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QUASIVECTOR TOPOLOGIES

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A topology on a vector space for which the vector operations are only separately continuous is called a quasivector topology. Some version of most of the usual results for topological vector spaces is obtained for these topologies.

Convergence structures which are more general than topologies can be used to obtain results about quasivector topologies and this relationship is described and used.

The techniques are motivated by certain quasivector topologies which occur in functional analysis and references are given to these occurrences.

Topologies on vector spaces with respect to which the vector operations are only separately continuous do occur in functional analysis. If (E, τ) is a locally convex space and E' is the space of all continuous linear functionals, let ρ be the finest topology on E' which agrees with $\sigma(E', E)$ on τ-equicontinuous sets. It can happen that the vector addition on E' is only separately continuous with respect to ρ ([4]). Another interesting example is given in [8, page 5]. We refer to such topologies as quasivector topologies.

Some of the results which are true for topological vector spaces are true in this more general setting. It is the purpose of this paper to establish such results. For an application of the results of this paper see [6], [7].

Some quasivector topologies are related to general convergences with respect to which the vector operations are jointly continuous. Indeed, two examples mentioned above are of this type. General convergence theories are very useful in the study of quasivector topologies as they often allow one to avoid having to work with the complicated neighborhoods which are associated with the quasivector topologies.

Such a convergence theory has been defined in [3] and [5], and in [3] the theory has been developed sufficiently to offer effective application.[1] We begin with a definition and a theorem from [3].

Let X be any set. If $\mathscr{F}$ and $\mathscr{G}$ are two filters on X we write $\mathscr{F} \leqq \mathscr{G}$ to mean $\mathscr{F} \subseteqq \mathscr{G}$. If $\mathscr{B}$ is a filterbase or subbase we write $[\mathscr{B}]$ for the filter generated by $\mathscr{B}$. If A is a subset of X and $\mathscr{F}$ is a filter on X such that $A \cap F \neq \varnothing$ for each F in $\mathscr{F}$, we denote the filter on A, $\{A \cap F \mid F \in \mathscr{F}\}$, by $tr_A\mathscr{F}$. If there is an F in $\mathscr{F}$ such that $A \cap F = \varnothing$ we say that $tr_A\mathscr{F}$ does not exist.

[1] In [5] the theory was used to study compactifications of topological spaces.

DEFINITION 1. A convergence structure on X is a function τ defined on X such that, for x in X

(1) $\tau(x)$ is a collection of filters on X.

(2) $\dot{x} = \{F \mid x \in F\}$ is in $\tau(x)$.

(3) $\mathscr{F} \in \tau(x)$ and $\mathscr{G} \geqq \mathscr{F}$ implies that $\mathscr{G} \in \tau(x)$.

(4) $\mathscr{F} \in \tau(x)$ and $\mathscr{G} \in \tau(x)$ implies that $\mathscr{F} \wedge \mathscr{G} \in \tau(x)$.[2]

If $\mathscr{F} \in \tau(x)$ we say that $\mathscr{F}$ converges to x with respect to τ. For each subset $A \subseteqq X$ we define $\bar{A} = \{x \mid$ there is an $\mathscr{F}$ in $\tau(x)$ such that $A \in \mathscr{F}\}$, and we say that A is τ-closed if $A = \bar{A}$. If (X, τ) and (Y, σ) are two convergence spaces and $f: X \rightarrow Y$ is any mapping we say that f is $\tau - \sigma$ continuous if $[f(\mathscr{F})] \in \sigma(f(x))$ for each $\mathscr{F}$ in $\tau(x)$ and each x in X.

If τ and σ are two convergence structures on X we say that τ is finer than σ (or σ is coarser than τ) and write $\sigma \leqq \tau$ if $\tau(x) \subseteqq \sigma(x)$ for each x in X. Clearly we can consider the set of all topologies on X to be a subset of the set of all convergence structures on X. If τ is any convergence structure on X we will use $\omega\tau$ to denote the finest topology coarser than τ. A set V is $\omega\tau$-open if and only if $\mathscr{F} \in \tau(x)$ and $x \in V$ implies that $V \in \mathscr{F}$. Also, τ and $\omega\tau$ have the same closed sets.

