $p = 0$ and 1 yields

$$L = (cz^2 + az + \bar{c})d/dz + (2cz + b) ,$$

where a and b are real.

3. Self-adjoint extensions. The operator S has another characterization which will be of use in the study of self-adjoint extensions. For f and g in $\mathscr{D}$ consider the bilinear form

$$\langle fg \rangle = (Lf, g) - (f, Lg) , \tag{3.1}$$

and let $\tilde{\mathscr{D}}$ be the set of those f in $\mathscr{D}$ for which $\langle fg \rangle = 0$ for all g in $\mathscr{D}$. Since $S = T^*$ and $\mathscr{D}(T^*) = \tilde{\mathscr{D}}$, S has domain $\tilde{\mathscr{D}}$.

Let $\mathscr{D}^+$ and $\mathscr{D}^-$ denote the set of all solutions of the equations $Lu = iu$ and $Lu = -iu$ respectively, which are in $\mathscr{H}$. It is known from the general theory of Hilbert space [3, p. 1227–1230] that

$$\mathscr{D} = \tilde{\mathscr{D}} + \mathscr{D}^+ + \mathscr{D}^- , \tag{3.2}$$

and every $f \in \mathscr{D}$ has the unique representation

$$f = \tilde{f} + f^+ + f^-, \ (\tilde{f} \in \tilde{\mathscr{D}}, f^+ \in \mathscr{D}^+, f^- \in \mathscr{D}^-) .$$

Let the dimensions of $\mathscr{D}^+$ and $\mathscr{D}^-$ be m^+ and m^- respectively. Clearly, m^+ and m^- cannot exceed the order of L. These integers are referred to as the deficiency indices of S, and S has self-adjoint extensions if and only if $m^+ = m^-$. Moreover S is itself self-adjoint if and only if $m^+ = m^- = 0$.

We assume that $m^+ = m^- = m$ and seek to characterize all self-adjoint extensions of S. Von Neumann has shown that the self-adjoint extensions of S are in a one-to-one correspondence with the unitary operators U of $\mathscr{D}^+$ onto $\mathscr{D}^-$. Corresponding to any such U there exists a self-adjoint extension A of S whose domain is the set of all $f \in \mathscr{D}$ which are of the form

$$f = \tilde{f} + (I - U)f^+ , \quad (f \in \tilde{\mathscr{D}}, f^+ \in \mathscr{D}^+) ,$$

where I is the identity operator on $\mathscr{D}^+$. Conversly every such A has a domain of this type.

We now introduce the notion of abstract boundary conditions and indicate how the domain of any self-adjoint extension of S can be obtained. A boundary condition is a condition on $f \in \mathscr{D}$ of the form

$$\langle fh \rangle = 0 ,$$

where h is a fixed function in $\mathscr{D}$. The conditions

$$\langle fh_j \rangle = 0 , \qquad j = 1, \cdots, n ,$$

are said to be linearly independent if the only set of complex numbers $\alpha_1, \cdots, \alpha_n$ for which

$$\sum_{j=1}^{n} \alpha_j \langle fh_j \rangle = 0$$

identically in $f \in \mathscr{D}$ is $\alpha_1 = \cdots = \alpha_n = 0$. A set of n linearly independent boundary conditions $\langle fh_j \rangle = 0$, $j = 1, \cdots, n$, is said to be self-adjoint if $\langle h_j h_k \rangle = 0$, $j, k = 1, \cdots, n$.

The following theorem follows directly from the proof of Theorem 3 in the paper of Coddington [1].

