PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 3, 1971

HOW TO RECOGNIZE HOMEOMORPHISMS AND ISOMETRIES

CARLOS R. BORGES

We study necessary and sufficient conditions for a given function $f; X \to X$ to be one-to-one, or a homeomorphism or a topological isometry[1], by analyzing the family $\{f^1, f^2, \cdots\}$ of its composition iterates.

In § 1, 2 and 3 we develop the key results. (See Theorems 2.1 and 2.3).

In § 4 we obtain a variety of applications, most of which are known, by various different techniques. They all follow easily from our key results:

(a) Conditions which imply that locally nonexpansive maps (in metric spaces or uniform spaces) become local isometries (see Theorem 4.2, Corollary 4.3 and Theorem 4.5, and also [2], [5] and [6]).

(b) Conditions which imply that a given map on a *metrizable* space is a *topological* isometry[1] (see Theorems 4.6 and 4.7, and also [8]).

(c) Some results on compact commutative groups of homeomorphisms (or topological isometries) of X onto X (see Theorems 4.11 and 4.12, and also [4], [10] and [11]).

1. **Auxiliary results.** For each space X we let id_X denote the identity function on X and, for each continuous function $f: X \to X$, we let

(a) $f^1 = f, f^n = f \circ f^{n-1}$ for $n > 1$

(b) $P = \bigcap_{n=1}^{\infty} f^n(X)$

(c) $\Gamma(f) = \{f^n \mid n = 1, 2, \cdots\}^-$, the closure being with respect to the co. (i.e. compact-open) topology on the function space X^X.

PROPOSITION 1.1. *If X is locally compact and Hausdorff then X^X (and therefore also $\Gamma(f)$) is a topological semigroup with respect to composition.*

Proof. Immediate from Theorem 2.2 (p. 259) of Dugundji [3].

LEMMA 1.2. *Suppose X is compact Hausdorff and $\{f^1, f^2, \cdots\}$ is*

[1] We wish to thank Professors J. Dugundji, W. Comfort and A. D. Wallace for bringing to our attention references [6], [2] and [10] respectively. They strongly motivated us to sharp improvements of our original results.

evenly continuous. Then $\Gamma(f)$ is compact.

Proof. By Theorem 19 (p. 235) of [7], $\Gamma(f)$ is evenly continuous since $\Gamma(f)$ is contained in the pointwise (topology) closure of $\{f^1, f^2, \cdots\}$. By Ascoli's Theorem (p. 236 of [7]), $\Gamma(f)$ is compact.

LEMMA 1.3 *Suppose X is compact Hausdorff and $\{f^1, f^2, \cdots\}$ is evenly continuous. Then $\Gamma(f)$ has an idempotent e* (i.e. $e^2 = e$) *such that $e(X) = P$ and $e/P = id_P$.*

Proof. By our Lemma 1.2 and Theorem 3.1 of [9], $\Gamma(f)$ has an idempotent e (the unit of the maximal nonempty subgroup $K(f)$ of $\Gamma(f)$).

We show $P \subset e(X)$: Let $y \in P$. Then there exists $x_n \in X$ such that $f^n(x_n) = y$, for each n. Pick a subnet $\{(f^\nu, x_\nu)\}$ of $\{(f^n, x_n)\}$ such that $\lim_\nu f^\nu = e$. Then there exists $w \in X$ and a subnet $\{(f^\lambda, x_\lambda)\}$ of the net $\{(f^\nu, x_\nu)\}$ such that $\lim_\lambda (f^\lambda, x_\lambda) = (e, w)$, because $\Gamma(f) \times X$ is compact. Consequently $e(w) = \lim_\lambda f^\lambda(x_\lambda) = \lim_\lambda y = y$, because of Theorems 4.71 and 2.5 of [1].

We show $e(X) \subset P$: Let $x \in X$ and pick subnet $\{f^\nu\}$ of $\{f^n\}$ such that $\lim f^\nu = e$. Then $e(x) = \lim_\nu f^\nu(x)$ and, since each $f^n(X)$ is a closed subset of X (because X is compact), $e(x) \in P$.

Clearly $e|P = id_P$ since $e(e(x)) = e(x)$ for each $x \in X$.

It appears that our main results become more meaningful when stated in terms of a generalization of the concept of "an almost periodic[2] function" introduced by Whyburn [12], especially because of Lemma 1.5 below.

