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ON THE BERGMAN INTEGRAL OPERATOR FOR AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION WITH A SINGULAR COEFFICIENT

P. ROSENTHAL

Let $P_2(f)$ be Bergman's integral operator of the second kind. In this paper it is shown (1) $P_2(f)$ can be uniformly approximated by a linear combination of particular solutions; (2) $P_2(f)$ can be analytically continued; (3) $P_2(f)$ admits singular points if f is meromorphic.

In the study of functions of one complex variable one derives various relations between properties of the coefficients a_ν of the series development

$$(1) \qquad f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu ,$$

of the function $f(z)$ and various properties of $f(z)$ in the large, such as the location and character of the singularities, growth of the function, etc. The method of integral operators enables one to generalize these theorems to the theory of linear partial differential equations

$$(2) \qquad L(\psi) = \frac{\partial^2 \psi}{\partial z \partial z^*} + A_1(z, z^*)\psi_z + A_2(z, z^*)\psi_{z^*} + A_3(z, z^*)\psi = 0 ;$$

$4(\partial^2\psi\backslash\partial z\partial z^*) = \Delta_1\psi = (\partial^2\psi\backslash\partial\lambda^2 + \partial^2\psi\backslash\partial\theta^2)$, z, z^* are complex variables, $\lambda = z + z^*\backslash 2$, $\theta = z - z^*\backslash 2i$, $A_\nu(z, z^*)$, $\nu = 1, 2, 3$, are regular functions of z and z^* in a sufficiently large domain. The situation changes in the case when the A_ν admit singularities. In this paper we consider the equation

$$(3) \qquad L(\psi) = \Delta_1\psi + 4F(\lambda)\psi \equiv 0 ,$$

where $F(s) = s^{-3}(a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n + \cdots)$, $s = (-\lambda)^{2/3}$, $a_0 = 5\backslash 144$, $a_1 = 0$, while the a_n, $n \geqq 2$ are such that $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 0$.

The integral operator

$$(4) \qquad \psi(z, z^*) \equiv P_2(f) = \int_l E(z, z^*, t) f\Big(\frac{z}{2}(1 - t^2)\Big)\frac{dt}{\sqrt{1 - t^2}}$$

(where l is some rectifiable Jordan path in the upper complex t-plane connecting the points -1 and 1), transforming analytic functions $f(z)$ in the neighborhood of the origin into solutions of (3), has been introduced and investigated by S. Bergman in [1, 2, 5, 6], see also

[8], [10]. E, $\neq 0$, called a generating function, is analytic in the three variables z, z^* and t providing $|z + z^*| < |t^2 z|$.

In analogy to (1) we write

$$\psi(z, z^*) = \sum_{\nu=0}^{\infty} a_\nu \psi_\nu(z, z^*) , \tag{5}$$

where

$$\psi_\nu(z, z^*) = \int_l E(z, z^*, t)\left(\frac{z}{2}(1 - t^2)\right)^\nu \frac{dt}{\sqrt{1 - t^2}} . \tag{6}$$

In § 2 it is shown every solution ψ regular in the wedge domain $W = \{(\lambda, y) | 3^{1/2}|\lambda| < y, \lambda \leqq 0, y > 0\}$ (a case which arises in the study of two-dimensional nonviscous compressible fluid flow problems) can be uniformly approximated by finite linear combinations $\sum_{\nu=0}^{N} a_\nu \psi_\nu(z, z^*)$, where $z = \lambda + iy$, $z^* = \bar{z} = \lambda - iy$, on certain compact sets $Q \subset W$.

In § 3 an extension and summation method is applied to derive an extension of the operator ψ defined by (5). In § 4 it is shown that the Borel theorem on the multiplication of poles can be extended to (5).

2. Uniform approximation of a solution ψ by finite linear combinations of the particular solutions (6) in W. Consider any domain $D \subset W$ which is bounded by the closed segments $0A_1, 0A_2, 0$ the origin, and the arc $\widehat{A_1 A_2}$, $0A_1, 0A_2$ lie on the respective lines $\alpha = \alpha_1, \alpha_2$, $\pi - \tan^{-1}\sqrt{3} \geqq \alpha_1 > \alpha_2 \geqq \pi \backslash 2$. Let $W \supset R = \{(\lambda, y) | y \backslash -\lambda > [((1 - t_1^4)^{1/2}) \backslash t_1^2], 0 < 2t_1^2 < t_0^2, 0 < t_0 < 1, 0 < t_0 \leqq |t|, t \in l\}$, l will be specified in what follows. Let Q be compact and $\subset R$.

