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LINEAR SEMIPRIME $(p;q)$ RADICALS

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This paper introduces McKnight's $(p; q)$-regularity and $(p; q)$ radicals, a collection of radicals which contains the Jacobson radical and the radicals of regularity and strong regularity among its members. The linear semiprime $(p; q)$ radicals are classified canonically and, as a result of this classification, these radicals can be distinguished by the fields $GF(p)$ and are shown to form a lattice. The semiprime $(p; q)$ radicals are found to be hereditary and the linear semiprime $(p; q)$ radical of a complete matrix ring of a ring R is determined to be the complete matrix ring over the $(p; q)$ radical of R. More generally, the $(p; q)$ radical of a complete matrix ring over R is contained in the matrix ring over the $(p; q)$ radical of R for all $(p; q)$ radicals.

A function ρ which assigns to each ring R an ideal ρR of the ring is called a *radical function* in the sense of Amitsur and Kurosh [1; 5] if it has the following properties:

R1: If $\phi: R \rightarrow S$ is a ring epimorphism and $\rho R = R$, then $\rho S = S$.

R2: $\rho(\rho R) = \rho R$ for all rings R and if $\rho I = I$ for any ideal I of R, then $I \subseteqq \rho R$.

R3: $\rho(R/\rho R) = 0$ for all rings R.

If ρ is a radical function, then the ideal ρR is called the *radical* of R. If $\rho R = R$ for some ring R, then R is called a *ρ-radical ring* while if $\rho R = 0$ we call R a *ρ-semisimple ring*. If I is an ideal (right ideal) of a ring R, then I is called a *ρ-radical ideal* (right ideal) if I is a ρ-radical ring.

Now let $p(x)$ and $q(x)$ be polynomials with integer coefficients. An element r of a ring R is called *$(p; q)$-regular* if $r \in p(r)Rq(r)$, that is, $r = p(r)sq(r)$ for some $s \in R$ where an integer multiple of a ring element has its usual meaning. If every element of an ideal I of R is $(p; q)$-regular, that is, if $r \in p(r)Iq(r)$ for all $r \in I$, then I is said to be a *$(p; q)$-regular ideal*. Examples of $(p; q)$-regularity are quasi-regularity, $(x + 1; 1)$ [4], von Neumann regularity, $(x; x)$ [7] and strong regularity, $(x^2; 1)$ [2].

LEMMA 1. *If I and R/I are $(p; q)$-regular, then R is $(p; q)$-regular.*

Proof. Let $r \in R$. Then $r + I \in R/I$, which implies

$$r + I = p(r + I)(s + I)q(r + I) = p(r)sq(r) + I$$

for some $s + I \in R/I$. Thus $r - p(r)sq(r) \in I$ and, since I is $(p; q)$-

regular, $r - p(r)sq(r) = p[r - p(r)sq(r)]tq[r - p(r)sq(r)]$ for some $t \in I$. Moreover there exist $u, v \in R$ such that

$$\begin{aligned} r - p(r)sq(r) &= p[r - p(r)sq(r)]tq[r - p(r)sq(r)] \\ &= [p(r) - p(r)u]t[q(r) - vq(r)] \end{aligned}$$

or $r = p(r)(s + t - ut - tv + utv)q(r)$. Therefore R is $(p; q)$-regular.

LEMMA 2. *If I and J are $(p; q)$-regular ideals of R, then $I + J$ is a $(p; q)$-regular ideal of R.*

Proof. Immediate from Lemma 1, since the homomorphic image of a $(p; q)$-regular ring is a $(p; q)$-regular ring.

COROLLARY 1. *The sum of all $(p; q)$-regular ideals of a ring R is a $(p; q)$-regular ideal of R.*

Proof. This follows from Lemma 2, since $(p; q)$-regularity is defined elementwise.

We shall let $(p(x)Rq(x))$ denote the largest $(p; q)$-regular ideal of the ring R. Then we have

THEOREM 1. (J. D. McKnight, Jr.) *If a function ρ is defined by $\rho R = (p(x)Rq(x))$ for all rings R, then ρ is a radical function.*

Proof. We only need to show R3 holds. Let $I/\rho R$ be a $(p; q)$-regular ideal of $\rho(R/\rho R)$. Then by Lemma 1, I is a $(p; q)$-regular ideal of R and $I \subseteq \rho R$.