DEFINITION 1.4. Let $f: X \to X$ be a function. Then f is said to be *net periodic* if there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = id_X$ (in X^X).

Clearly net periodicity is equivalent to almost periodicity whenever X^X is metrizable.

LEMMA 1.5. *Let $f: X \to X$ be net periodic. Then f is one-to-one.*

Proof. Suppose $f(x) = f(y)$. Then $f^n(x) = f^n(y)$. for all n.Pick subnet $\{f^\nu\}$ of $\{f^n\}$ such that $\lim_\nu f^\nu = id_X$. Then $id_X(x) = \lim_\nu f^\nu(x) = \lim_\nu f^\nu(y) = id_X(y)$ implies that $x = y$. This does the trick.

[2] For any metric space (X, ρ) a function $f: X \to X$ is said to be *almost periodic* (see [12]) if for each $\varepsilon > 0$ there exists a positive integer n such that $\rho(f^n(x), x) < \varepsilon$ for all $x \in X$. This is clearly equivalent to saying that there exists a subsequence $\{f^{n_i}\}$ of $\{f^n\}$ which converges uniformly to the identity function of X.

LEMMA 1.6. *Let X be compact Hausdorff and $f: X \to X$ a continuous function. If f is net periodic then f is an onto homeomorphism.*

Proof. Because of Lemma 1.5 we only need prove that f is onto: Suppose not. Then $X - f(X)$ is an open nonempty subset of X. Let $\{f^\nu\}$ be a subnet of $\{f^n\}$ such that $\lim f^\nu = id_X$. Then for any $a \in X - f(X)$, $\lim_\nu f^\nu(a) = a$ while $f^\nu(X) \subset f(X)$ (i.e. $f^\nu(X) \cap [X - f(X)] = \emptyset$) for all ν. This is a contradiction.

2. Main results.

THEOREM 2.1 *Let X be compact Hausdorff and $f: X \to X$ a function. If $\{f^1, f^2, \cdots\}$ is evenly continuous then $f|P$ is net periodic* (and therefore a homeomorphism).

Proof. By Lemma 1.3, there exists an idempotent $e \in \Gamma(f)$ such that $e|P = id_P$. Consequently there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = e$ (even if $e = f^n$ for some n) which implies that $\lim_\nu f^\nu|P = e|P = id_P$. Since $f(P) = P$ (because $f^n(X) \supset f^{n+1}(X) \supset P$ for all n) we get that $f^\nu|P = (f|P)^\nu$. Therefore $\lim_\nu (f|P)^\nu = id_P$ and $f|P$ is net periodic.

COROLLARY 2.2. *Let X be compact Hausdorff and $f: X \to X$ an onto function. If $\{f^1, f^2, \cdots\}$ is evenly continuous then f is net periodic* (and therefore a homeomorphism).

Since the converse of Theorem 2.1 is false (for example, let $f: I \to I$ be defined by $f(t) = t^2$, where I denotes the closed unit interval), the following seems worthy of mention.

THEOREM 2.3. *Let X be any space and $f: X \to Y$ a continuous function. Then f is one-to-one if and only if $\Gamma(f)$ contains a one-to-one function.*

Proof. The "only if" part is clear. To prove the "if" part let $e: X \to X$ be any one-to-one function such that $e \in L(f)$. Assume $f(x) = f(y)$. Then $f^n(x) = f^n(y)$ for all n and one easily sees that $e(x) = e(y)$. (This is clear if $e = f^n$ for some n. Otherwise there exists a subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = e$. Therefore $e(x) = \lim_\nu f^\nu(x) = \lim_\nu f^\nu(y) = e(y)$). This shows that f is one-to-one.)

3. Generalizations. With some additional restrictions on f, Theorem 2.1 remains valid for a larger class of spaces.

Throughout this section we *let* E be a *noncompact, locally compact, space,* and $\hat{E} = E \cup \{\infty\}$ be its one-point compactification. Also for each onto map $f: E \to E$ we consider the three conditions:

(A) For each sequence $\{x_n\}$ in E, $\lim_n x_n = x \neq \infty$ and $\lim_n f^{i_n}(x) = \infty$ implies that $\lim_n f^{i_n}(x_u) = \infty$,

(B) For each sequence $\{x_n\}$ in E, $\lim_n x_n = \infty$ implies that $\lim_n f^{i_n}(x_n) = \infty$,

(C) For each sequence $\{x_n\}$ in E, $\lim_n x_n = \infty$ implies that $\lim_n f(x_n) = \infty$.