THEOREM 3.1. *If A is a self-adjoint extension of S with domain $\mathscr{D}_A$, then there exists a set of m self-adjoint boundary conditions,*

$$\langle fh_j \rangle = 0 \qquad j = 1, \cdots, m , \tag{3.3}$$

such that $\mathscr{D}_A$ is the set of all $f \in \mathscr{D}$ satisfying these conditions. Conversly, if (3.3) *is a set of m self-adjoint boundary conditions, there exists a self-adjoint extension A of S whose domain is the set of all $f \in \mathscr{D}$ satisfying* (3.3)

Let $\phi_1, \cdots, \phi_m$ and $\psi_1, \cdots, \psi_m$ be orthonormal sets for $\mathscr{D}^+$ and $\mathscr{D}^-$ respectively and (u_{jk}) a unitary matrix representing U, then the h_j are given by

$$h_j = \phi_j - \sum_{k=1}^{m} u_{jk} \psi_k , \qquad j = 1, \cdots, m . \tag{3.4}$$

Let A be a self-adjoint operator associated with L and $E(\lambda)$ the corresponding resolution of the identity. We shall show the projection E_Δ corrresponding to $\Delta = (a, b]$ can be expressed as an integral operator with a kernel given in terms of a basis of solutions for $Lu - \lambda u = 0$ and a certain spectral matrix. Our work was inspired by the treatment of E. A. Coddington [2] of the case when A arises from a formal differential operator in the space $L_2(I)$, I an open interval. We begin by showing that the resolvent operator of A,

$$R(\ell) = (A - \ell)^{-1} , \quad \mathrm{Im}\,(\ell) \neq 0 ,$$

is an integral operator with a nice kernel.

THEOREM 3.2. *$R(\lambda)$ is an integral operator with kernel K,*

$$R(\lambda)f(z) = \iint_{|w|<1} K(z, w, \lambda)f(w)dudv\ , \quad f \in \mathscr{H}\ . \tag{3.5}$$

K is jointly analytic in z, $\bar{w}$, and λ on the region $|z| < 1$, $|w| < 1$, $\mathrm{Im}\,(\lambda) \neq 0$.

Moreover, $K(z, w, \lambda) = \overline{K(w, z, \bar{\lambda})}$ and

$$(L - \lambda)K(w, z, \lambda) = K_z(w),\ \textit{for fixed } z \textit{ and } \lambda. \tag{3.6}$$

Proof. Since $R(\lambda)f(z) = (R(\lambda)f, K_z)$ and $R^*(\lambda) = R(\bar{\lambda})$, it follows that (3.1) holds with $K(z, w, \lambda) = \overline{R(\bar{\lambda})K_z(w)}$. Hence K is analytic in $\bar{w}$ for fixed z and λ. That $K(z, w, \lambda) = \overline{K(w, z, \bar{\lambda})}$ can be seen from the following computations,

$$K(z, w, \lambda) = \overline{(R(\bar{\lambda})K_z, K_w)} = \overline{(K_z, R(\lambda)K_w)} = \overline{K(w, z, \bar{\lambda})}\ .$$

Hence $K(z, w, \lambda)$ is analytic in z for fixed w and λ. It follows from the analyticity of $R(\lambda)$ for $\mathrm{Im}\,(\lambda) \neq 0$ that $K(z, w, \lambda) = (R(\lambda)K_w, K_z)$ is analytic in λ for fixed z and w on any region for which $\mathrm{Im}\,(\lambda) \neq 0$. Since analyticity in each of the variables separately implies joint analyticity it only remains to verify (3.6). This follows from the fact that $K(w, z, \lambda) = \overline{K(z, w, \bar{\lambda})} = R(\lambda)K_z(w)$.

We now split the kernel $K(z, w, \lambda)$ into two parts one of which satisfies the homogeneous equation $(L-\lambda)u=0$. Since the coefficients of L are polynomials, p_n has at most a finite number of zeros in the unit disk. Introducing radial branchcuts at these zeros, we obtain the region $\tilde{D}$, simply connected relative to D, in which p_n never vanishes. Let $z_0 \in \tilde{D}$, it follows from standard theorems that there exists a basis of solutions for the equation $(L - \lambda)\phi = 0$ such that:

(i) $\phi_i(\lambda)$, $i = 1, \cdots, n$, are single-valued analytic functions on $\tilde{D}$

(ii) $\phi_i^{(j-1)}(z_0, \lambda) = \delta_{ij}$, $i, j = 1, \cdots, n$,

(iii) $\phi_i(w, \lambda)$, $i = 1, \cdots, n$, is entire in λ for each $w \in \tilde{D}$.