We shall call $(p(x)Rq(x))$ the $(p; q)$ radical of the ring R. Thus the Jacobson radical and the radicals of regularity and strong regularity of R are given by $((x + 1)R)$, (xRx) and (x^2R) respectively.

1. A canonical representation for linear semiprime $(p; q)$ radicals.

A radical function ρ is called *semiprime* if ρR is a semiprime ideal, equivalently, if ρR contains the prime (Baer-lower) radical [6; 3]. Now we shall determine the form of the semiprime $(p; q)$ radicals.

LEMMA 3. *ρ is a semiprime radical function if and only if $\rho R = R$ for all zero rings R.*

Proof. The necessity is clear. Now if $I^2 \subseteq \rho R$ for some ideal I of R, then $\rho[(I + \rho R)/\rho R] = (I + \rho R)/\rho R$ since $(I + \rho R)/\rho R$ is isomorphic to the zero ring $I/(I \cap \rho R)$. Also $\rho(R/\rho R) = 0$ implies

$$\rho[(I + \rho R)/\rho R] = 0$$

and therefore $I \subseteqq \rho R$.

THEOREM 2. (A. H. Ortiz) *$(p(x)Rq(x))$ is semiprime for all rings R if and only if the constant terms of $p(x)$ and $q(x)$ are 1 or -1.*

Proof. Let $p(x)$ and $q(x)$ have constant terms 1 or -1 and R be any zero ring. Then for $r \in R$, we have $r = p(r)(\pm r)q(r)$ and $R \subseteqq (p(x)Rq(x))$. Thus $R = (p(x)Rq(x))$. Conversely, if a_0 and b_0 are the constant terms of $p(x)$ and $q(x)$ respectively, then suppose $a_0 \neq \pm 1$ or $b_0 \neq \pm 1$. Since we are assuming the $(p; q)$ radical is semiprime, it follows from Lemma 3 that for the zero ring with additive group $\boldsymbol{Z}/(a_0 b_0)$ we have $(p(x)[\boldsymbol{Z}/(a_0 b_0)]q(x)) = \boldsymbol{Z}/(a_0 b_0)$ where $\boldsymbol{Z}$ denotes the ring of integers and $(a_0 b_0)$ the ideal generated by $a_0 b_0$. However if $r \in (p(x)[\boldsymbol{Z}/(a_0 b_0)]q(x))$, then $r \in p(r)[\boldsymbol{Z}/(a_0 b_0)]q(r)$ and $r = 0$. Hence $\boldsymbol{Z}/(a_0 b_0) = 0$, which is a contradiction.

Henceforth we shall be considering semiprime $(p; q)$ radicals and, since $(p(x)R) = (p(-x)R) = (-p(x)R)$, we shall assume that the constant term of $p(x)$, similarly the constant term of $q(x)$, is 1.

LEMMA 4. *If the constant term of $p(x)$ is 1, then for all $r \in R$ we have $r \in p(r)R$ if and only if $R = p(r)R$.*

Proof. The sufficiency is obvious. Now let $r \in p(r)R$. Since $p(r) = rf(r) + 1$ for some integral polynomial $f(x)$, for any $s \in R$ we have, $p(r)s = rf(r)s + s$. Since $r \in p(r)R$ we have $s \in p(r)R$ and $R \subseteqq p(r)R$. Therefore, $R = p(r)R$.