THEOREM 1. ([3]). *If $f: X \rightarrow Y$ is $\tau - \sigma$ continuous then f is $\omega\tau - \omega\sigma$ continuous. Thus if (Y, σ) is a topological space the set of $\tau - \sigma$ continuous mappings is the same as the set of $\omega\tau - \sigma$ continuous mappings.*

Given convergence spaces (X, τ) and (Y, σ) the product space $(X \times X, \tau \times \sigma)$ is defined by $\tau \times \sigma((x, y)) = \{\mathscr{F} \mid p_1(\mathscr{F}) \in \tau(x)$ and $p_2(\mathscr{F}) \in \sigma(y)\}$ where p_1 and p_2 are the two projections. If X and Y are also vector spaces we will denote $(X \times Y, \tau \times \sigma)$ with the usual vector operations by $(X \oplus Y, \tau \oplus \sigma)$ or $(X, \tau) \oplus (Y, \sigma)$.

DEFINITION 2. Let E be a vector space[3], and τ be a convergence structure on E.

(1) We say that τ is a vector convergence if the vector operations are jointly continuous.

(2) We say that τ is a quasivector convergence if the vector operations are separately continuous.

(3) We say that τ is balanced if $\mathscr{F} \in \tau(0)$ implies that $b(\mathscr{F}) = [\{b(F) \mid F \in \mathscr{F}\}] \in \tau(0)$ where $b(F) = \{kv \mid k \in K, v \in F$, and $|k| \leqq 1\}$.

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[2] By $\wedge$ we mean greatest lower bound with respect to the order defined in the preceding paragraph.

[3] We will denote the scalars by K which refers to either the real or complex numbers. For each positive real number r, $S(r)$ will donote the set $\{k \mid k \in K$ and $|k| < r\}$.

Using theorem 1 we see that if τ is a quasivector convergence then $\omega\tau$ is a quasivector topology. It can happen that τ is a vector convergence but $\omega\tau$ is only a quasivector topology.

The balanced property is what is need to characterized continuous linear functionals by their kernels.

THEOREM 2. *Let τ be a quasivector balanced convergence structure on the vector space E. A linear functional $f: E \to K$ is τ-continuous if and only if Ker f is τ-closed.*

Proof. Necessity is clear. Suppose that Ker f is closed. If $f = 0$ it is continuous. If not, choose x such that $f(x) = 1$. Suppose that $\mathscr{F} \in \tau(0)$. Let s be any positive real number. Then $H = sx + \operatorname{Ker} f$ is a τ-closed set and $0 \notin H$. Now $b(\mathscr{F}) \in \tau(0)$ and thus there is a set $F \in \mathscr{F}$ such that $b(F) \subseteq E\backslash H$ since $E\backslash H$ is $\omega\tau$-open. Now $f(b(F))$ is a balanced set of scalars which does not contain the real number s. Thus $f(F) \subseteq f(b(F))$ is bounded in norm by s. Since s was arbitrary this proves that $f(\mathscr{F})$ converges to 0; f therefore is continuous at 0, hence continuous since τ is a quasivector convergence.

If τ is any topology and α is a convergence structure such that $\tau = \omega\alpha$ we say that τ can be approximated by α. (In this case the τ-closure of a set can be obtain by a transfinite iteration of α-closures.) Since it is not true that α balanced implies that τ is balanced, the following is pleasing.

COROLLARY 1. *Suppose that τ is a quasivector topology on E which can be approximated by a balanced quasivector convergence structure. A linear functional is τ-continuous if and only if its kernel is τ-closed.*

Proof. Let α be the approximating convergence structure. The α-closed sets and the τ-closed sets are the same. Now use Theorem 1 and Theorem 2.

LEMMA 1. *Let α be a balanced vector convergence structure on E, let F be an α-closed vector subspace of E, and $x \notin F$. Then $Kx + F$ is α-closed and $(Kx + F, \alpha \mid Kx + F) \cong (Kx, \alpha \mid Kx) \bigoplus (F, \alpha \mid F)$.*

Proof. Define $f: Kx + F \to K$ by $f(kx + v) = k$. Then $\operatorname{Ker} f = F$ and thus f is α-continuous. Suppose that $\mathscr{F}$ is a filter on $Kx + F$ such that $[\mathscr{F}] \to y$. Then $\mathscr{F} - \mathscr{F}$ converges to 0 with respect to α and thus $f(\mathscr{F}) - f(\mathscr{F})$ converges to 0 in K. Thus there is a number s such that $f(\mathscr{F})$ converges to s. Now $\mathscr{F} - f(\mathscr{F})x$ converges to $y - sx$. Since $tr_{\operatorname{Ker} f}(\mathscr{F} - f(\mathscr{F})x)$ exists we can conclude that $y - sx$ is in $\overline{\operatorname{Ker} f} = F$ and thus y is in $Kx + F$. Thus $Kx + F$

is α-closed. Finally, the isomorphism with the direct sum is given by the mapping $T(x) = f(x)x + (x - f(x)x)$.

