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RESIDUAL FINITENESS OF FINITELY GENERATED COMMUTATIVE SEMIGROUPS

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This paper gives a short semigroup theoretic proof that finitely generated commutative semigroups are residually finite. The approach is to use Schein's classification of subdirectly irreducible commutative semigroups and show that a finitely generated subdirectly irreducible commutative semigroup is finite.

In [6] Malcev gave an outline of a proof that finitely generated commutative semigroups are residually finite. Lallement has reproduced Malcev's proof (which is ring theoretic) and has also given a constructive semigroup theoretic proof of this result [5]. In the following a shorter proof is obtained based on the results of Schein [8]. The author wishes to express his appreciation to Professor G. Lallement and Professor T. Evans for their suggestions.

We note first that the property of being subdirectly irreducible is shared simultaneously by semigroups S and S^1 (S and S^0) where S^1 denotes S with adjoined unit (unless S already has a unit), S^0 denotes S with adjoined zero (unless S already has a zero). The above is also true if "being subdirectly irreducible" is replaced by "being finitely generated" or "having every element of finite order" or "having ascending chain condition on ideals". An ideal will be called nonnull if it contains at least two elements; the least nonnull ideal (if it exists) will be called the core. The following theorem is proved by Schein [8].

THEOREM 1. *A subdirectly irreducible semigroup has a core.*

Note that if S is a subdirectly irreducible semigroup with unit, K the core of S, $s \in S$, $k \in K$, s, $k \neq 0$, then there exist x, y such that $xsy = k$, for otherwise, the set SsS is a nonnull ideal not containing the core. Theorem 2 is a restatement of Theorem 5.1 and Corollaries 5.3.1, 5.3.2 in Schein [8].

THEOREM 2. *A subdirectly irreducible commutative semigroup S with unit and zero is either a subdirectly irreducible group with zero or has the following structure: the set A of all nondivisors of zero forms a subdirectly irreducible abelian group, the set F of divisors of zero is an ideal containing the core K, and $KF = \{0\}$.*

THEOREM 3. *If S is a subdirectly irreducible commutative semigroup with ascending chain condition on ideals, then every element of S is of finite order.*

Proof. Let S denote a subdirectly irreducible commutative semigroup. Because of the properties shared by S and S^1 (S and S^0) the theorem is true if and only if it is true of $(S^1)^0 = S^*$. By Theorem 2, if S^* is a subdirectly irreducible abelian group with 0 the theorem follows since a subdirectly irreducible abelian group is a subgroup of a p-quasicyclic group for some prime p. So let F denote the ideal of all divisors of zero, K the core, and A the set of all nondivisors of zero. The theorem is true for the elements of A as above, so we need only show that all elements of F have finite order. Assuming there is an element of infinite order in F we produce an infinite proper ascending chain of ideals.

Let f denote an element in F of infinite order, $Z_i = \{x \mid xf^i \in K\}$. For all i, Z_i is an ideal in S^*, and since $KF = \{0\}$, $Z_i \subseteqq Z_{i+1}$. We now show that the inclusion is proper. Consider the element f^{i+1} and let k be a nonzero element in the core. Then there is an element x such that $xf^{i+1} = k$ (x is not the unit for if this is the case f^{i+1} is in the core and f^{i+2} would be 0). Hence the element x is an element of Z_{i+1}. If x is an element of Z_i then $xf^i \in K$ so $xf^{i+1} \in KF = \{0\}$. But $xf^{i+1} = k \neq 0$. Thus there is no element of infinite order in F and the theorem is proved.

COROLLARY 1. *A finitely generated subdirectly irreducible commutative semigroup is finite.*

Proof. Redei [7] has shown that all finitely generated commutative semigroup are finitely presented (see also Freyd [4]), or equivalently, all have ascending chain condition on congruences, hence on ideals. The corollary follows.

A semigroups is residually finite if for any $x \neq y$ there is a homomorphism θ of S onto a finite semigroup such that $x\theta \neq y\theta$.

COROLLARY 2. *Malcev* [6]. *Finitely generated commutative semigroups are residually finite.*

Proof. Let S be a finitely generated commutative semigroup. By Birkhoff [1] S is isomorphic to a subdirect product of subdirectly irreducible commutative semigroups. Each factor in the subdirect product is finitely generated and hence S is a subdirect product of finite semigroups, which is equivalent to being residually finite.

COROLLARY 3. [2], [6]. *The word problem for finitely generated commutative semigroups is solvable.*

Proof. Finitely generated commutative semigroups are finitely presented and residually finite. By Theorem 2 of Evans [3], if a finitely presented algebra A in a variety V is residually finite, then the word problem is solvable for A.

REFERENCES

1. G. Birkhoff, *Subdirect unions in universal algebra*, Bull. Amer. Math. Soc. **50** (1944), 764-768.
2. A. P. Biryukov, *Some algorithmic problems for finitely defined commutative semigroups*, Sibirsk. Mat. Z. **8** (1967), 525-534.
3. T. Evans, *Some connections between residual finiteness, finite embeddability and the word problem*, J. London Math. Soc. (2) **1** (1969), 399-403.
4. P. Freyd, *Finiteness theorem for commutative semigroups*, Proc. Amer. Math. Soc. **19** (1968), 1003.
5. G. Lallement, *On a theorem of Malcev* (to appear)
6. A. I. Malcev, *On homomorphisms onto finite groups*, Uch. Zap. Ivanov. Gos. Ped. Inst. **18** (1958), 49-60.
7. L. Redei, *Theorie der endlich erzeugbaren kommutativen Halbgruppen*, Hamburger Mathematische Einzelschriften, Heft 41, Physica-Verlag, Wurzburg, 1963.
8. B. M. Schein, *Homomorphisms and subdirect decompositions of semigroups*, Pacific J. Math. **17** (1966), 529-547.

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