THEOREM 3.3. *The kernel $K(z, w, \lambda)$ has the representation*

$$K(z, w, \lambda) = \sum_{i,j=1}^{n} \psi_{ij}(\lambda)\phi_i(z, \lambda)\overline{\phi_j(w, \bar{\lambda})} + G(z, w, \lambda)\ , \tag{3.7}$$

where $G(z, w, \lambda)$ is entire in λ for fixed z and w.

Proof. For fixed $z \in \tilde{D}$ and $\mathrm{Im}\,(\lambda) \neq 0$ it follows from (3.6) that

$$K(w, z, \bar{\lambda}) = \sum_{j=1}^{n} \psi_j(z, \lambda)\phi_j(w, \bar{\lambda}) + \Omega(z, w, \bar{\lambda}) , \tag{3.8}$$

where $\Omega(z, w, \bar{\lambda})$ is the particular solution furnished by the variation of parameters method and is entire in $\bar{\lambda}$ for fixed z, w. Moreover,

$$\frac{\partial^{i-1}}{\partial w^{i-1}}\Omega(z, z_0, \bar{\lambda}) = 0 , \qquad i = 1, \cdots, n . \tag{3.9}$$

Now consider the differential equation $(L_z - \lambda)K(z, w, \lambda) = K_w(z)$, where L_z denotes the fact that L is applied with respect to z. Differentiating with respect to $\bar{w}$ and making use of the symmetry of K we obtain

$$(L_z - \lambda)\frac{\partial^{j-1}}{\partial \bar{w}^{j-1}}\overline{K(w, z, \bar{\lambda})} = \frac{\partial^{j-1}}{\partial \bar{w}^{j-1}}K_w(z) , \quad j = 1, \cdots, n .$$

Using (3.8), (3.9) and the relationships

$$\phi_i^{(j-1)}(z_0, \lambda) = \delta_{ij}$$

we obtain

$$(L_z - \lambda)\overline{\psi_j(z, \lambda)} = \frac{\partial^{j-1}}{\partial \bar{w}^{j-1}}K_{z_0}(z) .$$

Variation of parameters yields

$$\psi_j(z, \lambda) = \sum_{i=1}^{n} \bar{\psi}_{ij}(\lambda)\overline{\phi_i(z, \lambda)} + \overline{\Omega_j(z, \lambda)} , \quad j = 1, \cdots, n , \tag{3.10}$$

where the $\Omega_j(z, \lambda)$ are entire in λ for fixed z and satisfy

$$\frac{\partial^{i-1}}{\partial z^{i-1}}\Omega_j(z_0, \lambda) = 0 , \qquad i, j = 1, \cdots, n . \tag{3.11}$$

It follows from (3.8) and (3.10) that (3.7) holds where

$$G(z, w, \lambda) = \overline{\Omega(z, w, \bar{\lambda})} + \sum_{j=1}^{n} \Omega_j(z, \lambda)\overline{\phi_j(w, \bar{\lambda})}$$

is entire in λ for each z, $w \in \tilde{D}$.

Concerning the matrix $\psi = (\psi_{ij})$ we have the following.

THEOREM 3.4. *The matrix* ψ *is analytic for* $\mathrm{Im}(\lambda) \neq 0$, $\psi^*(\lambda) = \psi(\bar{\lambda})$, *and* $\mathrm{Im}\,\psi(\lambda)/\mathrm{Im}(\lambda) \geqq 0$, *where* $\mathrm{Im}\,\psi = (\psi - \psi^*)/2i$.

Proof. It follows from (3.9) and (3.10) that

$$\psi_{ij}(\ell) = \frac{\partial^{i+j-2}}{\partial z^{i-1}\partial w^{j-1}} K(z_0, z_0, \ell) \,, \qquad i, j = 1, \cdots, n \,, \tag{3.12}$$

and hence ψ is analytic for $\mathrm{Im}\,(\ell) \neq 0$. Using (3.12) and the symmetry of K we obtain $\psi_{ij}(\ell) = \overline{\psi_{ji}(\ell)}$.