COROLLARY 2. *If the constant terms of $p(x)$ and $q(x)$ are both 1, then for all $r \in R$ we have $r \in p(r)Rq(r)$ if and only if $R = p(r)Rq(r)$.*

THEOREM 3. *If $(p(x)Rq(x))$ and $(p'(x)Rq'(x))$ are semiprime for all rings R, then $(p(x)Rq(x)) \cap (p'(x)Rq'(x)) = (p(x)p'(x)Rq'(x)q(x))$.*

Proof. Clearly $(p(x)p'(x)Rq'(x)q(x)) \subseteqq (p(x)Rq(x)) \cap (p'(x)Rq'(x))$. Now let $r \in (p(x)Rq(x)) \cap (p'(x)Rq'(x))$. Then $r \in (p'(x)Rq'(x))$ implies $r \in p'(r)Rq'(r)$ and, by Corollary 2, $R = p'(r)Rq'(r)$. Now $r \in p(r)Rq(r)$ and $R = p'(r)Rq'(r)$ implies $r \in p(r)p'(r)Rq'(r)q(r)$. The product polynominals $p(x)p'(x)$ and $q(x)q'(x)$ have constant terms 1, hence $r = p(r)p'(r)sq'(r)q(r)$ implies that $s \in (p(x)Rq(x)) \cap (p'(x)Rq'(x))$. Therefore $(p(x)Rq(x)) \cap (p'(x)Rq'(x))$ is $(pp'; q'q)$-regular and

$$(p(x)Rq(x)) \cap (p'(x)Rq'(x)) \subseteqq (p(x)p'(x)Rq'(x)q(x)) .$$

In what follows we shall determine a canonical representation for all *linear semiprime* $(p; q)$ *radicals*, that is, $(p; q)$ radicals determined by integral polynomials $p(x)$ and $q(x)$ which are products of linear polynomials having constant term 1.

LEMMA 5. $((ax + 1)(bx + 1)R) \subseteqq ([(a + b)x + 1]R)$ *for all integers* a, b.

Proof. Let $r \in ((ax + 1)(bx + 1)R)$. Then $r = (ar + 1)s$ for

$$s \in ((ax + 1)(bx + 1)R) .$$

Thus $r = (ar + 1)(bs + 1)t = (br + ar + 1)t = ((a + b)r + 1)t$, where $t \in ((ax + 1)(bx + 1)R)$, implies that

$$((ax + 1)(bx + 1)R) \subseteqq ([(a + b)x + 1]R .$$

COROLLARY 3. $((ax + 1)R) \subseteqq ((\max + 1)R)$ *for all integers* m.

Proof. By Theorem 3 we have $((ax + 1)^m R) = ((ax + 1)R)$.

COROLLARY 4. $((ax + 1)R) \subseteqq ((a^k x + 1)R)$ *for* $k = 1, 2, 3, \cdots$.

LEMMA 6. $((ax + 1)(bx + 1)R) \subseteqq ([(ma + nb)x + 1]R)$ *for all integers* m, n.

Proof. This is immediate from Corollary 3, Lemma 5 and Theorem 3.

Now Corollary 3, Lemma 6 and Theorem 3 yield

THEOREM 4. $((ax + 1)(bx + 1)R) = ([(a, b)x + 1]R)$ *where* (a, b) *is the greatest common divisor of* a *and* b.

We shall now show that the converse of Corollary 4 is true.

LEMMA 7. $((a^k x + 1)R) \subseteqq ((ax + 1)R)$ *for* $k = 1, 2, 3, \cdots$.

Proof. We first show that $((a^2x + 1)R) \subseteqq ((ax + 1)R$. For this inclusion it is sufficient to show that $((a^2x + 1)R) = 0$ whenever $((ax + 1)R) = 0$ so suppose $((ax + 1)S) = 0$ for some ring S. Then if $r \in ((a^2x + 1)S)$ we have $r = (a^2r + 1)s$ or $ar = (a(ar) + 1)as$. Thus $a((a^2x + 1)S) \subseteqq ((ax + 1)S)$ and $ar = 0$ for all $r \in ((a^2x + 1)S)$. Therefore $r = (a^2r + 1)s = (ar + 1)s$ implies that $((a^2x + 1)S) \subseteqq ((ax + 1)S) = 0$. The result now follows by induction.

Combining Corollary 4 and Lemma 7 we have

THEOREM 5. *$((ax + 1)R) = ((a^k x + 1)R)$ for $k = 1, 2, 3, \cdots$.*

Our next lemma and Theorem 3 permit us to represent each linear semiprime $(p; q)$ radical as a $(pq; 1)$ radical.