THEOREM. *Suppose that f is continuous on $\bar{D}$, closure of D, and analytic in D as well as on the boundary segments $0A_1, 0A_2$ including the end points. Then the function*

$$\psi(z, z^*) = \int_l E(z, z^*, t) f\left(\frac{1}{2} z(1 - t^2)\right) \frac{dt}{\sqrt{1 - t^2}} \qquad z^* = \bar{z} , \tag{7}$$

can be uniformly approximated in Q by finite linear combinations of the particular solutions defined in (6).

Proof. We choose for the integration curve $l \equiv C = C_1 \cup C_2 \cup C_3$, $C_1 = (-1 \leqq t < t_0)$, $C_3 = (t_0 < t \leqq 1)$, $C_2 = (t = t_0 e^{i\varphi}, \pi \geqq \varphi \geqq 0, 1 > t_0 > 0)$. The existence of C for our case follows by modifying the proof of Lemma 7.1 of [4], namely, by replacing the inequality $\theta \backslash \Delta > 1 - t_1^2 \backslash t_1^2$ by $y \backslash -\lambda > (1 - t_1^4)^{1/2} \backslash t_1^2$, substituting y for θ and λ for Δ. This also determines R. Our hypotheses about f permit us to rotate the sides of the domain D through a small angle $\pi\backslash 2 > \Delta\alpha > 0$ to obtain a wedge-shaped domain S such that $\bar{D} \subset \bar{S}$ and $\bar{S}$ is contained in the domain of regularity of f.

LEMMA. *There exists a* $1 > t_0(\Delta\alpha) > 0$ *such that if* $z \in \bar{D}, t \in C$, *then* $z(1-t^2)\backslash 2 \in \bar{S}$.

Proof. For $t \in C_1 \cup C_3$, $1 - t^2 < 1$. Hence $z(1-t^2)\backslash 2 \in \bar{D} \subset \bar{S}$. We next consider the case $t \in C_2$. We choose for $t_0 = (\tan^2 \Delta\alpha/1 + \tan^2 \Delta\alpha)^{1/4}$. This choice of t_0 gives then $\Delta\alpha$ for the maximum argument of $1 - t^2$. Since the maximum of $|1 - t^2| = 1 + t_0^2$, the lemma follows. Since the domain W was obtained by taking l to be the semi-circle path in the upper half of the t-plane, $\psi(z, z^*)$ for $l = C$ is the regular restriction of $\psi(z, z^*)$ for $l = (t, t = e^{i\theta}, 0 \leqq \theta \leqq \pi)$. This is a known property of the operator defined by (4).

By our assumptions on $f(q)$, we can uniformly approximate f by polynomials $P_N(q) = \sum_{n=0}^{N} a_\nu q^\nu$, $q \in \bar{S}$, see [12, p. 36]. Let $q = 1\backslash 2(z(1-t^2))$, where $z \in Q, t \in l \equiv C$. By the above lemma, $q \in \bar{S}$. Then

$$\left| \int_{l=C} E(z, \bar{z}, t) f\left(\frac{z}{2}(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} - \int_{l=C} E(z, \bar{z}, t) P_N\left(\frac{1}{2} z(1-t^2)\right) \frac{dt}{\sqrt{1-t^2}} \right| < \varepsilon L M ,$$

L is the length of C, $M = \max_{z \in Q, t \in C} |E(z, \bar{z}, t)|$, $\varepsilon > 0$, and arbitrary. This completes the proof of the theorem.

3. Summation and extension methods applied to the operator $P_2(f)$. In the case of analytic functions of one complex variable when considering the series development $f(q) = \sum_{n=0}^{\infty} a_n q^n$ converging in the star domain, one can determine the values of f in a larger domain using various summation methods.