In order to demonstrate the positivity of $\mathrm{Im}\,\psi(\ell)/\mathrm{Im}\,(\ell) \geqq 0$ we consider the functionals ℓ_k defined by

$$\ell_k(f) = f^{(k-1)}(z_0) \,, \qquad f \in \mathscr{H}, k = 1, \cdots, n \,.$$

Since convergence in $\mathscr{H}$ implies uniform convergence on compact subsets, the ℓ_k are bounded linear functional on $\mathscr{H}$. Consequently there exist functions $K_1, \cdots, K_n$ in $\mathscr{H}$ for which

$$f^{(k-1)}(z_0) = (f, K_k) \,,$$

all f in $\mathscr{H}$. Let $\xi_1, \cdots, \xi_n$ be any set of n complex numbers and consider the function $f = \sum_{k=1}^{n} \xi_k K_k$. The inner product $(R(\ell)f, f) = \sum_{i,j=1}^{n} \xi_i \bar{\xi}_j (R(\ell)K_i, K_j)$. Now $R(\ell)K_i(z) = (K_i, K_{z\ell})$, where $K_{z\ell}(w) = \overline{K(z, w, \ell)} = K(w, z, \bar{\ell})$. Consequently,

$$R(\ell)K_i(z) = \overline{\frac{\partial^{i-1}}{\partial w^{i-1}} K(z_0, z, \bar{\ell})} \,,$$

and

$$(R(\ell)K_i, K_j) = \frac{\partial^{i+j-2}}{\partial^{i-1}\bar{w}\partial z^{j-1}} K(z_0, z_0, \ell) = \psi_{ji}(\ell) \,.$$

Using the resolvent equation it is not hard to see that

$$\mathrm{Im}\,(R(\ell)f, f)/\mathrm{Im}\,(\ell) = \| R(\ell)f \|^2 \geqq 0$$

and hence

$$\sum_{i,j=1}^{n} \frac{\mathrm{Im}\,\psi_{ji}(\ell)}{\mathrm{Im}\,(\ell)} \xi_i \bar{\xi}_j \geqq 0 \,.$$

This completes the proof.

It is shown in [2] that Theorem 3.4 implies the existence of a spectral matrix ρ for the resolvent R.

THEOREM 3.5. *The matrix ρ defined by*

$$\rho(\lambda) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_0^{\lambda} \mathrm{Im}\,(\nu + i\varepsilon) d\nu$$

exists, is nondecreasing, and is of bounded variation on any finite interval.

We now consider the projections E_Δ corresponding to the interval $\Delta = (a, b]$. It follows from the proof of Theorem 3.2, that E_Δ is an integral operator with kernel $e_\Delta(z, w) = \overline{E_\Delta K_z(w)}$. The following theorem shows how $e_\Delta(z, w)$ can be described in terms of the basis $\phi_1, \cdots, \phi_n$ and the spectral matrix given by Theorem 3.5.

THEOREM 3.6. *If a and b are continuity points of E then*

$$e_\Delta(z, w) = \int_\Delta \sum_{i,j=1}^{n} \phi_i(z, \nu)\overline{\phi_j(w, \nu)}d\rho_{ij}(\nu) , \tag{3.13}$$

where $\rho = (\rho_{ij})$ is the spectal matrix given by Theorem 3.5.

Proof. The idea is to use the inversion formula

$$(E_\Delta f, g) = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_\Delta ((R(\nu + i\varepsilon)f, g) - (R(\nu - i\varepsilon)f, g))d\nu ,$$

for all f and g in $\mathscr{H}$, a and b continuity points of E_λ. Since E_Δ is self-adjoint $e_\Delta(z, w) = (E_\Delta K_w, K_z)$ and hence

$$\begin{aligned} e_\Delta(z, w) &= \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_\Delta \{(R(\nu + i\varepsilon)K_w, K_z) - (R(\nu - i\varepsilon)K_w, K_z)\}d\nu . \\ &= \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_\Delta K(z, w, \nu + i\varepsilon) - K(z, w, \nu - i\varepsilon)d\nu . \end{aligned}$$

For $z, w \in \tilde{D}$, this becomes

$$\begin{aligned} &\lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_\Delta \sum_{i,j=1}^{n} \psi_{ij}(\nu + i\varepsilon)\phi_i(z, \nu + i\varepsilon)\overline{\phi_j(w, \nu - i\varepsilon)} \\ &\quad - \psi_{ij}(\nu - i\varepsilon)\phi_i(z, \nu - i\varepsilon)\overline{\phi_j(w, \nu + i\varepsilon)}d\nu \\ &\quad + \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_\Delta G(z, w, \nu + i\varepsilon) - G(z, w, \nu - i\varepsilon)d\nu . \end{aligned}$$

Since $G(z, w, \ell)$ is entire in ℓ the later integral tends to zero as $\varepsilon \to +0$.