In Example 5.1, it is shown that conditions (A) and (C) do not imply (B).

LEMMA 3.1. *Condition* (C) *is satisfied if f is either* (a) *uniformly continuous with respect to some compatible metric on* $\hat{E}$ (assuming E is separable metrizable), *or* (b) *a homeomorphism.*

Proof. (a) Since $\hat{E}$ is completely metrizable we can continuously extend f to $\hat{f}: \hat{E} \to \hat{E}$. Since E is not compact, $\hat{f}(\infty) = \infty$. This shows that f satisfies (C).

(b) Suppose $\lim_\nu x_\nu = \infty$ but $\lim_\nu f(x_\nu) \neq \infty$. Then there exists a subsequence $\{x_\sigma\}$ of $\{x_\nu\}$ such that $\lim_\sigma x_\sigma = \infty$ $\lim_\sigma f(x_\sigma) = y$ for some $y \in E$. Then $f^{-1}(y) = \lim_\sigma f^{-1} f(x_\sigma) = \lim_\sigma x_\sigma = \infty$, which is impossible.

LEMMA 3.2. *Condition (B) implies (C).* (The converse is false-see Example 5.1.)

Proof. Simply let all $i_n = 1$ in (B).

LEMMA 3.3. *If E is metrizable and $f: E \to E$ is nonexpansive* (with respect to some compatible metric d on E) *then f satisfies conditions* (A) *and* (C).

Proof. (That f satisfies (A) is essentially proved in the proof of Theorem 1.1 of [8]. We present a simpler argument.) Suppose that $\lim_n x_n = x \in E$, $\lim_n f^{i_n}(x) = \infty$ but $\lim_n f^{i_n}(x_n) \neq \infty$. Then there exists compact $C \subset E$, $z \in C$ and subsequence $\{\beta\}$ of the integers such that $f^{i_\beta}(x_\beta) \in C$ and $\lim_\beta f^{i_\beta}(x_\beta) = z$. Since $d(f^{i_\beta}(x_\beta), f^{i_\beta}(x)) \leqq d(x_\beta, x)$, we get that $\lim_\beta f^{i_\beta}(x_\beta) = z$, a contradiction. Similarly, one can easily see that f satisfies (C).

THEOREM 3.4. *Let $f: E \to E$ be an onto function such that*

$\{f^1, f^2, \cdots\}$ *is evenly continuous and* f *satisfies conditions* (A) *and* (B). *Then* f *extends continuously to an onto function* $\hat{f}: \hat{E} \to \hat{E}$ *such that* $\{f^1, f^2, \cdots\}$ *is evenly continuous.* (Therefore, f and $\hat{f}$ are net periodic.)

Proof. Since f satisfies (C), by Lemma 3.2, we can extend f to $\hat{f}: \hat{E} \to \hat{E}$ continuously, by letting $\hat{f}(\infty) = \infty$. Because of (A) and (B) it is easily seen that $\{\hat{f}^1, \hat{f}^2, \cdots\}$ is evenly continuous. Therefore, by Theorem 2.1, $\hat{f}$ is net periodic. This completes the proof.

Example 5.1 shows that if f does not satisfy (B) then f is not necessarily net periodic.

4. Applications.

DEFINITION 4.1. Let (X, ρ) be a metric space. A function $f: X \to X$ is said to be *locally nonexpansive* if each $p \in X$ has a neighborhood N_p such that $\rho(f(x), f(y)) \leqq \rho(x, y)$ for all $x, y \in N_p$. If $N_n = X$ then f is said to be *nonexpansive*. If, for come $\varepsilon > 0$, $\rho(x, y) < \varepsilon$ implies that $\rho(f(x), f(y)) \leqq \rho(x, y)$, then f is said to be *ε-nonexpansive*. Replacing "$\leqq$" by "$=$" above, we get the definitions of *local isometry* and of *ε-isometry*.

