PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 3, 1970

NOTES ON COMMUTATIVE POWER JOINED SEMIGROUPS

RICHARD G. LEVIN AND TAKAYUKI TAMURA

Let S be a commutative semigroup. The main theorem in this paper is to prove that the following two conditions are equivalent: (1) For all $a, b \in S$ there are positive integers m, n such that $a^m = b^n$. (2) For all $a, b \in S$, $a^l = a^m b^n$, $b^r = b^s a^t$ for some l, m, n, r, s, t. As a consequence of the theorem, the authors prove that a commutative archimedean semigroup S without idempotent is power joined if and only if the structure group of S is a torsion group.

Let S be a commutative archimedean semigroup without idempotent. Consider the following question: "Under what condition on the structure group (defined below) of S will S be power joined?" Levin proved in [4] that if S is finitely generated, equivalently if the structure group of S is finite, then S is power joined. Also he obtained a necessary and sufficient condition for S to be power joined. The following is Theorem 2 in [4]:

THEOREM 1. *Let S be a commutative, archimedean semigroup without idempotent. Let $G_a = S/\rho_a$ be the structure group of S determined by a. Then S is power joined if and only if G_a is periodic and the congruence class containing a modulo ρ_a is power joined.*

If we assume that S is additionally cancellative, that is, S is an $\mathfrak{N}$-semigroup, then the answer is simple. The following is due to Chrislock [1, 2].

THEOREM 2. *An $\mathfrak{N}$-semigroup S is power joined if and only if G_a is periodic for some $a \in S$, equivalently for all $a \in S$.*

Naturally the following question is raised: Can Theorem 1 be improved such that Theorem 2 is extended to S in Theorem 1? The question is affirmative. In this paper we study the problem for more general case, i.e., for commutative archimedean semigroups. The main theorem of this paper asserts that a commutative semigroup S is power joined if and only if it is archimedean and its group homomorphic images are periodic. As a corollary we can answer the above question.

Semigroups are assumed to be commutative throughout this paper.

DEFINITION 1. A semigroup S is called power joined if and only if for all $a, b \in S$, there are positive integers n, m such that

$$a^n = b^m .$$

DEFINITION 2. A semigroup S is called archimedean if and only if for all $a, b \in S$, there exist $u, v \in S$ and positive integers n, m such that

$$a^n = bu \quad \text{and} \quad b^m = av .$$

DEFINITION 3. Let S be an archimedean semigroup without idempotent. We define a congruence ρ_b on S for fixed $b \in S$ as follows. We define $x\rho_b y$ if and only if there are positive integers n and m such that

$$b^n x = b^m y .$$

REMARK. More information on commutative, archimedean semigroups without idempotent can be found in [1], [6] and [7]. In particular a proof that ρ_b (as defined above) is a congruence relation and that $S/\rho_b = G_b$ is a group can be found in [7]. S/ρ_b is called the structure group of S determined by b. Also notice $xy \neq y$ for all $x, y \in S$.

THEOREM 3. *The following statements are equivalent.*

(3.1) *The semigroup S is power joined.*

(3.2) *The semigroup S is archimedean and its group homomorphic images are periodic.*

(3.3) *The semigroup S satisfies the conditions: for all pairs $a, b \in S$, there are positive integers l, m, n, s, t, p such that*

$$a^l = a^m b^n \quad \textit{and} \quad b^s = b^t a^p .$$

Proof. We will prove: $(3.1) \Rightarrow (3.2) \Rightarrow (3.3) \Rightarrow (3.1)$. Let S be a power joined semigroup. It is trivial to show that S is archimedean. Let G be a group homomorphic image of S with $\varphi\colon S \to G$ the homomorphism. We will show that G is a periodic group. Let $a \in G$ and let e be the identity of G. There exist $x, y \in S$ such that $\varphi(x) = a$, $\varphi(y) = e$. Since S is power joined, there exist positive integers n, m such that $x^n = y^m$. Then

$$a^n = [\varphi(x)]^n = \varphi(x^n) = \varphi(y^m) = [\varphi(y)]^m = e^m = e .$$

We see that G is periodic and this completes the proof that $(3.1) \Rightarrow (3.2)$.

We next prove that (3.2) $\Rightarrow$ (3.3). Let S be an archimedean semigroup whose group homomorphic images are periodic.