We now rewrite the first integrand as

$$\begin{aligned} &\sum_{i,j=1}^{n} [\psi_{ij}(\nu + i\varepsilon) - \psi_{ij}(\nu - i\varepsilon)]\phi_i(z, \nu)\overline{\phi_j(w, \nu)} + \\ &\sum_{i,j=1}^{n} \psi_{ij}(\nu + i\varepsilon)[\phi_i(z, \nu + i\varepsilon)\overline{\phi_j(w, \nu - i\varepsilon)} - \phi_i(z, \nu)\overline{\phi_j(w, \nu)}] + \\ &\sum_{i,j=1}^{n} \psi_{ij}(\nu - i\varepsilon)[\phi_i(z, \nu)\overline{\phi_j(w, \nu)} - \phi_i(z, \nu - i\varepsilon)\overline{\phi_j(w, \nu + i\varepsilon)}] , \end{aligned}$$

and denote the three sums by $I_1(\nu, \varepsilon)$, $I_2(\nu, \varepsilon)$, and $I_3(\nu, \varepsilon)$ respectively.

Consider $I_1(\nu, \varepsilon)$,

$$\lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{\Delta} I_1(\nu, \varepsilon) d\nu = \lim_{\varepsilon+0} \frac{1}{\pi} \int_{\Delta} \sum_{i,j=1}^{n} \operatorname{Im} \psi_{ij}(\nu + i\varepsilon) \phi_i(z, \nu) \overline{\phi_j(w, \nu)} d\nu .$$

Now

$$\rho(\lambda) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\Delta} \operatorname{Im} \psi(\nu + i\varepsilon) d\nu$$

and it follows from a theorem of Helly that

$$\lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{\Delta} I_1(\nu, \varepsilon) d\nu = \int_{\Delta} \sum_{i,j=1}^{n} \phi_i(z, \nu) \overline{\phi_j(w, \nu)} d\rho_{ij}(\nu) . \tag{3.14}$$

As is shown in [2] we have the following estimate

$$\sum_{i,j=1}^{n} \int_{\Delta} | \psi_{ij}(\nu \pm i\varepsilon) | d\nu = O\Big(\log \frac{1}{\varepsilon}\Big) \qquad (\varepsilon \to +0) . \tag{3.15}$$

Since the $\phi_i(z, \ell)$ are entire in ℓ for fixed z there exists a constant $M > 0$ such that for ε sufficiently small

$$| \phi_i(z, \nu + i\varepsilon) \overline{\phi_j(w, \nu - i\varepsilon)} - \phi_i(z, \nu) \overline{\phi_j(w, \nu)} | < M\varepsilon \tag{3.16}$$

for all $\nu \in \Delta$.

Combining (3.15) and (3.16) we see that

$$\frac{1}{\pi} \int_{\Delta} I_2(\nu, \varepsilon) d\nu = O\Big(\varepsilon \log \frac{1}{\varepsilon}\Big) \qquad (\varepsilon \to +0) ,$$

which tends to zero as $\varepsilon \to +0$. A similar result holds for

$$\frac{1}{\pi} \int_{\Delta} I_3(\nu, \varepsilon) d\nu .$$

Consequently we have

$$e_{\Delta}(z, w) = \int_{\Delta} \sum_{i,j=1}^{n} \phi_i(z, \nu) \overline{\phi_j(w, \nu)} d\rho_{ij}(\nu) . \tag{3.13}$$

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UNIVERSITY OF CALIFORNIA, LOS ANGELES
AND
SAN DIEGO STATE COLLEGE