Many quasivector topologies can be approximated by vector convergences so the following is a useful corollary.

COROLLARY 2. *Let τ be a quasivector topology for E which can be approximated by a balanced vector convergence. If F is a τ-closed vector subspace of E and G is any finite dimensional subspace, then $F + G$ is τ-closed.*

A word about internal direct sums is in order. Suppose the quasivector topology τ can be approximated by the quasivector convergence α. Suppose that F and G are disjoint τ-closed (hence α-closed) vector subspaces and that $\overline{F + G} = F + G \cong F \oplus G$ with respect to α. It does not follow that the isomorphism holds with respect to τ. In general it is not true that $\omega(\alpha \mid M) = (\omega\alpha) \mid M$, however in [6] it is shown that this does hold if M is α-closed or $\omega\alpha$-open. Using this fact and Theorem 1 we have

$$\begin{aligned}(F + G, \tau \mid F + G)' &= (F + G, \omega(\alpha \mid F + G))' = (F + G, \alpha \mid F + G)' \\ &= (F, \alpha \mid F)' \oplus (G, \alpha \mid G)' = (F, \omega(\alpha \mid F))' \oplus (G, \omega(\alpha \mid G))' \\ &= (F, \tau \mid F)' \oplus (G, \tau \mid G)'\end{aligned}$$

where $'$ denotes the continuous linear functionals.

The following is routine.

THEOREM 3. *Let α be a convergence structure on the vector space E and let F be a vector subspace. If α is a vector convergence or if α is a quasivector topology then $\bar{F}$ is a vector subspace.*

The uniqueness of finite dimensional Hausdorff topologies remains true.

THEOREM 4. *Suppose that τ is a balanced T_1 quasivector topology on R^n and ρ is the usual Hausdorff vector topology for R^n. Then $\tau \leqq \rho$. If τ is T_2 then $\tau = \rho$.*

Proof. Assume τ is T_1. The proof is by induction on n. For $n = 1$ Theorem 2 and the separate continuity give equality ($\tau = \rho$). Suppose the result ($\tau \leqq \rho$) is true for dimension $n - 1$. Let $\mathscr{N}$ be the usual neighborhood filter at 0 for the real numbers, let $\mathscr{R}$ be the ρ neighborhood filter at 0, and let $\mathscr{T}$ be the τ neighborhood filter at 0.

Let V be any τ-open balanced neighborhood of 0. Let R^* be the extended real number system and let B be the set of all vectors in R^n having length 1. For each x in B, $V \cap Rx$ is a balanced set of real numbers. Thus it is of the form $(-s(x), s(x))x$ where $s(x)$ is a positive element of R. We therefore have a mapping $s: B \rightarrow R^*$. For each $\delta > 0$ let $A_\delta = \{x \mid s(x) > \delta\}$. Suppose that $x \in A_\delta$. Choose v in V such that $v = rx$ with $r > \delta$. Let F be the $n-1$ dimensional subspace of R^n consisting of all vectors perpendicular to x. By the induction hypothesis $\tau \mid F \leqq \rho \mid F$ and thus the filter $\mathscr{N}S$ converges to 0 with respect to τ, where S is the set of all vectors in F which have length less than 1. Now $v + \mathscr{N}S$ converges to v and since V is τ-open there is $k > 0$ such that $v + kS \subseteqq V$. Projecting the set $v + kS$ onto B gives a ρ-open neighborhood of x in which s is greater than δ. Thus A_δ is ρ-open in B for each $\delta > 0$. Since B is ρ-compact and s is positive this shows that s is bounded away from 0. This shows that V contains a member of $\mathscr{R}$ and thus we have that $\tau \leqq \rho$.

Assume that τ is Hausdorff. Since $\tau \leqq \rho$ and Hausdorff we know that τ and ρ agree on ρ-bounded sets. Now suppose that $\tau \neq \rho$. Then every member of $\mathscr{T}$ is unbounded with respect to ρ and since τ is balanced we know that $tr_B\mathscr{T}$ exists. Let x be a ρ-cluster point of this filter. Now using the fact that $\tau \leqq \rho$ we see that it is impossible to separate x from $2x$ by τ-open sets. This contradicts the fact that τ is Hausdorff and thus we must have $\tau = \rho$.

