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A CLASS OF COUNTEREXAMPLES ON PERMANENTS

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A method is described to construct a strictly positive doubly stochastic matrix A of order $3k$ such that per$(xE - A)$ has at least k real zeros.

Let A be an irreducible doubly stochastic matrix. de Oliveira conjectured [1] that $per(xE - A)$ has no real zeros or exactly one real zero depending on the parity of the order of A. We prove that the number of real zeros can be arbitrarily large for matrices of sufficiently large order, even or odd. We denote by E the identity matrix, always assuming its order to be such that the formulae make sense.

LEMMA. *There exist an infinite sequence $A_1, A_2, \cdots$ of doubly stochastic matrices of order 3 and a strictly incresing sequence of real numbers $x_1, x_2, \cdots$ such that $per(x_tE - A_i) < 0$, for $t \leqq i$, $per(x_tE - A_i) > 0$, for $t > i$, all i.*

Proof. Let $0 < d < 1$,

$$A_d = \begin{bmatrix} 0 & d & 1-d \\ 1-d & 0 & d \\ d & 1-d & 0 \end{bmatrix} \text{ and } P_d(x) = per(xE - A_d) .$$

Then $P_d(x) = x^3 + 3d(1 - d)(x + 1) - 1$, and we have $P_d(-1) = -2 < 0$, $P_d(1) = 6d(1 - d) > 0$ and $P_d'(x) = 3x^2 + 3d(1 - d) > 0$. Hence P_d is strictly increasing and has precisely one real zero which lies in the interval $(-1, 1)$. To each infinite sequence $\{d_i\}$ $(0 < d_i < 1)$ we associate the sequence $\{y_i\}$ where y_i(real) is defined by $P_{d_i}(y_i) = 0$. Since $\lim_{d \to 1} P_d(x) = x^3 - 1$, there exists a strictly increasing sequence $d_1 < d_2 < \cdots$ such that the associated sequence of the y_i is strictly increasing. Setting $x_1 = -1$, $x_{i+1} = (y_i + y_{i+1})/2$ and $A_i = A_{d_i}$ our lemma follows.

THEOREM. *For arbitrary positive integer k there exists a strictly positive doubly stochastic matrix A of order $3k$ such that $per(xE - A)$ has at least k distinct real zeros.*

Proof. Let us consider a pair of sequences $\{A_n\}$ and $\{x_n\}$ of our lemma and let B_k be the direct sum of $A_1, A_2, \cdots, A_k$. Then sgn $[\text{per}(x_iE - B_k)] = (-1)^{k-i+1}$ for $i \leqq k$. Let $\varepsilon > 0$ and $B_{k,\varepsilon} = (1 + 3k\varepsilon)^{-1}$

$[B_k + \varepsilon J]$ where J is a matrix of ones. Since $\lim_{\varepsilon \to 0} B_{k,\varepsilon} = B_k$ there exists a positive ε_0 such that

$$\operatorname{sgn}[\operatorname{per}(x_i E - B_{k,\varepsilon_0})] = \operatorname{sgn}[\operatorname{per}(x_i E - B_k)] = (-1)^{k-i+1}$$

for $i = 1, 2, \cdots, k + 1$. Then $A = B_{k,\varepsilon_0}$ satisfies the requirements of the theorem.

Strictly positive matrices being irreducible, the above proof provides a method for actually constructing counterexamples for de Oliveira's conjecture. Choosing ε_0 sufficiently small, one can even guarantee that $per(xE - A)$ has precisely k real zeros.

Reference

1. G. N. de Oliveira, *A conjecture and some problems on permanents*, Pacific J. **32** (1970), 495-499.

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