Case 1. Assume that S has an idempotent e. Then the set Se is a group and is the homomorphic image of S (see [3] or [5]). Let $a, b \in S$. Then ae and be are elements of Se. Since Se is a periodic group with e as its identity element, there exist positive integers n and m such that

$$(ae)^n = e \quad \text{and} \quad (be)^m = e \, .$$

That is,

$$a^n e = e = b^m e \, . \tag{1}$$

Since S is archimedean, there exist positive integers k and t and $u, v \in S$ such that

$$a^k = ev \quad \text{and} \quad b^t = eu \, . \tag{2}$$

From equations (1) and (2) we derive

$$\begin{aligned} a^n eu &= b^m eu, \\ \text{or } a^n b^t &= b^m b^t, \\ \text{or } a^n b^t &= b^r \quad \text{where} \quad \mathrm{r} = m + t \, . \end{aligned}$$

Similarly, we derive $a^l = a^k b^m$ for some positive integers l, k and m.

Case 2. Assume that S does not have an idempotent. Let $a, b \in S$. Consider the congruence ρ_a of Definition 3. Then S/ρ_a is a group homomorphic image of S and, therefore, is a periodic group. Also

$$S = \bigcup_{\lambda \in S/\rho_a} S_\lambda$$

and $a \in S_\varepsilon$, where ε is the identity of S/ρ_a. There is $\lambda \in S/\rho_a$ such that $b \in S_\lambda$. There exists a positive integer k such that $\lambda^k = \varepsilon$. Thus,

$$b^k \in S_{\lambda^k} = S_\varepsilon \, .$$

That is, a and b^k are ρ_a related. By definition of ρ_a, there are positive integers n and m such that

$$\begin{aligned} a^n a &= a^m b^k \, , \\ \text{or } a^l &= a^m b^k \, , \quad \text{where} \quad l = n + 1 \, . \end{aligned}$$

Similarly, we can derive the equation

$$b^s = b^t a^p \, .$$

The proof that (3.2) $\Rightarrow$ (3.3) is now complete.

We now prove that (3.3) $\Rightarrow$ (3.1).

Case 1. Assume that S has an idempotent e. Let $a \in S$. Then there are positive integers l, m, n, s, t and p such that

$$e^l = e^m a^n \quad \text{and} \quad a^s = a^t e^p , \tag{3}$$

$$\text{or } e = ea^n \quad \text{and} \quad a^s = a^t e . \tag{4}$$

Using the equations of (4) we derive

$$e = e^t = (ea^n)^t = e^t(a^t)^n = (ea^t)^n = (a^s)^n .$$

Thus, we have $e = a^r$ for a positive integer r.
It is now obvious that if $a, b \in S$, there are positive integers u and v such that $a^u = b^v$. Therefore S is power joined.

Case 2. Assume that S has no idempotent. Again we have for any pair $a, b \in S$, positive integers l, m, n, s, t and p such that

$$a^l = a^m b^n \quad \text{and} \quad b^s = b^t a^p . \tag{5}$$

We will prove that there are positive integers l' and n' such that $a^{l'} = a^m b^{n'}$, and $n'p \geqq mt$. Since S does not have an idempotent, $l > m$ in (5). Then

$$a^{2l-m} = a^{l-m}a^l = a^{l-m}a^m b^n = a^l b^n = (a^m b^n)b^n = a^m b^{2n} .$$

Now assume that for some integer $k \geqq 1$, we have

$$a^{kl-(k-1)m} = a^m b^{kn} .$$

We will prove that

$$a^{(k+1)l-km} = a^m b^{(k+1)n} .$$

Now we have

$$\begin{aligned} a^{(k+1)l-km} &= a^{kl-km}a^l = a^{kl-km}(a^m b^n) = (a^{kl-km}a^m)b^n \\ &= a^{kl-(k-1)m}b^n = (a^m b^{kn})b^n = a^m b^{(k+1)n} . \end{aligned}$$

Thus, by induction we have the relation: for every $k \geqq 1$

$$a^{kl-(k-1)m} = a^m b^{kn} .$$

Now choose k such that $knp \geqq mt$. Set $n' = kn$, $l' = kl - (k-1)m$. We replace the equations of (5) by the equations

$$a^{l'} = a^m b^{n'} \quad \text{and} \quad b^s = b^t a^p . \tag{6}$$

From (6) we derive

$$\begin{aligned} a^{l'tp} &= (b^{n'})^{tp}(a^m)^{tp} = (b^t)^{n'p}(a^p)^{mt} , \\ \text{or } a^{l'tp} &= (b^t)^{mt+(n'p-mt)}(a^p)^{mt} \\ &= (b^t)^{mt}(b^t)^{n'p-mt}(a^p)^{mt} \\ &= (b^t a^p)^{mt}(b^t)^{n'p-mt} \\ &= (b^s)^{mt}(b^t)^{n'p-mt} . \end{aligned}$$

Set $u = l'tp$ and $v = smt + t(n'p - mt)$. We see that we have derived the equation $a^u = b^v$. Therefore S is power joined. This concludes the proof that $(3.3) \Rightarrow (3.1)$.