THEOREM. *Consider a sequence of particular solutions* $(\psi_\nu(z, z^*))$. *Let* $f(q) = \sum_{n=0}^{\infty} a_n q^n$ *in some neighborhood of the origin. Let* $\psi(z, z^*) = \sum_{n=0}^{\infty} a_n \psi_n(z, z^*)$ *be the solution determined by* $f(q)$ (see (6)). *Suppose further that a sequence* $(\sigma_n(\delta))$ *is given such that*

(1) $\sigma_n(\delta)$ *is real for* $\delta > 0$

(2) $\lim_{\delta \to 0^+} \sigma_n(\delta) = 1$

(3) $\overline{\lim}_{n \to \infty} |\sigma_n(\delta)|^{1/n} = 0, \ \delta > 0$

(4) $\varphi_\delta(z) = \sum_{n=0}^{\infty} \sigma_n(\delta) z^n \to 1\backslash 1 - z$ *as* $\delta \to 0^+$

uniformly in z in any compact set containing no point of the line $(1, \infty)$. *Then* $\lim_{\delta \to 0^+} \sum_{n=0}^{\infty} \sigma_n(\delta) a_n \psi_n(z, z^*)$ *will give the value of* $\psi(z, z^*)$

at any point (z, z^*), *where* ψ *exists.*

Proof. Let

$$\psi(z, z^*) = \int_l E(z, z^*, t) f\left(\frac{z}{2}(1 - t^2)\right) \frac{dt}{\sqrt{1 - t^2}} ,$$

where $f(z/2(1 - t^2))$ is the analytic function given at the origin by the series development $f(q) = \sum_{n=0}^{\infty} a_n q^n$. Since our hypotheses satisfy the known summation theorem, see [9, pp. 190–191], we conclude $\sum_{n=0}^{\infty} \sigma_n(\delta) a_n q^n \to f(q)$ as $\delta \to 0^+$ uniformly in q in every star domain with respect to the origin in which $f(q)$ is analytic. Because of the uniform convergence we are permitted to interchange the order of summation and integration to obtain

$$(8) \quad \int_l E(z, z^*, t) \sum_{n=0}^{\infty} \sigma_n(\delta) a_n \left(\frac{z}{2}(1 - t^2)\right)^n \frac{dt}{\sqrt{1 - t^2}} = \sum_{n=0}^{\infty} \sigma_n(\delta) a_n \psi_n(z, z^*) .$$

Also by our hypotheses we are permitted to interchange the limit and integration operations to obtain,

$$(9) \quad \lim_{\delta \to 0^+} \int_l E(z, z^*, t) \sum_{n=0}^{\infty} \sigma_n(\delta) a_n \left(\frac{1}{2} z(1 - t^2)\right)^n \frac{dt}{\sqrt{1 - t^2}} = \psi(z, z^*) .$$

(8) and (9) give us the result as was to be shown.

4. Application of a theorem of Borel. Bergman's theory of integral operators enables one to apply results in the theory of functions of one complex variable about the relations between coefficients of a_ν of the development $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and location and properties of singularities of $f(z)$ to the case of solutions of $L(\psi) = 0$. That singularities can occur for the operator $P_2(f)$, we note the following, which is an immediate consequence of a result in [5]: Let the associate function $f(q)$ be meromorphic with poles at $q = q_i \neq 0$, $1 \leqq i \leqq k$. Then $\psi(z, z^*) = P_2(f)$ will be singular, i.e., will not admit a Taylor series about the points (z, z^*), $z^* = -z$, $z = 2q_1, \cdots, 2q_k$.

THEOREM. *Assume that* $\psi(z, z^*)$ *has the development* $\sum_{n=0}^{\infty} a_n b_n \psi_n(z, z^*)$, *where* a_n, b_n *are the coefficients respectively of the meromorphic functions* $a(q)$, $b(q)$ *with series development about the origin* $\sum_{n=0}^{\infty} a_n q^n$, $\sum_{n=0}^{\infty} b_n q^n$, *and poles at* α_i, $i = 1, \cdots, p$, β_k, $k = 1, \cdots, r$, *respectively. Then* $\psi(z, z^*)$ *is singular at the points* $z^* = -z = -2\alpha\beta_k$, *providing* $\alpha_i \beta_k \neq \alpha\beta$, where α, β *are any other singular points or external points of* $a(q)$ *and* $b(q)$.

Proof. By a theorem of Borel (see [7, p. 106]) the function

$f(q) = \sum_{n=0}^{\infty} a_n b_n q^n$ has poles at the points $\alpha_i \beta_k$. By the result mentioned in § 4 the theorem follows.

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STANFORD UNIVERSITY