LEMMA 8. *$((ax + 1)R) = (R(ax + 1))$.*

Proof. First, for $r, s \in R$ define a circle product by $r \circ s = r + s + ars$. Then $(r \circ s) \circ t = r \circ (s \circ t)$. Now if $r \in ((ax + 1)R)$, then $r \circ s = 0$ for some $s \in ((ax + 1)R)$. Since $s \circ t = 0$ for some $t \in ((ax + 1)R)$, we have that $r = t$ and $s \circ r = 0$. Therefore, $((ax + 1)R) \subseteqq (R(ax + 1))$. A similar argument yields the opposite inclusion, hence equality.

We can now give a canonical representation for all linear semiprime $(p; q)$ radicals.

THEOREM 6. *Every linear semiprime $(p; q)$ radical can be uniquely represented by a radical of the form $((ax + 1)R)$ where the nonnegative integer a is a finite product of distinct prime factors.*

Proof. Theorem 3 and Lemma 8 show that

$$(p(x)Rq(x)) = (p(x)q(x)R)$$

for the linear semiprime radical $(p(x)Rq(x))$. Then Theorems 3, 4 and 5 show that $(p(x)q(x)R) = ((ax + 1)R)$ for some nonnegative integer a where a is a finite product of distinct prime factors.

To distinguish between the linear semiprime radicals observe that if $a = \pi_{i=1}^{n} p_i$ for primes p_i, then $((ax + 1)R) = R$ for $R = GF(p_i)$, $i = 1, 2, \cdots, n$ and $((ax + 1)R) = 0$ for $R = GF(p)$ for all primes $p \neq p_i$, $i = 1, 2, \cdots, n$.

2. **The lattice of linear semiprime $(p; q)$ radicals.** Let $(p; q)$ denote the radical function defined by $(p; q)(R) = (p(x)Rq(x))$ for all rings R. We partially order the linear semiprime $(p; q)$ radical functions by defining $(ax + 1; 1) \leqq (bx + 1; 1)$ if $((ax + 1)R) \subseteqq ((bx + 1)R)$ for all rings R. Then we have

THEOREM 7. *The collection of all linear semiprime $(p; q)$ radicals form a lattice with respect to the partial order $\leqq$ where the infimum and supremum are given by the canonical representatives:*

(i) $(ax + 1; 1) \wedge (bx + 1; 1) = ((a, b)x + 1; 1)$
(ii) $(ax + 1; 1) \vee (bx + 1; 1) = ([a, b]x + 1; 1)$
where $[a, b]$ *denotes the least common multiple of* a *and* b.

Proof. (i) By Corollary 3 we have $((a, b)x + 1; 1) \leqq (ax + 1; 1)$, $(bx + 1; 1)$. Now if $(cx + 1; 1) \leqq (ax + 1; 1), (bx + 1; 1)$, then $((cx + 1)R) \subseteqq ((ax + 1)R) \cap ((bx + 1)R) = ((ax + 1)(bx + 1)R) = ([(a, b)x + 1]R)$.

(ii) First let a and b be relatively prime. Since $(abx + 1; 1)$ is clearly an upper bound of $(ax + 1; 1)$ and $(bx + 1; 1)$, we show that for all rings R, $((abx + 1)R) \subseteqq ((cx + 1)R)$ for any other upper bound $(cx + 1; 1)$. Again it is enough to show that this inclusion holds for any ring S for which $((cx + 1)S) = 0$. As in the proof of Lemma 7, $a((abx + 1)S) \subseteqq ((bx + 1)S) \subseteqq ((cx + 1)S) = 0$ and similarly $b((abx + 1)S) = 0$. Therefore, since $(a, b) = 1$, for all $r \in ((abx + 1)S)$ we have integers m, n such that $r = m(ar) + n(br) = 0$. Therefore $((abx + 1)S) = 0$ and $((abx + 1)R) \subseteqq ((cx + 1)R)$. Thus when $(a, b) = 1$, we have $(ax + 1; 1) \vee (bx + 1; 1) = ([a, b]x + 1; 1)$. Using this result it is easy to see that the statement is true for arbitrary integers a and b.