REMARK. Each of (3.1), (3.2) and (3.3) is equivalent to one of (3.4) and (3.5) below:

(3.4) The semigroup S satisfies the following condition: there is an element a_0 of S such that for all $b \in S$ there are positive integers l, m, n, s, t, p satisfying

$$a_0^l = a_0^m b^n \text{ and } b^s = b^t a_0^p .$$

(3.5) The semigroup S satisfies the condition: for all pairs $a, b \in S$ there are positive integers l, m, s, t such that

$$a^l = (ab)^m \quad \text{and} \quad b^s = (ba)^t .$$

Proof. We define a relation τ on S as follows: $a\tau b$ if and only if $a^l = a^m b^n$ and $b^s = b^t a^p$ for some l, m, n, s, t, p. Then τ is an equivalence on S. Reflexivity and symmetry are obvious. Transitivity is proved as follows: suppose $a^l = a^m b^n$ and $b^k = b^q c^v$. First we have

$$a^{lk} = a^{mk} b^{nk} = a^{mk} b^{nq} c^{nv}$$

and then

$$a^{l'} = a^{m'}(a^{mq} b^{nq}) c^{nv} = a^{m'+lq} c^{nv}$$

where

$$l' = lk, m' = mk - mq \quad \text{if} \quad k \geqq q$$

$$l' = lk = mq - mk, \; m' = 0 \quad \text{if} \quad k < q .$$

Therefore $(3.4) \Rightarrow (3.3)$ is obtained as an immediate consequence; $(3.3) \rightarrow (3.4)$, $(3.1) \rightarrow (3.5)$ and $(3.5) \rightarrow (3.3)$ are obvious.

If S is a nil-semigroup, i.e., a semigroup in which some power of every element is zero, Theorem 3 is trivial since every nil-semigroup is power joined.

If S is an archimedean semigroup whose idempotent is not zero, then $G = Se$ is the kernel, i.e., the minimal ideal and the unique maximal subgroup. Then we have

COROLLARY 5. *S is power joined if and only if the kernel G is periodic.*

The essense of Theorem 3 is in the case where S is an archimedean semigroup without idempotent.

THEOREM 6. *An archimedean semigroup without idempotent is power joined if and only if the structure group $G_a = S/\rho_a$ of S is periodic for some $a \in S$, equivalently for all $a \in S$.*

Proof. Let S be an archimedean semigroup without idempotent. Then the statement (3.3) is equivalent to:

$$S/\rho_a \text{ is periodic for all } a \in S .$$

(3.4) is equivalent to:

$$S/\rho_a \text{ is periodic for some } a \in S .$$

The first statement is obvious. To see the second we will prove the following:

If S/ρ_{a_0} is periodic, then for all $b \in S$ there are positive integers l, m, n, s, t, p such that

$$a_0^l = a_0^m b^n \ , \quad b^s = b^t a_0^p \ . \tag{7}$$

The first of (7) is immediately obtained. Since S is archimedean there is a positive integer k and an element $c \in S$ such that

$$b^k = a_0 c$$

which implies $b^{kl} = a_0^l c^l$. Since S has no idempotent, $l > m$ in the first of (7). Now we have

$$b^{kl} a_0^{l-m} = a_0^{l-m} a_0^l c^l = a_0^{l-m} a_0^m b^n c^l = b^n a_0^l c^l = b^n b^{kl} = b^{n+kl} \ .$$

This completes the proof.

BIBLIOGRAPHY

1. James Chrislock, *The structure of archimedean semigroups*, Thesis, University of California, Davis, 1966.
2. ———, *On medial semigroups*, J. of Algebra **12** (1969), 1-9.
3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 1, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.
4. Richard Levin, *On commutative, nonpotent, archimedean semigroups*, Pacific J.

Math. **27** (1968), 365-371.
5. Takayuki Tamura, *Note on unipotent inversible semigroups*, Kodai Math. Sem. Reports (1954), 93-95.
6. ———, *Commutative nonpotent archimedean semigroups with cancellation law* 1, J. of Gakugei, Tokushima Univ. **8** (1951), 5-11.
7. ———, *Construction of trees and commutative archimedean semigroups*, Math. Nachr. **36** (1968), 255-287.

Received October 9, 1969, and in revised form June 9, 1970. The research for this paper was supported in part by NSF, GP-11964; a part of the results of this paper was presented by R. Levin in the meeting of the American Mathematical Society which was held in Santa Cruz, California, on April 26, 1969.

WESTERN WASHINGTON STATE COLLEGE, BELLINGHAM, WASHINGTON
AND
UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA