

LINEAR IDENTITIES IN GROUP RINGS, II

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In this paper we continue our study of linear identities satisfied by group rings. An algebra E is said to have a polynomial part if E has an idempotent e such that eEe satisfies a polynomial identity. Let $K[G]$ be the group ring of G over K and suppose that this ring is semiprime. If Δ denotes the finite conjugate subgroup of G , then we show that $K[G]$ has a polynomial part if and only if $[G: \Delta] < \infty$ and $|\Delta'| < \infty$.

The notation here and the list of references (with a few exceptions) are the same as in [I] and are therefore not repeated. Results referred to from that paper have an "I" placed before the appropriate numbers so that, for example, Theorem I.4.2 is Theorem 4.2 of [I].

1. Twisted group rings. Let K be a field and let G be a (not necessarily finite) group. We let $K^t[G]$ denote a twisted group ring of G over K . That is $K^t[G]$ is an associative K -algebra with basis $\{\bar{x} \mid x \in G\}$ and with multiplication defined by

$$(*) \quad \bar{x}\bar{y} = \gamma(x, y)\overline{xy}, \quad \gamma(x, y) \in K - \{0\}.$$

The associativity condition is equivalent to $\bar{x}(\bar{y}\bar{z}) = (\bar{x}\bar{y})\bar{z}$ for all $x, y, z \in G$ and this is equivalent to

$$(**) \quad \gamma(x, yz)\gamma(y, z) = \gamma(x, y)\gamma(xy, z).$$

We call the function $\gamma: G \times G/K - \{0\}$ the factor system of $K^t[G]$. If $\gamma(x, y) = 1$ for all $x, y \in G$ then $K^t[G]$ is in fact the ordinary group ring $K[G]$. In this section we briefly consider the possibility that $K^t[G]$ satisfies a polynomial identity.

LEMMA 1.1. *If $x \in G$, then in $K^t[G]$ we have*

$$\begin{aligned} (i) \quad & 1 = \gamma(1, 1)^{-1}\bar{1} \\ (ii) \quad & \bar{x}^{-1} = \gamma(x, x^{-1})^{-1}\gamma(1, 1)^{-1}\overline{x^{-1}} \\ & = \gamma(x^{-1}, x)^{-1}\gamma(1, 1)^{-1}\overline{x^{-1}}. \end{aligned}$$

Proof. Setting $y = 1$ in (**) yields $\gamma(1, z) = \gamma(x, 1)$. Thus for all $x, z \in G$ we have $\gamma(1, z) = \gamma(x, 1) = \gamma(1, 1)$. Therefore by (*), $\gamma(1, 1)^{-1}\bar{1}$ is the identity of $K^t[G]$.

Now by (*) and the above $\gamma(x^{-1}, x)^{-1}\gamma(1, 1)^{-1}\overline{x^{-1}}$ is a left inverse

for \bar{x} and $\gamma(x, x^{-1})^{-1} \gamma(1, 1)^{-1} \overline{x^{-1}}$ is a right inverse for \bar{x} . Thus \bar{x} is invertible and \bar{x}^{-1} is equal to both of these expressions.

THEOREM 1.2. *Let $K'[G]$ be a twisted group ring. Suppose that G has a subgroup A with $[G: A] = n$ and with $K'[A]$ commutative. Then $K'[G]$ satisfies a polynomial identity of degree $2n$.*

Proof. Using the above lemma, we see as in the proof of Theorem I. 4.2 that $K'[G] \subseteq (K'[A])_n$, the ring of $n \times n$ matrices over $K'[A]$. The result follows from the main theorem of [2].

We now consider the reverse direction.

LEMMA 1.3. *Let D be division algebra over a field F and suppose that*

$$\dim_F D < \text{cardinality of } F.$$

Then D is algebraic over F . If, in addition, F is algebraically closed, then $D = F$.

Proof. Let $x \in D - F$. Then for all $z \in F$ we have $1 - zx \neq 0$ and hence $(1 - zx)$ is invertible. Clearly all such terms $(1 - zx)^{-1}$ commute. Since $\dim_F D$ is less than the cardinality of the set $\{(1 - zx)^{-1}\}$ there must be a nontrivial linear dependence

$$\sum_{i=1}^n \frac{w_i}{1 - z_i x} = 0$$

with $w_i, z_i \in F$, $w_i \neq 0$ and the z_i distinct. Multiplying the above by $\prod (1 - z_i x)$ yields a nontrivial polynomial satisfied by x . Thus D is algebraic over F . Since F is central in D , the second statement is clear.

PROPOSITION 1.4. *Let A be an abelian group and suppose that $K'[A]$ satisfies a polynomial identity of degree n . Then A has a subgroup Z with $[A: Z] \leq (n/2)^2$ and with $K'[Z]$ central in $K'[A]$.*

Proof. Let F be a field extension of K which is algebraically closed and with the cardinality of F bigger than that of A . By Lemma I. 5.3, $K'[A]$ satisfies a multilinear polynomial identity of degree n and hence $F'[A] = F \otimes_K K'[A]$ also satisfies this identity.

Now $F'[A]$ has a unit element by Lemma 1.1 and thus $F'[A]$ has an irreducible representation ρ . Then $\rho(F'[A])$ is a primitive F -algebra satisfying a polynomial identity of degree n and hence by a

theorem of Kaplansky ([14], Th. 6.3.1) $\rho(F^t[A]) \cong D_m$ where D is a division algebra over F and $m \leq n/2$. Now F is algebraically closed and

$$\dim_F D \leq \text{cardinality of } A < \text{cardinality of } F$$

and hence by Lemma 1.3 we have $D = F$. We choose a fixed isomorphism and write $\rho(F^t[A]) = F_m$. Let tr denote the usual trace of a matrix in F_m .

Let $Z = \{x \in A \mid \bar{x} \text{ is central in } K^t[A]\}$. Then by (*) and Lemma 1.1 we see that Z is a subgroup of A and $K^t[Z]$ is central in $K^t[A]$.

Let $x \in A - Z$. Since \bar{x} is not central in $K^t[A]$ we conclude that for some $y \in A$, \bar{x} and \bar{y} do not commute. Since A is abelian we then have by (*), $\bar{y}^{-1}\bar{x}\bar{y} = a\bar{x}$ where $a \in K \subseteq F$ and $a \neq 1$. Thus

$$\rho(\bar{y})^{-1} \rho(\bar{x}) \rho(\bar{y}) = a \rho(\bar{x}).$$

Since similar matrices have the same trace we obtain

$$\begin{aligned} \text{tr } \rho(\bar{x}) &= \text{tr } \rho(\bar{y})^{-1} \rho(\bar{x}) \rho(\bar{y}) \\ &= \text{tr } a \rho(\bar{x}) = a \text{tr } \rho(\bar{x}) \end{aligned}$$

and hence $\text{tr } \rho(\bar{x}) = 0$ since $a \neq 1$.

Now let $x \in Z$. Then \bar{x} is also central in $F^t[A]$ so $\rho(\bar{x})$ is central in F_m and hence $\rho(\bar{x}) = aI$ where a is a nonzero scalar and I is the identity matrix. This yields $\text{tr } \rho(\bar{x}) = a \cdot m$. If m were equal to zero in F then we would have $\text{tr } \rho(\bar{x}) = 0$ for all $x \in A$ and hence

$$\text{tr } (F_m) = \text{tr } \rho(F^t[A]) = 0,$$

certainly a contradiction. Thus $m \neq 0$ in F .

Let x_1, x_2, \dots, x_r be a finite set of elements in distinct cosets of Z . We show that $\rho(\bar{x}_1), \rho(\bar{x}_2), \dots, \rho(\bar{x}_r)$ are linearly independent over F . Suppose that

$$a_1 \rho(\bar{x}_1) + a_2 \rho(\bar{x}_2) + \dots + a_r \rho(\bar{x}_r) = 0$$

for $a_i \in F$. Multiplying the above on the left by $\rho(\bar{x}_i^{-1})$ we obtain

$$\sum_{j=1}^r a_j \rho(\bar{x}_i^{-1} \bar{x}_j) = 0.$$

Now for $i \neq j$, $\bar{x}_i^{-1} \bar{x}_j = b\bar{x}$ for some $b \in F$ and $x \in A - Z$ and hence $\text{tr } \rho(\bar{x}_i^{-1} \bar{x}_j) = 0$. Thus taking traces of the above expression we obtain $a_i m = a_i \text{tr } I = 0$ and hence $a_i = 0$ since $m \neq 0$ in F . This shows that the $\rho(\bar{x}_i)$'s are linearly independent and thus $r \leq m^2$. This clearly yields $[A: Z] \leq m^2 \leq (n/2)^2$ and the result follows.

If R is a ring, we let R^0 denote the opposite ring of R . Thus $R^0 = R$ as sets and addition is the same. Moreover, for $a, b \in R^0$ we

have $a \circ b = ba$.

LEMMA 1.5. *Let E be a semiprime algebra over K and suppose that E satisfies a polynomial identity of degree n . Then $E \otimes_K E^0$ satisfies a polynomial identity of degree $\leq n^2/2$.*

Proof. Let P be a prime ideal in E . Then E/P is a prime ring satisfying a polynomial identity of degree n and hence, by a theorem of Posner ([14], Th. 7.3.2), $E/P \subseteq (F_P)_m$ where F_P is a field depending on P and $m = [n/2]$. Since E is semiprime it follows easily that $E \subseteq R_m$ where R is the commutative algebra $\prod_P F_P$. Thus clearly $E^0 \subseteq R_m$ also and

$$E \otimes_K E^0 \subseteq R_m \otimes_K R_m = (R \otimes_K R)_{m^2}.$$

Thus by [2], $E \otimes_K E^0$ satisfies a polynomial identity of degree $2m^2 \leq n^2/2$.

LEMMA 1.6. *Let $K^t[G]$ be a twisted group ring. Then $K[G]$ is K -isomorphic to a subalgebra of $K^t[G] \otimes K^t[G]^0$.*

Proof. Let $x, y \in G$. In $K^t[G]$ we have $\overline{xy} = \gamma(x, y)\overline{xy}$ so taking inverses yields $\overline{y}^{-1}\overline{x}^{-1} = \gamma(x, y)^{-1}\overline{xy}^{-1}$. Thus in $K^t[G]^0$ we have

$$\overline{x}^{-1} \circ \overline{y}^{-1} = \gamma(x, y)^{-1}\overline{xy}^{-1}.$$

Finally in $K^t[G] \otimes K^t[G]^0$ we have

$$\begin{aligned} (\overline{x} \otimes \overline{x}^{-1}) (\overline{y} \otimes \overline{y}^{-1}) &= (\overline{xy}) \otimes (\overline{x}^{-1} \circ \overline{y}^{-1}) \\ &= \gamma(x, y) \gamma(x, y)^{-1} \overline{xy} \otimes \overline{xy}^{-1} \\ &= \overline{xy} \otimes \overline{xy}^{-1}. \end{aligned}$$

This shows that the K -linear map $K[G] \rightarrow K^t[G] \otimes_K K^t[G]^0$ defined by $x \mapsto \overline{x} \otimes \overline{x}^{-1}$ is an injective isomorphism and the result follows.

THEOREM 1.7. *There exists a finite valued function J'' with the following property. Let K be a field and let $K^t[G]$ be a twisted group ring which satisfies a polynomial identity of degree n . Suppose that both $K^t[G]$ and $K[G]$ are semiprime. Then G has a subgroup A with $[G:A] \leq J''(n)$ and with $K^t[A]$ commutative.*

Proof. Since $K^t[G]$ is semiprime, Lemma 1.5 implies that

$$K^t[G] \otimes_K K^t[G]^0$$

satisfies a polynomial identity of degree $\leq n^2/2$. Thus by Lemma 1.6, $K[G]$ is a semiprime group ring which satisfies a polynomial identity

of degree $\leq n^2/2$. Hence by Theorem I. 9.4, G has an abelian subgroup B with $[G: B] \leq J'(n^2/2)$. Now $K^t[B]$ satisfies a polynomial identity of degree n so by Proposition 1.4, B has a subgroup A with $[B: A] \leq (n/2)^2$ and with $K^t[A]$ commutative. The result follows with $J''(n) = (n/2)^2 J'(n^2/2)$.

Two final remarks are in order. First, the examples of § 4 of [3] show that even in the case of ordinary group rings, an exact converse to Theorem 1.2 does not hold without the semiprime assumption. Second, it follows easily from Theorem 3.7 of [11] that if $K[G]$ is semiprime, then so is $K^t[G]$. Thus the two semiprime assumptions of Theorem 1.7 can be reduced to the single assumption that $K[G]$ is semiprime.

2. Weak polynomial identities. Let E be an algebra over K . A generalized polynomial over E is, roughly speaking, a polynomial in the indeterminates $\zeta_1, \zeta_2, \dots, \zeta_n$ in which elements of E are allowed to appear both as coefficients and between the indeterminates. A precise definition appears in [13]. We say that E satisfies a generalized polynomial identity if there exists a nonzero generalized polynomial $f(\zeta_1, \zeta_2, \dots, \zeta_n)$ such that $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$ for all $\alpha_1, \alpha_2, \dots, \alpha_n \in E$. The problem here is precisely what does it mean for f to be nonzero. For example, suppose the center of E is bigger than K and let α be a central element not in K . Then E satisfies the identity $f(\zeta_1) = \alpha\zeta_1 - \zeta_1\alpha$ but surely this must be considered trivial. Again, suppose E is not prime. Then we can choose nonzero $\alpha, \beta \in E$ such that E satisfies the identity $f(\zeta_1) = \alpha\zeta_1\beta$ and this must also be considered trivial. We avoid these difficulties by restricting the allowable form of the polynomials.

We say that f is a weak polynomial of degree n if

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma \in S_n} \alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \cdots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}$$

where $\alpha_{i,\sigma} \in E$. This form is of course motivated by Lemma I. 5.3. The above f is said to be nondegenerate if for some σ

$$\alpha_{1,\sigma} E \alpha_{2,\sigma} E \cdots \alpha_{n,\sigma} E \alpha_{n+1,\sigma} \neq 0.$$

Otherwise f is degenerate.

LEMMA 2.1. *Suppose $K[G]$ satisfies a nondegenerate weak polynomial identity of degree n . Then $K[G]$ satisfies a weak polynomial identity*

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma \in S_n} \alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \cdots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}$$

with $\theta(\alpha_{1,1})\theta(\alpha_{2,1})\cdots\theta(\alpha_{n+1,1}) \neq 0$.

Proof. Suppose $K[G]$ satisfies f with the above form. Since f is nondegenerate, by reordering the ζ 's if necessary, we may assume that

$$\alpha_{1,1} K[G] \alpha_{2,1} K[G] \cdots \alpha_{n,1} K[G] \alpha_{n+1,1} \neq 0.$$

Thus there exists $x_1, x_2, \dots, x_n \in G$ with

$$\alpha_{1,1} x_1 \alpha_{2,1} x_2 \cdots \alpha_{n,1} x_n \alpha_{n+1,1} \neq 0.$$

If we replace ζ_i in f by $x_i \zeta_i$ we see clearly that $K[G]$ satisfies a weak identity f with $\alpha_{1,1} \alpha_{2,1} \cdots \alpha_{n+1,1} \neq 0$.