It is interesting to observe that $((x + 1); 1)$, the Jacobson radical, is the least element in this lattice.

3. Hereditary $(p; q)$ radicals. A radical function ρ is called *hereditary* if every ideal of a ρ-radical ring is ρ-radical. Equivalently, if for any (associative) ring R and any ideal I of R we have the equation $\rho I = I \cap \rho R$, then ρ is hereditary [3, p. 125]. The linear semiprime $(p; q)$ radical functions are hereditary. Moreover we have

THEOREM 8. *If* $(p; q)$ *is semiprime, then it is hereditary.*

Proof. Let I be an ideal of R. For any radical function ρ we have $\rho I \subseteqq I \cap \rho R$ [3, p. 125]. Now $r \in I \cap (p(x)Rq(x))$ implies $r \in I$ and $r = p(r)sq(r)$ for some $s \in (p(x)Rq(x))$. Since the constant terms of $p(x)$ and $q(x)$ are 1 we have $s \in I$ and $I \cap (p(x)Rq(x)) \subseteqq (p(x)Iq(x))$.

It is easy to see that if the polynomial $p(x)q(x)$ has x^2 as a factor, then $(p; q)$ is also hereditary. Thus the radicals of von Neumann regularity and strong regularity are hereditary. The radical given by (xR) is not hereditary for if R is the ring of integers modulo 4 and I is the ideal $\{0, 2\}$, then $(xI) = 0$ while $I \cap (xR) = I \cap R = I$.

4. $(p; q)$ radicals of matrix rings. Let R_n denote the ring of all $n \times n$ matrices whose elements are taken from the ring R. We shall show that for all $(p; q)$ radicals and all rings R the inclusion

$(p(x)R_nq(x)) \subseteqq (p(x)Rq(x))_n$ holds while for linear semiprime $(p; q)$ radicals we have equality. D. M. Morris has shown

LEMMA 9. *If* $p(x) = \pm 1$ *or* $p(x) = \pm x$, *then* $(p(x)\boldsymbol{Z}) = \boldsymbol{Z}$; *otherwise* $(p(x)\boldsymbol{Z}) = 0$

Proof. Clearly $(p(x)\boldsymbol{Z}) = \boldsymbol{Z}$ when $p(x) = \pm 1$ or $p(x) = \pm x$. Suppose that $(p(x)\boldsymbol{Z}) \neq 0$ and that $p(x) \neq \pm 1$. Then $(p(x)\boldsymbol{Z}) = (r)$ where (r) is the ideal generated by some positive integer r. Let m be any prime not dividing r. Since $mr \in (p(x)\boldsymbol{Z})$ we have $mr = p(mr)m'r$ for some $m' \in \boldsymbol{Z}$. Since $p(x) \neq \pm 1$ we must have $p(mr) = \pm m$ for infinitely many primes m. It follows that $p(x) = \pm x$.

COROLLARY 5. $(1\boldsymbol{Z}1) = (x\boldsymbol{Z}) = (\boldsymbol{Z}x) = \boldsymbol{Z}$ *and* $(p(x)\boldsymbol{Z}q(x)) = 0$ *for all other choices of* $p(x)$ *and* $q(x)$.

Proof. Clearly $(x\boldsymbol{Z}x) = 0$ and since $(p(x)\boldsymbol{Z}q(x)) \subseteqq (p(x)\boldsymbol{Z})$, the corollary is established.

LEMMA 10. *Let* ρ *be any radical function such that* $\rho\boldsymbol{Z} = 0$. *Then any ring* R *can be embedded in a ring* S *with unity such that* $\rho R = \rho S$.

Proof. Let ϕ be the usual embedding of a ring R into the ring S with unity and identify R with ϕR, [6, p. 8]. Then $S/R \cong \boldsymbol{Z}$, which implies that $\rho(S/R) = 0$. Therefore $\rho S \subseteqq R$ and $\rho S \subseteqq \rho R$. But since R is an ideal of S we always have $\rho R \subseteqq \rho S$, [3, p. 124]. Therefore $\rho R = \rho S$.