It is worth nothing that in the absence of joint continuity of vector addition, the properties T_1 and T_2 are not equivalent. Indeed the last part of the preceding theorem is not true if T_2 is replaced by T_1.

The extension of a continuous linear function from a dense vector subspace to the entire space is usually accomplished using uniform continuity which does not make sense for quasivector topologies. Of course there is always a topological extension (see [2, page 216]; [1] for a generalization). The problem is whether or not this extension is large enough. The following establishes an affirmative answer for linear functionals.

THEOREM 5. *Suppose that τ is a topology for a vector space E such that*

(1) *For each x in E the mapping $T_x(y) = x + y$ is continuous*
(2) *The mapping $R(x) = -x$ is continuous.*

Let F be a dense vector subspace and $f: F \rightarrow K$ be a continuous linear functional. Then for each x in E $f(tr_F\mathscr{N}(x))$ converges, where $\mathscr{N}(x)$ is the τ-neighborhood filter at x.

Proof. Let $x \in E \backslash F$. Denote $tr_F\mathscr{N}(x)$ by $\mathscr{F}$. Now $\mathscr{F} - x$

converges to 0. Since f is continuous there is a τ-open set N such that $0 \in N \cap F \subseteqq f^{-1}(S(1))$. Choose $V \in \mathscr{F}$ such that $V - x \subseteqq N$. Choose $v \in V$. Since $v - x \in N$ and $(-\mathscr{F} + x) + v - x$ converges to $v - x$ we can choose W in $\mathscr{F}$ so that $(-W + x) + v - x \subseteqq N$. We thus get that $v - W \subseteqq N \cap F \subseteqq f^{-1}(S(1))$. This shows that the set $f(W)$ is bounded.

Thus the set of cluster points of $f(\mathscr{F})$ is not empty. To complete the proof we need only show that there is exactly one cluster point. Suppose that r and s are two cluster points and further assume that $r \neq s$. Let $\varepsilon = \frac{1}{2}|r - s|$. Let N be a τ-open neighborhood of 0 such that $N \cap F \subseteqq f^{-1}(S(\varepsilon))$. Choose V in $\mathscr{F}$ such that $V - x \subseteqq N$, Since $r \in \overline{f(V)}$ we can choose $v \in V$ so that $|f(v) - r| < \frac{1}{4}|r - s|$. Now choose W in $\mathscr{F}$ so that $(-W + x) + v - x \subseteqq N$. But $s \in \overline{f(W)}$ so that we can choose $w \in W$ so that $|f(w) - s| < \frac{1}{4}|r - s|$. Now $v - W \subseteqq N \cap F \subseteqq f^{-1}(S(\varepsilon))$ and $|f(v - w)| = |f(v) - f(w)| > \frac{1}{2}|r - s| = \varepsilon$, a contradiction.

COROLLARY 3. *Let τ be a quasivector topology for a vector space E, F be a τ-dense vector subspace, and $f\colon F \to K$ a τ-continuous linear functional. Then f has a τ-continuous linear extension to all of E.*

Proof. By the preceding theorem and [2, page 216] f has a continuous extension. It is given by $f(x) = \operatorname{limit} f(tr_F \mathscr{N}(x))$ for each x in E. Since τ is a quasivector topology it follows that $\mathscr{N}(kx) = k\mathscr{N}(x)$ for each x in E and each k in K. Thus $f(kx) = kf(x)$. Now let G be any vector subspace containing F. Since f is continuous we know that $f(x) = \operatorname{limit} f(tr_G \mathscr{N}(x))$ for each x in E. Let x be any element in $E \backslash G$ and y be any element of G. Then $\mathscr{N}(x + y) = y + \mathscr{N}(x)$ and, since y is in G, $tr_G(y + \mathscr{N}(x)) = y + tr_G \mathscr{N}(x)$. This shows that $f(x + y) = f(x) + f(y)$ which shows that f is linear on $Kx + G$. By Zorn's lemma f is linear on all of E.

The use of the boundedness of neighborhoods and the relative compactness of bounded sets prevent the extension of the above theorem to a class of infinite dimensional range spaces. It might be hoped to extend linear transformations with dense domains and values in a complete topological vector space if the quasivector topology on the initial space can be approximated by a vector convergence structure. It seems that the concept of regular Cauchy structure, as described in [6] will be useful in working with this problem.

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