For $i = 1, 2, \dots, n+1$ write

$$\alpha_{i,1} = \sum \beta_{ij} y_j$$

where $\beta_{ij} \in K[A]$ and $\{y_j\}$ is a set of coset representatives for A in G . Since $\alpha_{1,1} \alpha_{2,1} \cdots \alpha_{n+1,1} \neq 0$ we conclude that

$$\beta_{1,j_1} y_{j_1} \beta_{2,j_2} y_{j_2} \cdots \beta_{n+1,j_{n+1}} y_{j_{n+1}} \neq 0$$

for some j_1, j_2, \dots, j_{n+1} . Thus if $z_i = y_{j_1} y_{j_2} \cdots y_{j_{i-1}}$ with $z_1 = 1$ then

$$\beta_{1,j_1}^{-1} \beta_{2,j_2}^{-1} \cdots \beta_{n+1,j_{n+1}}^{-1} \neq 0.$$

Now $\beta_{i,j_i} = \theta(\alpha_{i,1} y_{j_i}^{-1})$ so

$$\beta_{i,j_i}^{-1} = \theta(z_i \alpha_{i,1} y_{j_i}^{-1} z_i^{-1}) = \theta(z_i \alpha_{i,1} z_{i+1}^{-1}).$$

It therefore follows that if we replace ζ_i in f by $z_{i+1}^{-1} \zeta_i z_{i+1}$ and if, in addition, we multiply f on the left by z_1 and on the right by z_{n+2}^{-1} , then this new weak identity obtained has the required property.

The following is a slight extension of Lemma I. 1.3.

LEMMA 2.2. *Let $S = \bigcup H_r u_{r_s}$ be a finite union of cosets of the subgroups H_r of G all of infinite index and let H be a subgroup of finite index in G . Let $\alpha_1, \alpha_2, \dots, \alpha_t, \beta_1, \beta_2, \dots, \beta_t \in K[G]$ and suppose that for all $x \in H - S$ we have*

$$\alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_t x \beta_t = 0.$$

Then $\theta(\alpha_1)\beta_1 + \theta(\alpha_2)\beta_2 + \cdots + \theta(\alpha_t)\beta_t = 0$ and the above equation holds for all $x \in H$.

Proof. We show first that $\theta(\alpha_1)\beta_1 + \theta(\alpha_2)\beta_2 + \cdots + \theta(\alpha_t)\beta_t = 0$.

Suppose by way of contradiction that this is not the case and let $v \in G$ be in the support of this element of $K[G]$.

Let W be the intersection of H and the centralizers of all elements in $\text{Supp } \theta(\alpha_i)$ for $i = 1, 2, \dots, t$. By Lemma I. 1.1, $[G: W] < \infty$. Clearly if $x \in W \subseteq H$ then x centralizes all $\theta(\alpha_i)$.

Write $\alpha_i = \theta(\alpha_i) + \alpha'_i$ where $\text{Supp } \alpha'_i \cap \Delta = \emptyset$ and then write the finite sums

$$\begin{aligned}\alpha'_i &= \sum a_{ij} y_j, & y_j \in \Delta \\ \beta_i &= \sum b_{ij} z_j.\end{aligned}$$

If y_i is conjugate to some $v z_j^{-1}$ in G choose $g_{ij} \in G$ with $g_{ij}^{-1} y_i g_{ij} = v z_j^{-1}$.

Let $x \in W - S$. Then we must have

$$\begin{aligned}0 &= x^{-1} \alpha_1 x \beta_1 + x^{-1} \alpha_2 x \beta_2 + \dots + x^{-1} \alpha_t x \beta_t \\ &= [\theta(\alpha_1) \beta_1 + \theta(\alpha_2) \beta_2 + \dots + \theta(\alpha_t) \beta_t] \\ &\quad + [\alpha'_1 x \beta_1 + \alpha'_2 x \beta_2 + \dots + \alpha'_t x \beta_t].\end{aligned}$$

Since v occurs in the support of the first term it must also occur in the second and hence there exists y_i, z_j with $v = y_i^x z_j$ or

$$x^{-1} y_i x = v z_j^{-1} = g_{ij}^{-1} y_i g_{ij}.$$

Thus $x \in C_G(y_i) g_{ij}$. We have therefore shown that

$$W \subseteq S \cup (\bigcup C_G(y_i) g_{ij}) = (\bigcup H_r u_{rs}) \cup (\bigcup C_G(y_i) g_{ij}).$$

Since $[G: W] < \infty$ this implies that G is a finite union of cosets of the subgroups H_r and $C_G(y_i)$ all of infinite index in G . This contradicts Lemma I. 1.2 and the first result follows.

Now write $\alpha_i = \sum y_j \alpha_{ij}$ where $\alpha_{ij} \in K[\Delta]$ and the y_j are coset representatives for Δ in G . Then $\alpha_{ij} = \theta(y_j^{-1} \alpha_i)$ so $\alpha_i = \sum y_j \theta(y_j^{-1} \alpha_i)$. Since

$$y_j^{-1} \alpha_i x \beta_1 + y_j^{-1} \alpha_i x \beta_2 + \dots + y_j^{-1} \alpha_i x \beta_t = 0$$

for all $x \in H - S$ we conclude from the above that

$$\theta(y_j^{-1} \alpha_i) \beta_1 + \theta(y_j^{-1} \alpha_i) \beta_2 + \dots + \theta(y_j^{-1} \alpha_i) \beta_t = 0.$$

Thus clearly $\alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_t \beta_t = 0$.

Finally, let $y \in H$. Then for all $x \in H - S y^{-1}$ we have

$$\alpha_1 x(y \beta_1) + \alpha_2 x(y \beta_2) + \dots + \alpha_t x(y \beta_t) = 0.$$

Since $S y^{-1}$ has the same form as S we conclude from the preceding paragraph that

$$\alpha_1 y \beta_1 + \alpha_2 y \beta_2 + \dots + \alpha_t y \beta_t = 0.$$

Since $y \in H$ was arbitrary, the result follows.

As an amusing corollary of this lemma we have

THEOREM 2.3. *Suppose that K is algebraically closed and non-denumerable and suppose that $K[G]$ is semiprime. Then $K[G]$ has a minimal right ideal if and only if G is finite.*

Proof. Certainly if G is finite then $K[G]$ has a minimal right ideal. We now consider the converse.

Suppose I is a minimal right ideal in $K[G]$. If $I^2 = 0$ then $K[G]I$ is a nonzero ideal with square zero and this contradicts Lemma I. 3.1 and the fact that $K[G]$ is semiprime. Thus $I^2 \neq 0$ and it follows easily that $I = eK[G]$ for some idempotent e . By Schur's lemma, $D = eK[G]e$ is a division algebra over K .

If G is not finite let H be a countably infinite subgroup of G with $H \supseteq \text{Supp } e$. We show now that $eK[H]e$ is also a division algebra. Let $\alpha \in K[H]$ and suppose that $e\alpha e \neq 0$. Then there exists $\beta \in K[G]$ with $(e\alpha e)(e\beta e) = e$. Write $\beta = \beta_0 + \beta_1$ where $\beta_0 \in K[H]$ and $\text{Supp } \beta_1 \cap H = \emptyset$. Then

$$(e\alpha e)(e\beta_0 e) + (e\alpha e)(e\beta_1 e) = e.$$

Since $e \in K[H]$ we have $\text{Supp } (e\alpha e)(e\beta_1 e) \cap H = \emptyset$ and thus

$$(e\alpha e)(e\beta_0 e) = e.$$

Now $eK[H]e$ is a division algebra of countable dimension over K . Since K is algebraically closed and nondenumerable we have $eK[H]e = Ke$ by Lemma 1.3. Thus for all $x \in H$ we have $exe = a_x e$ for some $a_x \in K$. Now clearly there are only finitely many $x \in H$ say x_1, x_2, \dots, x_i with $\text{Supp}(exe) \cap \text{Supp } e \neq \emptyset$. Thus $exe = 0$ for all $x \in H - \bigcup \{x_i\}$. We think of the $\{x_i\}$ as cosets of $\langle 1 \rangle$. Since $[H: \langle 1 \rangle] = \infty$ we conclude by Lemma 2.2 applied to the group ring $K[H]$ that $exe = 0$ for all $x \in H$. Thus $e^2 = 0$, a contradiction and the result follows.

3. Reduction to Δ . In this section we prove the following result.

THEOREM 3.1. *Suppose that $K[G]$ satisfies a nondegenerate weak polynomial identity of degree n . Then $[G: \Delta] < \infty$ and $K[\Delta]$ satisfies a nondegenerate weak polynomial identity of degree n .*

Proof. We show first that $[G: \Delta] < \infty$ following the proof of Theorem I. 6.2. Assume by way of contradiction that $[G: \Delta] = \infty$. By Lemma 2.1 we can assume that $K[G]$ satisfies the weak identity

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma \in S_n} \alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \cdots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}$$

with $\theta(\alpha_{1,1}) \theta(\alpha_{2,1}) \cdots \theta(\alpha_{n+1,1}) \neq 0$.

For $j = 1, 2, \dots, n + 1$ define $T_j \subseteq S_n$ by

$$T_j = \{\sigma \in S_n \mid \sigma(1) = 1, \sigma(2) = 2, \dots, \sigma(j-1) = j-1\}.$$

Then $T_1 = S_n$, $T_{n+1} = \{1\}$ and T_j is just an imbedding of S_{n+1-j} in S_n . Define the weak polynomials f_j of degree $n + 1 - j$ by

$$\begin{aligned} & f_j(\zeta_j, \zeta_{j+1}, \zeta_n) \\ &= \sum_{\sigma \in T_j} \theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{j-1,\sigma}) \alpha_{j,\sigma} \zeta_{\sigma(j)} \alpha_{j+1,\sigma} \zeta_{\sigma(j+1)} \cdots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}. \end{aligned}$$

Thus $f_1 = f$ and

$$f_{n+1} = \theta(\alpha_{1,1}) \theta(\alpha_{2,1}) \cdots \theta(\alpha_{n,1}) \alpha_{n+1,1}$$

is a nonzero constant function since

$$\theta(\alpha_{1,1}) \theta(\alpha_{2,1}) \cdots \theta(\alpha_{n+1,1}) \neq 0.$$

Let \mathcal{M} be the set of monomial polynomials obtained as follows. For each σ we start with

$$\alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \cdots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}$$

and we modify it by (1) deleting some but not all of the ζ_i ; (2) replacing some of the $\alpha_{i,\sigma}$ by $\theta(\alpha_{i,\sigma})$; and (3) replacing some of the $\alpha_{i,\sigma}$ by 1. Then \mathcal{M} consists of all such monomials obtained for all $\sigma \in S_n$ and clearly \mathcal{M} is a finite set. Note that \mathcal{M} may contain the zero monomial but it contains no nonzero constant monomial since in (1) we do not allow all the ζ_i to be deleted.

For $j = 1, 2, \dots, n + 1$ define $\mathcal{M}_j \subseteq \mathcal{M}$ by $\mu \in \mathcal{M}_j$ if and only if $\zeta_1, \zeta_2, \dots, \zeta_{j-1}$ do not occur as variables in μ . Thus $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_{n+1} \subseteq \{0\}$ where 0 is the zero monomial.

Under the assumption that $[G : \Delta] = \infty$ we prove by induction on $j = 1, 2, \dots, n + 1$ that for all $x_j, x_{j+1}, \dots, x_n \in G$ either

$$f_j(x_j, x_{j+1}, \dots, x_n) = 0$$

or there exists $\mu \in \mathcal{M}_j$ with $\text{Supp } \mu(x_j, x_{j+1}, \dots, x_n) \cap \Delta \neq \emptyset$. Since $f = f_1$ is an identity satisfied by $K[G]$, the result for $j = 1$ is clear.

Suppose the inductive result holds for some $j < n + 1$. Fix $x_{j+1}, x_{j+2}, \dots, x_n \in G$ and let $x \in G$ play the role of the j^{th} variable. Let $\mu \in \mathcal{M}_{j+1}$. If $\text{Supp } \mu(x_{j+1}, \dots, x_n) \cap \Delta \neq \emptyset$ we are done. Thus we may assume that $\text{Supp } \mu(x_{j+1}, \dots, x_n) \cap \Delta = \emptyset$ for all $\mu \in \mathcal{M}_{j+1}$. Set $\mathcal{M}_j - \mathcal{M}_{j+1} = \mathcal{N}_j$.

Now let $\mu \in \mathcal{M}_j$ so that μ involves the variable ζ_j . Write $\mu = \mu' \zeta_j \mu''$ where μ' and μ'' are monomials in the variables $\zeta_{j+1}, \dots, \zeta_n$. Then $\text{Supp } \mu(x, x_{j+1}, \dots, x_n) \cap \Delta \neq \emptyset$ implies that

$$x \in h'^{-1} \Delta h''^{-1} = \Delta h'^{-1} h''^{-1}$$

where $h' \in \text{Supp } \mu'(x_{j+1}, \dots, x_n)$ and $h'' \in \text{Supp } \mu''(x_{j+1}, \dots, x_n)$. Thus it follows that for all $x \in G - S$ where

$$S = \bigcup_{\substack{\mu \in \mathcal{M}_j \\ h', h''}} \Delta h'^{-1} h''^{-1}$$

we have $\text{Supp } \mu(x, x_{j+1}, \dots, x_n) \cap \Delta = \emptyset$ for all $\mu \in \mathcal{M}_j$. Thus by the inductive result for j we conclude that for all $x \in G - S$ we have $f_j(x, x_{j+1}, \dots, x_n) = 0$. Note that S is a finite union of cosets of Δ , a subgroup of infinite index in G .

Now clearly

$$\begin{aligned} & f_j(x, x_{j+1}, \dots, x_n) \\ &= \sum_{\sigma \in T_{j+1}} \theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{j-1,\sigma}) \alpha_{j,\sigma} x \alpha_{j+1,\sigma} x_{\sigma(j+1)} \cdots x_{\sigma(n)} \alpha_{n+1,\sigma} \\ &+ \sum_{\mu \in \mathcal{M}_{j+1}} \mu(x_{j+1}, \dots, x_n) x \eta(x_{j+1}, \dots, x_n) \end{aligned}$$

where the $\eta(\zeta_{j+1}, \dots, \zeta_n)$ are suitable monomials. Note that

$$\theta[\theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{j-1,\sigma}) \alpha_{j,\sigma}] = \theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{j,\sigma})$$

and that $\theta(\mu(x_{j+1}, \dots, x_n)) = 0$ since by assumption

$$\text{Supp } \mu(x_{j+1}, \dots, x_n) \cap \Delta = \emptyset.$$

Now $f_j(x, x_{j+1}, \dots, x_n) = 0$ for all $x \in G - S$ and hence by Lemma 2.2 and the above few remarks we obtain

$$\begin{aligned} 0 &= \sum_{\sigma \in T_{j+1}} \theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{j,\sigma}) \alpha_{j+1,\sigma} x_{\sigma(j+1)} \cdots x_{\sigma(n)} \alpha_{n+1,\sigma} \\ &= f_{j+1}(x_{j+1}, \dots, x_n) \end{aligned}$$

and the induction step is proved.

In particular, we conclude for $j = n + 1$ that either $f_{n+1} = 0$ or there exists $\mu \in \mathcal{M}_{n+1}$ with $\text{Supp } \mu \cap \Delta \neq \emptyset$. However, f_{n+1} is known to be a nonzero constant function and $\mathcal{M}_{n+1} \subseteq \{0\}$. Hence we have a contradiction and we must therefore have $[G : \Delta] < \infty$.

We show next that for any weak identity f we have

$$0 = \sum_{\sigma} \theta(\alpha_{1,\sigma}) \theta(\alpha_{2,\sigma}) \cdots \theta(\alpha_{n+1,\sigma}).$$

To do this it is convenient to multiply f on the right by a new variable

ζ_{n+1} so that if $S_n \subseteq S_{n+1}$ is the set of all permutations fixing the point $n+1$ then

$$\sum_{\sigma \in S_n} \alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)} \alpha_{n+1,\sigma} \zeta_{\sigma(n+1)}$$

is a weak identity satisfied by $K[G]$.