LEMMA 11. *Let* ρ *be any radical function satisfying* (i) $\rho\boldsymbol{Z} = 0$ *and* (ii) *if* S *is a ring with unity, then* $\rho(S_n) \subseteqq (\rho S)_n$. *Then* $\rho(R_n) \subseteqq (\rho R)_n$ *for all rings* R.

Proof. By Lemma 10 we can embed R as an ideal in a ring S with unity such that $\rho R = \rho S$. Therefore we have $\rho(R_n) \subseteqq \rho(S_n) \subseteqq (\rho S)_n = (\rho R)_n$.

THEOREM 9. $(p(x)R_nq(x)) \subseteqq (p(x)(Rq(x))_n$ *for all* $(p; q)$ *radicals.*

Proof. For the $(1; 1)$ radical equality is obvious. Now consider all other $(p; q)$ radicals except for the $(x; 1)$ and $(1; x)$ radicals. By Corollary 5, $(p(x)\boldsymbol{Z}q(x)) = 0$. If S has unity, then $(p(x)S_nq(x)) = I_n$ for some ideal I of S [6]. If $r \in I$, then $rE_{11} \in I_n$ and

$$rE_{11} = p(rE_{11})Mq(rE_{11}) = p(r)m_{11}q(r)E_{11}$$

where $M \in I_n$, $m_{11} \in M$ and E_{11} is the $n \times n$ matrix $|e_{ij}|$ where $e_{11} = 1$, $e_{ij} = 0$ otherwise. Therefore $r = p(r)mq(r)$, for $m \in I$, which implies that $I \subseteqq (p(x)Sq(x))$ and $I_n = (p(x)S_nq(x)) \subseteqq (p(x)Sq(x))_n$. Now Lemma 11 yields $(p(x)R_nq(x)) \subseteqq (p(x)Rq(x))_n$ for all $(p; q)$ radicals except the $(x; 1)$ and $(1; x)$ radicals. To show $(xR_n) \subseteqq (xR)_n$, let $A \in (xR_n)$. Then there exists a $B \in (xR_n)$ such that $A = AB$, where $A = |a_{ij}|$ and $B = |b_{ij}|$. Let A_1 denote the product matrix AC of (xR_n) where $C = |c_{ij}|$, $c_{i1} = b_{i1}$, $c_{ij} = 0$ for $j > 1$, that is, $A_1 = |a'_{ij}|$ where $a'_{i1} = a_{i1}$ and $a'_{ij} = 0$ for $j > 1$. $A_1 \in (xR_n)$ implies that $A_1 = A_1D$ or $a'_{i1} = a'_{i1}d_{11}$ where $d_{11} \in D$, $D \in (xR_n)$. Again, there is a matrix $D_1 \in (xR_n)$, $D_1 = |d'_{ij}|$, where $d'_{i1} = d_{i1}$ and $d'_{ij} = 0$ for $j > 1$. Therefore $D_1 = D_1F$ for $F \in (xR_n)$ and $d_{11} = d_{11}f_{11}$ where f_{11} is an element of F. Now for $G = |g_{ij}|$ where $g_{11} = d_{11}$ and $g_{ij} = 0$ otherwise, we have $G \in (xR_n)$ because $G = GF$. If we let $J = \{r \in R \,||r_{ij}| \in (xR_n), r_{11} = r, r_{ij} = 0 \text{ otherwise}\}$, then J is an ideal of R. It follows that for all $r \in J$ there exists an $s \in J$ such that $r = rs$. Therefore $J \subseteqq (xR)$ and $d_{11} \in (xR)$. But $d_{11} \in (xR)$ implies that $a_{i1} \in (xR)$ for $i = 1, 2, \cdots, n$. Similarly, $a_{ij} \in (xR)$ for

$$i = 1, 2, \cdots, n, j = 2, 3, \cdots, n .$$

Thus $A \in (xR)_n$ and $(xR_n) \subseteqq (xR)_n$. Similarly $(R_nx) \subseteqq (Rx)_n$.