THEOREM 4.2. *Let (X, ρ) be a compact metric space and $f: X \to X$ a locally nonexpansive onto function. Then f is almost periodic*[2] *and therefore a local isometry.*

Proof. By compactness of X, there exists $\varepsilon > 0$ such that f is ε-nonexpansive. By the net characterization of even continuity (cf. exercise L on p. 241 of [7]) it is easy to prove that $\{f^1, f^2, \cdots\}$ is evenly continuous (for each net (in this case, sequences suffice)

$$\{(f^\nu, x_\nu)\} \subset \{f^1, f^2, \cdots\} \times X, \lim_\nu x_\nu = x \quad \text{and} \quad \lim_\nu f^\nu(x) = y$$

implies that

$$\rho\,(f^\nu(x_\nu), y) \leqq \rho\,(f^\nu(x_\nu), f^\nu(x)) + \rho\,(f^\nu(x), y) \leqq \rho\,(x_\nu, x) + \rho\,(f^\nu(x), y)$$

for all ν with $\rho\,(x_\nu, x) < \varepsilon$, since each f^n is also ε-nonexpansive; therefore $\lim_\nu f^\nu(x_\nu) = y$, which does the trick).

Since f is onto, we get that $P = X$ and therefore that f is almost periodic, by Theorem 2.1 (and footnote 2).

It is now easily seen that f is a local isometry (pick subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = id_X$. Since

$$\rho(x, y) \leqq \rho(x, f^\nu(x)) + \rho(f^\nu(x), f^\nu(y)) + \rho(f^\nu(y), y)$$

we get that $\rho(x, y) \leqq \lim_\nu \rho(f^\nu(x), f^\nu(y))$ for all $x, y \in X$. If

$$\rho(f(a), f^\nu(b)) < \rho(a, b) < \varepsilon,$$

for some $a, b \in X$, then

$$\rho(f^\nu(a), f^\nu(b)) \leqq \rho(f(a), f(b)) < \rho(a, b), \qquad \text{for all } \nu,$$

which implies that $\lim_\nu \rho(f^\nu(a), f^\nu(b)) < \rho(a, b)$, a contradiction). This completes the proof.

COROLLARY 4.3 (Satze Ib of [6] and Theorem 1 of [5]). *Let (X, ρ) be a totally bounded metric space and $f: X \to X$ an ε-nonexpansive map such that $f(X)$ is dense in X. Then f is an ε-isometry and X is dense in $f(X)$.*

Proof. Let $\hat{X}$ be the usual metric completion (in terms of Cauchy sequences—see Theorem 27 on p. 196 of [7]). Since $f(X)$ is dense in X then $\hat{X}$ is also the completion of $f(X)$. Clearly $\hat{X}$ is compact metrizable by a metric $\hat{\rho}$ which extends ρ, and f is extendable to a continuous function (because f is uniformly continuous) $\hat{f}: \hat{X} \to \hat{X}$ such that $\hat{f}$ is an onto map (because of compactness). It is also easily seen that $\hat{f}$ is ε-nonexpansive (Let $x, y \in \hat{X}$ with $\hat{\rho}(x, y) < \varepsilon$. Pick nets (sequences suffice) $\{x_\nu\}$, $\{y_\nu\}$ in X with $\lim_\nu x_\nu = x$, $\lim_\nu y_\nu = y$ and $p(x_\nu, y_\nu) < \varepsilon$ for all ν (this can be done because of the definition of $\hat{\rho}$). Then $\hat{\rho}(\hat{f}(x), \hat{f}(y)) = \lim_\nu \rho(f(x_\nu), f(y_\nu)) \leqq \lim_\nu \rho(x_\nu, y_\nu) = \hat{\rho}(x, y)$, because $\rho(x, y) < \varepsilon$). Theorem 4.2 completes the proof.

DEFINITION 4.4. Let $(X, \mathscr{U})$ be a uniform space and $\theta = \{\rho_X\}_{\lambda \in \Lambda}$ a gage (see p. 188 of [7]) for $\mathscr{U}$. A function $f: X \to X$ is said to be *ε-nonexpansive with respect to θ* ($\varepsilon < 0$) if $\rho_\lambda(x, y) < \varepsilon$ implies that $\rho_\lambda(f(x), f(y)) \leqq \rho_\lambda(x, y)$, for each $\lambda \in \Lambda$. If f is ε-nonexpansive with respect to θ for all $\varepsilon > 0$ then f is said to be *nonexpansive with respect to θ*. Replacing "$<$" by "$=$" above we get the definitions of *ε-isometry* and of *isometry with respect to θ*.

THEOREM 4.5. *Let $(X, \mathscr{U})$ be a uniform compact Hausdorff space and $f: X \to X$ an onto ε-nonexpansive map with respect to some gage $\theta = \{\rho_\lambda\}_{\lambda \in \Lambda}$ for $\mathscr{U}$. Then f is net periodic and an ε-isometry with respect to θ.*

Proof. All the statements in the proof of Theorem 4.2 which involve ρ remain valid when ρ is replaced by any $\rho_\lambda \in \theta$. Therefore, by use of Theorem 19 on p. 189 of [7], one immediately gets that $\{f^1, f^2, \cdots\}$ is evenly continuous, and f is net periodic and and an ε-isometry with respect to θ. This completes the proof.

Brown and Comfort [2] have obtained a similar result for totally bounded Hausdorff uniform spaces.

With respect to topological isometries we get the following results, of which the first shows that the hypothesis that f be a homeomorphism in Corollary 1.2 of [8] is superfluous; the second complements Theorem 1.1 of [8].

THEOREM 4.6. *Let $f: M \to M$ be a function from the compact metrizable space M onto itself. Then f is a topological isometry if and only if $\{f^1, f^2, \cdots\}$ is evenly continuous.*

Proof. Immediate from Corollary 1.2 of [8] and our Theorem 2.1.

THEOREM 4.7. *Let $f: E \to E$ be a function from the separable locally compact metrizable space E onto itself. Then f is the restriction of a topological isometry $\hat{f}: \hat{E} \to \hat{E}$ if and only if $\{f^1, f^2, \cdots\}$ is evenly continuous and f satisfies conditions* (A) and (B).

Proof. Immediate from Theorems 4.6 and 3.4.

Finally we get some results on transformation groups. But first we need the following three lemmas.

LEMMA 4.8. *Let $f: X \to X$ be a function from a compact Hausdorff space X onto itself. If $\{f^1, f^2, \cdots\}$ is evenly continuous then $\{f^{-1}, f^{-2}, \cdots\} \subset \Gamma(f)$ and it is evenly continuous.*

Proof. By Corollary 2.2, f^{-1} is a well-defined function and there exists subnet $\{f^\nu\}$ of $\{f^n\}$ with $\lim_\nu f^\nu = 1_X$. Therefore

$$\lim_\nu f^{-1} f^\nu = \lim_\nu f^{\nu-1} = f^{-1} 1_X = f^{-1} \in \Gamma(f) .$$

Similarly one proves that $\{f^{-1}, f^{-2} \cdots\} \subset \Gamma(f)$. Therefore $\{f^{-1}, f^{-2} \cdots\}$ is evenly continuous, since $\Gamma(f)$ is compact by Lemma 1.2. This completes the proof.

LEMMA 4.9. *Let $f: X \to X$ be a function from a compact Hausdorff space X onto itself. If $\{f^1, f^2, \cdots\}$ is evenly continuous then, $\lim_\nu f^\nu(x) = y$ if and only if $\lim_\nu f^{-\nu}(y) = x$, for any given net $\{f^\nu\}$.*

Proof. (Essentially contained in the proof of Corollary 1.3 of [8].)

Let $\lim_\nu f^\nu(x) = y$ and assume $\lim_\nu f^{-\nu}(y) \neq x$. Then there exists a neighborhood $\mathscr{U}$ of x, some $z \in X - \mathscr{U}$, and a subnet $\{f^{-\alpha}(y)\}$ of $\{f^{-\nu}(y)\}$ with $\lim_\alpha f^{-\alpha}(y) = z$. Since $\lim_\alpha f^\alpha(x) = y$, by the net characterization of even continuity (of $\{f^{-1}, f^{-2}, \cdots\}$, by Lemma 4.8) we then get that

$$z = \lim_\alpha f^{-\alpha} f^\alpha(x) = \lim_\alpha x = x ,$$

a contradiction. This does the trick.