Let $\{y_j\}$ be a finite subset of G such that $\text{Supp } \alpha_{i,\sigma} \subseteq \{y_j\}$ for all i, σ . Let x_1, x_2, \dots, x_{n+1} be arbitrary elements of \mathcal{A} . If we write each $\alpha_{i,\sigma}$ in terms of the y 's and if we put $\zeta_i = x_i$ in the above then we obtain

$$0 = \sum_{\sigma, \tau} a_{\sigma, \tau} y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots y_{\tau_{n+1}} x_{\sigma(n+1)}$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_{n+1})$ runs through all $n+1$ -tuples with allowable entries and $a_{\sigma, \tau} \in K$.

Fix $x_2, \dots, x_{n+1} \in \mathcal{A}$. Then the above linear identity holds for all $x_1 \in \mathcal{A}$ and $[G: \mathcal{A}] < \infty$. Thus Lemma 2.2 applies with $H = \mathcal{A}$. Let us consider the effect of this lemma on a typical term of the above. Now

$$a_{\sigma, \tau} y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots x_1 \cdots y_{\tau_{n+1}} x_{\sigma(n+1)}$$

becomes

$$a_{\sigma, \tau} \theta(y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots) \hat{x}_1 \cdots y_{\tau_{n+1}} x_{\sigma(n+1)}$$

where \hat{x}_i indicates that x_i is deleted. Since $z = y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots \in G$ it follows that

$$\theta(z) = \begin{cases} z & \text{if } z \in \mathcal{A} \\ 0 & \text{if } z \notin \mathcal{A} \end{cases}.$$

Now \mathcal{A} is normal in G and $x_2, x_3, \dots, x_{n+1} \in \mathcal{A}$. Thus

$$z = y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots \in \mathcal{A}$$

if and only if $y_{\tau_1} y_{\tau_2} \cdots \in \mathcal{A}$. Therefore by applying Lemma 2.2 we obtain setting $x_1 = 1$

$$0 = \sum_{\sigma, \tau} a_{\sigma, \tau} y_{\tau_1} x_{\sigma(1)} y_{\tau_2} x_{\sigma(2)} \cdots y_{\tau_{n+1}} x_{\sigma(n+1)} \cdot \\ y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{\sigma^{-1}(1)}} \in \mathcal{A}$$

This identity holds for all $x_2, \dots, x_{n+1} \in \mathcal{A}$ so we apply Lemma 2.2 successively. At the last step, since all x_i will be 1, we will have

$$0 = \sum'_{\sigma, \tau} a_{\sigma, \tau} y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{n+1}}$$

where the sum is over all those σ, τ terms with

$$\begin{aligned}
y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{\sigma^{-1}(1)}} &\in \mathcal{A} \\
y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{\sigma^{-1}(2)}} &\in \mathcal{A} \\
&\vdots \\
y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{\sigma^{-1}(n+1)}} &\in \mathcal{A}.
\end{aligned}$$

Since σ is a permutation the above is equivalent to

$$y_{\tau_1} \in \mathcal{A}, y_{\tau_1} y_{\tau_2} \in \mathcal{A}, \dots, y_{\tau_1} y_{\tau_2} \cdots y_{\tau_{n+1}} \in \mathcal{A}$$

and hence equivalent to $y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_{n+1}} \in \mathcal{A}$. Thus clearly

$$\sum'_{\sigma, \tau} a_{\sigma, \tau} y_{\tau_1} y_{\tau_2} \cdots y_{\tau_n} = \sum_{\sigma \in S_n} \theta(\alpha_{1, \sigma}) \theta(\alpha_{2, \sigma}) \cdots \theta(\alpha_{n+1, \sigma})$$

and the latter sum is therefore equal to zero.

Now fix $x_1, x_2, \dots, x_n \in \mathcal{A}$ and replace ζ_i in f by $x_i \zeta_i$. This is a new weak identity satisfied by $K[G]$ and applying the above result to it we obtain

$$0 = \sum_{\sigma \in S_n} \theta(\alpha_{1, \sigma} x_{\sigma(1)}) \theta(\alpha_{2, \sigma} x_{\sigma(2)}) \cdots \theta(\alpha_{n, \sigma} x_{\sigma(n)}) \theta(\alpha_{n+1, \sigma}).$$

Since $x_i \in \mathcal{A}$ we have $\theta(\alpha_{i, \sigma} x_{\sigma(i)}) = \theta(\alpha_{i, \sigma}) x_{\sigma(i)}$ and hence

$$0 = \sum_{\sigma \in S_n} \theta(\alpha_{1, \sigma}) x_{\sigma(1)} \theta(\alpha_{2, \sigma}) x_{\sigma(2)} \cdots \theta(\alpha_{n, \sigma}) x_{\sigma(n)} \theta(\alpha_{n+1, \sigma}).$$

This shows immediately that $K[\mathcal{A}]$ satisfies the weak polynomial identity

$$g(\zeta_1, \dots, \zeta_n) = \sum_{\sigma \in S_n} \theta(\alpha_{1, \sigma}) \zeta_{\sigma(1)} \theta(\alpha_{2, \sigma}) \zeta_{\sigma(2)} \cdots \theta(\alpha_{n, \sigma}) \zeta_{\sigma(n)} \theta(\alpha_{n+1, \sigma}).$$

Moreover, g is nondegenerate since

$$\theta(\alpha_{1, 1}) \theta(\alpha_{2, 1}) \cdots \theta(\alpha_{n+1, 1}) \neq 0$$

and the result follows.

4. Polynomial parts. Let E be an algebra over K . We say that E has a polynomial part if and only if E has an idempotent e such that eEe satisfies a polynomial identity.

THEOREM 4.1. *Let $K[G]$ be a semiprime group ring. The following are equivalent:*

- (i) $[G: \mathcal{A}] < \infty$ and $|\mathcal{A}'| < \infty$;
- (ii) $K[G]$ has a polynomial part;
- (iii) $K[G]$ satisfies a nondegenerate weak polynomial identity.

Proof. (i) \Rightarrow (ii). Since $K[G]$ is semiprime it follows from

Theorem I. 3.6 that $|\Delta'|$ is not zero in K . Set

$$e = \frac{1}{|\Delta'|} \sum_{x \in \Delta'} x \in K[G].$$

It follows easily that e is a central idempotent in $K[G]$ since Δ' is normal in G . Moreover, it is also easy to show that $eK[G] \simeq K[G/\Delta']$. Now G/Δ' has an abelian subgroup Δ/Δ' of finite index. Hence by Theorem I. 4.2 $K[G/\Delta']$ satisfies a polynomial identity and thus $K[G]$ has a polynomial part

(ii) \Rightarrow (iii). Let e be an idempotent such that $E = eK[G]e$ satisfies a polynomial identity. By Lemma I. 5.3 E satisfies an identity of the form

$$g(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma} a_{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \dots \zeta_{\sigma(n)}.$$

If $\alpha \in K[G]$ then of course $e\alpha e \in E$. This shows immediately that $K[G]$ satisfies the weak polynomial identity

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma} a_{\sigma} e \zeta_{\sigma(1)} e \zeta_{\sigma(2)} e \dots e \zeta_{\sigma(n)} e.$$

Moreover, f is nondegenerate since $a_{\sigma} \neq 0$ for some σ and then $a_{\sigma} e \cdot e \dots e = a_{\sigma} e \neq 0$.