R. L. Snider gave the following example to show that the inclusion $\rho(R_n) \subseteqq \rho(R)_n$ is not true for all radicals. Let σR be the upper radical determined by declaring $GF(2)$ to be semisimple (In [3, p. 6] let $M = \{GF(2)\}$). Then since the ring of 2×2 matrices over $GF(2)$ cannot be mapped homomorphically onto $GF(2)$, $(GF(2))_2$ is not semisimple.

Finally we show that for all linear semiprime $(p; q)$ radicals the opposite inclusion holds; hence we have equality.

LEMMA 12. *The sum of two $(ax + 1; 1)$-regular right ideals of the ring R is an $(ax + 1; 1)$-regular right ideal of R.*

Proof. Let I and J be $(ax + 1; 1)$-regular right ideals of R and $r \in I$, $s \in J$. Then there exists an $r' \in I$ such that $r = (ar + 1)r'$. Now $s - asr' \in J$, which implies that there exists an $s' \in J$ such that $s - asr' = (a(s - asr') + 1)s'$. It is easy to see that $r + s = (a(r + s) + 1)(r' - ar's' + s')$, hence $I + J$ is an $(ax + 1; 1)$-regular right ideal.

COROLLARY 7. *The sum of all $(ax + 1; 1)$-regular right ideals of a ring R is an $(ax + 1; 1)$-regular right ideal of R.*

LEMMA 13. *The sum K of all $(ax + 1; 1)$-regular right ideals of a ring R is a two-sided ideal of R. Therefore $K \subseteqq ((ax + 1)R)$.*

Proof. [cf. 3, p. 93] Let $s \in K$ and $r \in R$. Then $sr \in K$ implies that $sr = (asr + 1)s'$ for some $s' \in K$. It is easy to see that $rs = (ars + 1)(-ars's + rs)$, hence rs is $(ax + 1; 1)$-regular. For $m \in \boldsymbol{Z}$, $t \in R$ we have $sm + st \in K$. Since from above $r(sm + st)$ must be $(ax + 1; 1)$-regular, $rs\boldsymbol{Z} + rsR$, the right ideal generated by rs, is an $(ax + 1; 1)$-regular right ideal and we have $rs\boldsymbol{Z} + rsR \subseteqq K$, therefore $rs \in K$. Since K is now an $(ax + 1; 1)$-regular ideal, $K \subseteqq ((ax + 1)R)$.

THEOREM 10. *If $(p; q)$ is a linear semiprime radical, then for all rings R, $(p(x)R_n q(x)) = (p(x)Rq(x))_n$.*

Proof. [cf. 4, p. 11] We only need to show that $((ax + 1)R)_n \subseteqq ((ax + 1)R_n)$ for all positive integers a. Let k be a fixed positive integer, $k \leqq n$, and $|r_{ij}| \in ((ax + 1)R_n)$ where $r_{ij} = 0$ for $i \neq k$. Then by Lemma 4, $r_{kk} \in (ar_{kk} + 1)R$ implies that $R = (ar_{kk} + 1)R$. Therefore for each r_{kj} there exists an s_{kj} such that $r_{kj} = (ar_{kk} + 1)s_{kj}$ for $j = 1, 2, \cdots, n$. Thus $|r_{ij}| = (a|r_{ij}| + 1)|s_{ij}|$ where $s_{ij} = 0$ for $i \neq k$. Hence the right ideal P_k of $n \times n$ matrices having elements of $((ax + 1)R)$ in the kth row and zeros elsewhere is an $(ax + 1; 1)$-regular right ideal, thus $P_k \subseteqq ((ax + 1)R_n)$. Since $((ax + 1)R)_n$ is the sum of the P_k, $k = 1, 2, \cdots, n$, we have $((ax + 1)R_n)$.

If R is a field we have $0 = (x^2 R_n) \subsetneqq (x^2 R)_n = R_n$, therefore the radical of strong regularity shows that we cannot have the matrix equality for all $(p; q)$ radicals.

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