LEMMA 4.10. *Let $f: X \to X$ be a function from the compact Hausdorff space X onto itself. If $\{f^1, f^2, \cdots\}$ is evenly continuous, then any $g \in \Gamma(f)$ is an onto homeomorpism of X.*

Proof. Pick net $\{f^\nu\}$ in the (evenly continuous family) $\{f^1, f^2, \cdots\}$ with $\lim_\nu f^\nu = g$ and assume $g(p) = \lim_\nu f^\nu(p) = \lim_\nu f^\nu(q) = g(q)$. Then, by Lemma 4.9, $p = \lim_\nu f^{-\nu}(g(p)) = \lim_\nu f^{-\nu}(g(q)) = q$, and this shows that g is one-to-one.

For any $p \in X$ and each ν, there exists $p_\nu \in X$ with $f^\nu(p_\nu) = p$. Pick subnet $\{f^\sigma\}$ of $\{f^\nu\}$ and $w, z \in X$ with $\lim_\sigma p_\sigma = z$ and $\lim_\sigma f^\sigma(z) = w$. Then, by even continuity, $w = \lim_\sigma f^\sigma(p_\sigma) = \lim_\sigma p = p$.

THEOREM 4.11. *Let $f: X \to X$ be a function from the compact Hausdorff space X onto itself. If $\{f^1, f^2, \cdots\}$ is evenly continuous then $\Gamma(f)$ is a compact commutative group of homeomorphisms of X onto X.*

Proof. Immediate from Lemma 4.10. ($\Gamma(f)$ is commutative because it is the closure of a commutative group.)

This result is similar to Theorem 2 of Wallace [10] and a result of Wallace [11].

THEOREM 4.12. *Let $f: M \to M$ be a function from the compact metrizable space M onto itself. If $\{f^1, f^2, \cdots\}$ is evenly continuous then there exists a compatible metric d for M such that $\Gamma(f)$ is a commutative group of d-isometries acting on M.*

Proof. By Theorem 4.6, f it a d-isometry for some compatible metric d for M. It is easily seen that also each $g \in \Gamma(f)$ is a d-isometry.

5. Counterexamples. The following example illustrates that most of the hypothesis in our main results are not superfluous.

EXAMPLE 5.1. There exists a nonexpansive onto function f: $E^1 \to E^1$ (E^1 is the real line) such that

(a) f is not net periodic and dose not satisfy (B),

(b) of course, f satisfies conditions (A) and (C), and $\{f^1, f^2, \cdots\}$ is evenly continuous.

Proof. Let $f: E^1 \to E^1$ be defined by

$$f(x) = x \qquad \text{for each } x < 0$$

$$f(x) = 0 \qquad \text{for all } 0 \leqq x \leqq 1$$

$$f(x) = x - 1 \qquad \text{for all } x > 1 .$$

Clearly f is onto and nonexpansive. (Note that $f^n(x) = x$ for $x < 0$, $f^n(x) = 0$ for $0 \leqq x \leqq n$ and $f^n(x) = x\text{-}n$ $x > n$. Therefore $\lim_n n = \infty$ but $\lim_n f^n(n) = 0$).

REFERENCES

1. R. Arens and J. Dugundji, *Topologies for function spaces*. Pacific J. Math., **1** (1951), 53-1.
2. T. Brown and W. Comfort, *New method for expansion and contraction maps in uniform spaces*, Proc. Amer. Math. Soc., **11** (1960), 483-486.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.
4. A. Edrei, *On iteration of mappings of a metric space onto itself*, J. London Math. Soc., **26** (1951), 96-103.
5. ———, *On mappings which do not increase small distances*, Proc. London Math. Soc., **2** (1952), 272-278.
6. H. Freudenthal and W. Hurewicz, *Dehnungen, Verkurzungen, Isometrien*, Fund. Math., **26** (1936), 120-122.
7. J. L. Kelley, *General Topology*, Van Nostrand, 1955.
8. M. M. Marjanovic, *On topological isometrics*, Indag. Math. **31** (1960), 184-189.
9. A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc., **61** (1955), 95-112.
10. ———, *The gebietstreue in semigroups*, Indag. Math., **18** (1956), 271-274.
11. ———, *Inverses in Euclidean mobs*, Math. J. Okayama, Univ., **3** (1953), 1-3.
12. G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publications, Vol. **28**.

Received May 4, 1970.

UNIVERSITY OF CALIFORNIA, DAVIS