(iii) \Rightarrow (i). By Theorem 3.1, $[G:\Delta] < \infty$ and $K[\Delta]$ satisfies a nondegenerate weak polynomial identity

$$f(\zeta_1, \dots, \zeta_n) = \sum_{\sigma} \alpha_{1,\sigma} \zeta_{\sigma(1)} \alpha_{2,\sigma} \zeta_{\sigma(2)} \dots \alpha_{n,\sigma} \zeta_{\sigma(n)} \alpha_{n+1,\sigma}.$$

By Lemma 2.1 applied to Δ we may assume that

$$\alpha_{1,1} \alpha_{2,1} \dots \alpha_{n,1} \alpha_{n+1,1} \neq 0.$$

Let C be the finitely generated subgroup of Δ , generated by $\text{Supp } \alpha_{i,\sigma}$ for all i, σ . Let H be the intersection of the centralizers in Δ of all the elements in $\text{Supp } \alpha_{i,\sigma}$ for all i, σ . By Lemma I. 1.1, $[\Delta:H] < \infty$. Clearly $K[H]$ centralizes $K[C]$. Thus if we restrict the variables of f to values in $K[H]$ we see that $K[H]$ satisfies the identity

$$g(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma} \beta_{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \dots \zeta_{\sigma(n)}$$

where $\beta_{\sigma} = \alpha_{1,\sigma} \alpha_{2,\sigma} \dots \alpha_{n+1,\sigma} \in K[C]$ and $\beta_1 \neq 0$. By multiplying all β_{σ} on the left by some $x \in C$ if necessary we may assume that $1 \in \text{Supp } \beta_1$. Write $\beta_{\sigma} = \beta'_{\sigma} + \beta''_{\sigma}$ where $\text{Supp } \beta'_{\sigma} \subseteq H' \cap C$ and

$$\text{Supp } \beta''_{\sigma} \cap (H' \cap C) = \emptyset.$$

Then by assumption $\beta'_1 \neq 0$ and we claim that

$$g'(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma} \beta'_{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

is an identity satisfied by $K[H]$. Since g' is multilinear it suffices to check this fact by evaluating g' on elements of H .

Let $x_1, x_2, \dots, x_n \in H$ and set $z = x_1 x_2 \cdots x_n$. Then since

$$x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} \equiv x_1 x_2 \cdots x_n \pmod{H'}$$

we have

$$x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = h_{\sigma} z$$

where $h_{\sigma} \in H'$. Since $K[H]$ satisfies the identity g we therefore have

$$0 = (\sum_{\sigma} \beta'_{\sigma} h_{\sigma}) z + (\sum_{\sigma} \beta''_{\sigma} h_{\sigma}) z.$$

Now $\text{Supp } \beta'_{\sigma} h_{\sigma} \subseteq H'$ and $(\text{Supp } \beta''_{\sigma} h_{\sigma}) \cap H' = \emptyset$. This shows that

$$0 = (\sum_{\sigma} \beta'_{\sigma} h_{\sigma}) z = g'(x_1, x_2, \dots, x_n).$$

Let Z be the subgroup of \mathcal{A} generated by the $\text{Supp } \beta'_{\sigma}$ for all σ . Then Z is a finitely generated subgroup of $H' \cap C$. Since H centralizes C it follows that Z is central in H . Since Z is finitely generated it follows that there is a finitely generated subgroup $H_1 \subseteq H$ with $Z \subseteq H_1$. By Lemma I. 2.2 H_1 is finite and hence Z is a finite central subgroup of H .

Let F be the algebraic closure of K . Then certainly $F[H]$ also satisfies the identity g' . Since $K[G]$ is semiprime and $Z \subseteq \mathcal{A}$, Theorem I. 3.6 implies that $|Z| \neq 0$ in F . Thus $F[Z]$ is a finite dimensional commutative semisimple algebra over an algebraically closed field and since $\beta'_1 \neq 0$ there exists a primitive idempotent $e \in F[Z]$ with $e\beta'_1 \neq 0$. Since $eF[Z] = eF$ we have $e\beta'_{\sigma} = ea_{\sigma}$ for some $a_{\sigma} \in F$. Thus since e is central in $F[H]$, the algebra $eF[H]$ satisfies the nontrivial ordinary polynomial identity

$$h(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma} a_{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}.$$

Since $K[G]$ is semiprime and $H \subseteq \mathcal{A}$ it follows easily from Theorem I. 3.3 and Theorem I. 3.6 that $F[H]$ is semiprime. Since $|Z|$ is finite there results also show that $F[H/Z]$ is semiprime. In addition $F[H] = eF[H] + (1-e)F[H]$ is a ring direct sum and thus Lemma I. 3.1 implies easily that $eF[H]$ is semiprime.

We now observe that $eF[H] \cong F'[H/Z]$ where the latter is some twisted group algebra of H/Z . This follows since

$$e \in F[Z], eF[Z] = eF$$

and e is a central idempotent in $F[H]$. In view of the above, Theorem

1.7 applies and we conclude that H/Z has an abelian subgroup A/Z with $[H/Z: A/Z] < \infty$. Since $[\Delta: H] < \infty$ we have $[\Delta: A] < \infty$. Moreover if B is the intersection of the finitely many conjugates of A in Δ then $[\Delta: B] < \infty$ by Lemma I. 1.1. Also B is a normal subgroup of Δ with $B' \subseteq A' \subseteq Z$ so B' is finite. Write

$$\Delta = \langle B, w_1, w_2, \dots, w_i \rangle.$$

Then by Lemma I. 1.1,

$$W = B \cap \left(\bigcap_1^t C_\Delta(w_i) \right)$$

has finite index in Δ . Since B/B' is abelian it then follows that WB'/B' is a central subgroup of finite index in Δ/B' and hence by Lemma I. 2.1, $(\Delta/B')'$ is finite. Since B' is finite and $(\Delta/B')' = \Delta'/B'$ we have $|\Delta'| < \infty$ and the theorem is proved.

It is interesting to note that, in contrast to Theorem I. 9.4, the degree of the weak polynomial identity does not play an important role here. For example, in the proof of (i) \Rightarrow (ii) above a weak polynomial was obtained whose degree was independent of $|\Delta'|$. Thus $|\Delta'|$ cannot be bounded by a function of the degree. The same is true of the index $[G: \Delta]$ as the following example indicates.

Suppose that $G = \Delta H$ where Δ is abelian, H is finite and $|H| \neq 0$ in K . For example, we could take $G = Z \wr Z_n$ where Z is infinite cyclic and Z_n is cyclic of order n . Set

$$e = \frac{1}{|H|} \sum_{x \in H} x$$

so that e is an idempotent in $K[G]$. If $h \in H$ then $he = e$ and thus $eK[G]e$ is spanned by the terms eye with $y \in \Delta$. Now

$$\begin{aligned} eye &= \frac{1}{|H|} e \sum_{x \in H} yx = \frac{1}{|H|} e \sum_{x \in H} xy^x \\ &= \frac{1}{|H|} e \sum_{x \in H} y^x. \end{aligned}$$

Since Δ is abelian and $G = \Delta H$ it follows easily that $\sum_x y^x$ is central in $K[G]$. This shows that $eK[G]e$ is commutative and hence satisfies a polynomial identity of degree 2. Thus $K[G]$ satisfies a nondegenerate weak polynomial identity of degree 2 independent of the index $[G: \Delta]$.

5. Miscellany. Let P be a nonabelian p -group of order p^3 . If K is a field of characteristic p then it is easy to see that $K[P]$ is Lie nilpotent of class p . Thus $K[P]$ satisfies a polynomial identity of degree $p + 1$. Now it would be interesting to find the minimal degree

of the polynomial identities satisfied by $K[P]$ and the following a mild step in that direction.

THEOREM 5.1. *Let K be a field and let P be a finite nonabelian p -group. If $K[P]$ satisfies a polynomial of degree n , then $n > \sqrt[3]{6p}$.*

Proof. Since $P' \neq \langle 1 \rangle$ we can choose N normal in P with $[P': N] = p$. If $\bar{P} = P/N$, then $K[\bar{P}]$ also satisfies a polynomial identity of degree n and \bar{P}' is central of order p . Thus it suffices to assume that $P = \bar{P}$ or equivalently that P' is central of order p . Choose $x, y \in P$ which do not commute and let $z = xyx^{-1}y^{-1} \neq 1$. Then $P' = \langle z \rangle$ and z is central of order p . We have $xy = zyx$ and hence by induction $xy^i = z^i y^i x$.

We may assume by Lemma I. 5.3 that $K[P]$ satisfies the multilinear identity

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \rho_1 \cdots \rho_n \rho_n + \sum_{\sigma \neq 1} a_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

and we set $\zeta_i = xy^i$. Consider the σ monomial evaluated here. We have

$$\mu_\sigma = xy^{\sigma(1)} xy^{\sigma(2)} \cdots xy^{\sigma(n)}.$$

Now shift all the y 's to the left starting with $y^{\sigma(1)}$. Since $y^{\sigma(i)}$ must pass precisely i x 's we have, using $xy^j = z^j y^j x$

$$\mu_\sigma = z^{\sum_1^{n-1} i \sigma(i)} y^{\sum_1^n \sigma(i)} x^n = z^{\sum_1^{n-1} i \sigma(i)} y^{n(n+1)/2} x^n.$$

For each σ we define the integral vector v_σ by

$$v_\sigma = \langle \langle \sigma(1), \sigma(2), \dots, \sigma(n) \rangle \rangle$$

so that $v_1 = \langle \langle 1, 2, \dots, n \rangle \rangle$ and

$$\mu_\sigma = z^{v_1 \cdot v_\sigma} y^{n(n+1)/2} x^n$$

where $v_1 \cdot v_\sigma$ is the usual dot product of vectors.

We consider the values $v_1 \cdot v_\sigma$. By the Cauchy-Schwarz inequality

$$(v_1 \cdot v_\sigma)^2 \leq (v_1 \cdot v_1) (v_\sigma \cdot v_\sigma)$$

with equality if and only if v_σ is a scalar times v_1 and hence in our case if and only if $\sigma = 1$. Since clearly $v_\sigma \cdot v_\sigma = v_1 \cdot v_1$ we have for all $\sigma \neq 1$, $(v_1 \cdot v_\sigma)^2 < (v_1 \cdot v_1)^2$ so

$$v_1 \cdot v_\sigma < v_1 \cdot v_1 = \sum_1^n i^2 = n(n+1)(2n+1)/6$$

since these are all positive quantities.

Clearly zero is a lower bound for all $v_1 \cdot v_\sigma$. A better bound is

obtained as follows. For each σ define σ' by $\sigma'(i) = n + 1 - \sigma(i)$. Then σ' is clearly also a permutation and $v_{\sigma'} = v - v_{\sigma}$ where

$$v = \langle\langle n+1, n+1, \dots, n+1 \rangle\rangle.$$

By the above

$$v_1 \cdot v_1 \geq v_1 \cdot v_{\sigma'} = v_1 \cdot (v - v_{\sigma}) = v_1 \cdot v - v_1 \cdot v_{\sigma}$$

and thus

$$v_1 \cdot v_{\sigma} \geq v_1 \cdot v - v_1 \cdot v_1 = (n+1) \sum_1^n i - \sum_1^n i^2 = n(n+1)(n+2)/6.$$

Returning to the polynomial identity we have

$$0 = f(xy^1, xy^2, \dots, xy^n) = \left\{ z^{v_1 \cdot v_1} + \sum_{\sigma \neq 1} a_{\sigma} z^{v_1 \cdot v_{\sigma}} \right\} y^{n(n+1)/2} x^n.$$

It therefore follows that the $z^{v_1 \cdot v_1}$ terms must be cancelled by other terms in the sum and thus for some $\sigma \neq 1$ we have $z^{v_1 \cdot v_1} = z^{v_1 \cdot v_{\sigma}}$. Hence $p \mid (v_1 \cdot v_1 - v_1 \cdot v_{\sigma})$. Now by the above

$$\begin{aligned} 0 < v_1 \cdot v_1 - v_1 \cdot v_{\sigma} &\leq n(n+1)(2n+1)/6 - n(n+1)(n+2)/6 \\ &= n(n-1)(n+1)/6 \end{aligned}$$

and since p divides $v_1 \cdot v_1 - v_1 \cdot v_{\sigma}$ we must have $n(n-1)(n+1)/6 \geq p$. Thus $n^3 > n(n^2-1) \geq 6p$ and $n > \sqrt[3]{6p}$. This completes the proof.

We will need the following few facts on locally finite groups.

LEMMA 5.2. *Let G be an arbitrary group.*

(i) *Let N be a normal subgroup of G and suppose that both N and G/N are locally finite. Then G is locally finite.*

(ii) *Let M be the subgroup of G generated by all locally finite normal subgroups of G . Then M is locally finite and G/M has no nonidentity finite normal subgroup.*

Proof. (i) Let H be a finitely generated subgroup of G . Then HN/N is a finitely generated subgroup of G/N and thus

$$HN/N \cong H/(H \cap N)$$

is finite. Since H is finitely generated and $[H: H \cap N] < \infty$ we see that $H \cap N$ is finitely generated by Lemma I. 7.1. Thus $H \cap N$ is finite and this result follows.

(ii) Suppose N_1 and N_2 are locally finite normal subgroups of G . Then $N_1 N_2$ is normal in G and $N_1 N_2 / N_1 \cong N_2 / (N_1 \cap N_2)$ is clearly a locally finite group. Thus by (i) above $N_1 N_2$ is locally finite. By induction we

see that any finite number of locally finite normal subgroups of G generate a locally finite group. Since any finitely generated subgroup of M is contained in the group generated by a finite number of locally finite normal subgroups we see that M is locally finite. Let H/M be a normal subgroup of finite order in G/M . Then H is normal in G and H is locally finite by (i) so $H = M$ and $H/M = \langle 1 \rangle$.

There are a number of interesting characteristic subgroups of G which are related to the question of $K[G]$ being prime or semiprime.

Let \mathcal{S} be the set of normal subgroups N of G such that $K[G/N]$ is prime. By Theorem I. 2.5 this condition is independent of the field K and in fact $N \in \mathcal{S}$ if and only if $\Delta(G/N)$ is torsion free abelian. Set

$$\Omega = \Omega(G) = \bigcap_{N \in \mathcal{S}} N.$$

Then Ω is clearly a characteristic subgroup of G .

Let p be a prime and let \mathcal{S}_p be the set of normal subgroups N of G such that $K[G/N]$ is semiprime for some field K of characteristic p . By Theorem I. 8.6 this condition is independent of the field K and in fact $N \in \mathcal{S}_p$ if and only if $\Delta(G/N)$ has no elements of order p . Set

$$\Omega_p = \Omega_p(G) = \bigcap_{N \in \mathcal{S}_p} N.$$

Then Ω_p is clearly a characteristic subgroup of G .

Let $\mathbf{O}^{p'}(G)$ denote the subgroup of G generated by all elements of order a power of p .

Theorem 5.3. *With the above notation we have*

(i) $K[G/\Omega]$ is prime for any field K and hence Ω is the unique minimal member of \mathcal{S} .

(ii) $K[G/\Omega_p]$ is semiprime for any field K of characteristic p and hence Ω_p is the unique minimal member of \mathcal{S}_p .

(iii) Ω is a characteristic locally finite subgroup of G .

(iv) Ω_p is a characteristic locally finite subgroup of G . Furthermore $\Omega_p = \mathbf{O}^{p'}(\Omega_p) \subseteq \mathbf{O}^{p'}(\Omega)$.

Proof. We start with a simple observation. Let $N \supseteq H$ be normal subgroups of G . Let $\bar{x} = xH$ be an element of order k in $\Delta(G/H)$. Then $x^k \in H \subseteq N$ and all conjugates of x in G are contained in finitely many cosets of H and hence of N . Thus $\tilde{x} = xN \in \Delta(G/N)$ and $\tilde{x}^k = 1$ so the order of \tilde{x} divides k .

(i) Let $\bar{x} = x\Omega$ be an element of finite order in $\Delta(G/\Omega)$ and let $N \in \mathcal{S}$. Then by the above $\tilde{x} = xN$ is an element of finite order in $\Delta(G/N)$ so $\tilde{x} = 1$ by Theorem I. 2.5. Thus $x \in \bigcap_{N \in \mathcal{S}} N = \Omega$ and $\bar{x} = 1$.

Hence $K[G/\Omega]$ is prime by Lemma I. 2.2 and Theorem I. 2.5.

(ii) The proof here is the same as (i) except that we let \bar{x} have order a power of p and we apply Theorem I. 3.1.

(iii) Let M be the subgroup of G given in Lemma 5.2. ii. Then G/M has no nonidentity finite normal subgroups so by Theorem I. 2.5., $M \in \mathcal{S}$. Since $\Omega \subseteq M$ and M is locally finite, this result follows.

(iv) Since any prime ring is also semiprime we have $\mathcal{S} \subseteq \mathcal{S}_p$ and hence $\Omega_p \subseteq \Omega$. Thus by (iii) Ω_p is locally finite. Let $N = \mathbf{O}^{p'}(\Omega_p)$ so that N is also normal in G . Since Ω_p is periodic we see that Ω_p/N has no elements of order p . Let xN be an element of order p in $\Delta(G/N)$. Then $x\Omega_p$ is an element of order p or 1 in $\Delta[G/\Omega_p]$. Since $K[G/\Omega_p]$ is semi-prime for any field K of characteristic p by (ii), Theorem I. 3.6 implies that $x\Omega_p$ has order 1 so $x \in \Omega_p$. Since $x^p \in N$ and Ω_p/N has no elements of order p we conclude that $x \in N$ and xN has order 1, a contradiction. Thus $\Delta(G/N)$ has no elements of order p and $N \in \mathcal{S}_p$ by Theorem I. 3.6. Hence $\Omega_p \subseteq N$ so $\Omega_p = N = \mathbf{O}^{p'}(\Omega_p)$. Finally $\Omega_p \subseteq \Omega$ implies $\mathbf{O}^{p'}(\Omega_p) \subseteq \mathbf{O}^{p'}(\Omega)$ and the result follows.

It is not necessarily true in (iv) above that $\Omega_p = \mathbf{O}^{p'}(\Omega)$. Let q be a prime distinct from p and let A be an infinite elementary abelian q group. Set $G = A \rtimes Z_p$, the semidirect product of A by Z_p , the cyclic group of order p , where Z_p acts on A by permuting a basis in cycles of length p . Clearly $A = \Delta(G)$ so by Theorem I. 3.6 we have $\Omega_p(G) = \langle 1 \rangle$. If N is normal in G with $K[G/N]$ prime then by Theorem I. 2.5 we must have $N \supseteq A$. Then G/N is finite so $G = N$. This shows that $\Omega(G) = G$ and hence $\mathbf{O}^{p'}(\Omega) \neq \Omega_p$ in this case.

6. Solvable subgroups. Probably the chief stumbling block in obtaining characteristic p analogues of Theorem 5.3 of [7] has been the use of Jordan's theorem on complex linear groups in its proof. In this section we offer a simple argument which can be used instead of it in characteristic 0. In addition we announce an appropriate analogue in characteristic p and we indicate some interesting consequences. Proofs of the latter results require a complete reformulation of the pertinent parts of [7] from a character-theoretic to a module-theoretic point of view and they will appear elsewhere.

LEMMA 6.1. *Let G be a finite group with r.b.n. (in the notation of [7]). If $G \neq \langle 1 \rangle$ then there exists an element $x \in G$, $x \neq 1$ with $[G: C(x)] \leq n^2$.*

Proof. Say G has m irreducible characters and hence also m conjugacy classes. Since $G \neq \langle 1 \rangle$ we have $m > 1$. Let the degrees of the characters be $1 = n_1, n_2, \dots, n_m$ where n_1 corresponds to the

principal character. Then

$$|G| = n_1^2 + n_2^2 + \cdots + n_m^2 \leq 1 + (m-1)n^2$$

since G has r.b.n.. Let the conjugacy classes of G have sizes $1 = c_1, c_2, \dots, c_m$ where c_1 corresponds to the identity class. If $c = \min_{i \neq 1} c_i$ then

$$|G| = c_1 + c_2 + \cdots + c_m \geq 1 + (m-1)c.$$

Taken together these two inequalities then yield $n^2 \geq c = [G: C(x)]$ for some $x \neq 1$.

LEMMA 6.2. *Let G be a finite group with r.b.n.. If $G \neq \langle 1 \rangle$ then G has a normal subgroup N with $[G: N] \leq (n^2)!$ and $C(N) \neq \langle 1 \rangle$.*

Proof. Let x be as in Lemma 6.1 and let N be the core of $C(x)$, that is, N is the intersection of all conjugates of $C(x)$. Then $[G: N] \leq (n^2)!$ since $[G: C(x)] \leq n^2$ and $x \in C(N)$ so $C(N) \neq \langle 1 \rangle$.

Jordan's theorem is used on page 896 of [7] to bound the order of a group E satisfying

- (1) E is nonabelian.
- (2) All proper homomorphic images of E are abelian.
- (3) E has no nonidentity normal abelian subgroup.

We use Lemma 6.2 instead as follows.

LEMMA 6.3. *Let E be a finite group with r.b.n. and let E satisfy (1), (2) and (3) above. Then $|E| \leq (n^2)!$*

Proof. Let N be the normal subgroup of E given by Lemma 6.2. Suppose $N \neq \langle 1 \rangle$. Then by (2), $N \cong E'$. Since $C(N)$ is normal in E and $C(N) \neq \langle 1 \rangle$ we have also $C(N) \cong E'$. Thus $N \cap C(N) = Z(N) \cong E'$ and E' is abelian, contradicting (1) and (3). Hence $N = \langle 1 \rangle$ and $|E| = [E: N] \leq (n^2)!$

The appropriate characteristic p analogue of bounded representation is of course that the group ring satisfy a polynomial identity. Using a combinatorial argument similar to that of Theorem I. 6.2 we can obtain

LEMMA 6.4. *Let G be finite and nonabelian and let K be any field. If $K[G]$ satisfies a polynomial identity of degree n then there exists an element $x \in G - Z(G)$ with $[G: C(x)] \leq (n!)^2$.*

Of course this immediately yields analogues of Lemmas 6.2 and 6.3. Define the integer valued function $\mathfrak{A}(n)$ as follows.

$$\mathfrak{A}(n) = (n!)^{n \cdot (n!)^2}.$$

Then using the above argument instead of Jordan's theorem we obtain the following replacements for Theorem I. 8.2 and I. 9.4 respectively.

THEOREM 6.5. *Let G be a finite group and let K be a field with $|G'| \neq 0$ in K . If $K[G]$ satisfies a polynomial identity of degree n , then G has an abelian subgroup A with $[G: A] \leq \mathfrak{A}(n)$.*

THEOREM 6.6. *Let $K[G]$ be a semiprime group ring which satisfies a polynomial identity of degree n . Then G has an abelian subgroup A with $[G: A] \leq n! \mathfrak{A}(n)$.*

If $K[G]$ is not semiprime we can still obtain results of interest. Let G be an arbitrary group and let $S(G)$ be the subgroup of G generated by all normal solvable subgroups of G . It follows easily that $S(G)$ is a characteristic locally solvable subgroup of G . In particular, if G is finite, then $S(G)$ is the unique maximal normal solvable subgroup of G .

THEOREM 6.7. *Let G be a finitely generated group and let K be any field. If $K[G]$ satisfies a polynomial identity of degree n then $S(G)$ is solvable and $[G: S(G)] \leq \mathfrak{A}(n)$.*

THEOREM 6.8. *Let G be an arbitrary group and let K be any field. If $K[G]$ satisfies a polynomial identity of degree n then $[G: S(G)] \leq n! \cdot \mathfrak{A}(n)$.*

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