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PRIMITIVE SUBALGEBRAS OF EXCEPTIONAL LIE ALGEBRAS

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The object of this paper is to classify (up to inner automorphism) the primitive, maximal rank, reductive subalgebras of the (complex) exceptional Lie algebras. By primitive we mean that the subalgebras correspond to (possibly disconnected) maximal Lie subgroups.

In [3], the corresponding classification for the (complex) classical Lie algebras was completed, as was the classification of the nonreductive maximal rank subalgebras of all the simple Lie algebras.

Using case by case techniques and some more general results proved in § 1, we prove the following theorem:

THEOREM 0. *The primitive maximal rank, reductive subalgebras of the exceptional* (complex, simple) *Lie algebras are listed* (up to conjugacy by an inner automorphism) *in the table below. Further, all subalgebras isomorphic to one of these are conjugate by an inner automorphism.*

We note that Theorem 5.5 (p. 148) in the reductive case of Dynkin [2] is incorrect. In particular $A_3 \oplus D_5$, $A_5 \oplus A_2 \oplus A_1$, $A_7 \oplus A_1$ in E_8, $A_3^2 \oplus A_1$ in E_7 and $A_3 \oplus A_1$ in F_4 are not maximal subalgebras.

Algebra	Subalgebra	*	Reference in text
E_8	$A_1 \oplus E_7$	maximal	§ 2 case 1, (a) (i)
	A_1^8	—	§ 2 case 1, (b) (i)
	$A_2 \oplus E_6$	maximal	§ 2 case 2, (a) (i)
	A_2^4	—	§ 2 case 2, (b), (i)
	A_4^2	maximal	§ 2 case 4, (i)
	D_4^2	—	§ 2 case 5, (i)
	D_8	maximal	§ 2 case 6,
	A_8	maximal	§ 2 case 7, (i)
	T^8	Cartan subalgebra	—
E_7	$A_1 \oplus D_6$	maximal	§ 2 case 1, (a), (ii)
	$A_1^3 \oplus D_4$	—	§ 2 case 1, (b), (ii)
	A_1^7	—	§ 2 case 1, (b), (ii)
	$A_2 \oplus D_5$	maximal	§ 2 case 2, (a), (ii)
	$A_2^3 \oplus T^1$	—	§ 2 case 2, (b), (ii)
	A_7	maximal	§ 2 case 7, (ii)
	$E_6 \oplus T^1$	maximal	§ 2 case 9,
	T^7	Cartan subalgebra	—

Algebra	Subalgebra	*	Reference in text
E_6	$A_1 \oplus A_5$	maximal	§ 2 case 1, (a) (iii)
	A_2^3	maximal	§ 2 case 2, (b), (iii)
	$D_4 \oplus T^2$	—	§ 2 case 5, (iii)
	$D_5 \oplus T^1$	maximal	§ 2 case 6,
	T^6	Cartan subalgebra	—
F_4	$A_1 \oplus C_3$	maximal	§ 3 case 1, (a)
	A_2^2	maximal	§ 3 case 2
	B_4	maximal	§ 3 case 7
	D_4	algebra of longer roots	§ 3 case 8
G_2	A_1^2	maximal	§ 4 case 1
	A_2	maximal	§ 4 case 3

T^k denotes the center of the subalgebra, where k is the dimension of the center. The other superscripts refer to the number of summands of the corresponding algebra.

(See Table 12, p. 150 of [12]).

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1. Preliminaries. We now present some basic notation and a characterization of primitivity from [3].

Let p be a maximal rank subalgebra of a simple Lie algebra g. By maximal rank, we mean that there exists a Cartan subalgebra h of g which is contained in p. We fix h.

Let W be the Weyl group relative to h. p is then decomposed by h into

$$p = h \oplus \sum_{\varphi} e_\varphi$$

where the φ's are roots in g determined by h, and the e_φ's are the corresponding one dimensional root spaces. p is then uniquely determined by the roots φ for which $e_\varphi \subset p$. Let $K_p = \{\varphi \text{ a root of } g \text{ relative to } h \mid e_\varphi \subset p\}$. Define $W_p = \{\alpha \in W \mid \alpha(K_p) = K_p\}$.

PROPOSITION 1.1. *Let p be a maximal rank subalgebra of g. Then p is primitive if and only if the following holds: If l is a subalgebra of g such that $p \subset l$, and $W_p \subset W_l$, then $p = l$ or $l = g$.*

Proof. This is just Proposition 3.2 of [3].

We will use this as our definition of primitivity.

In our notation K_g is the set of all roots in g. We introduce an operation [,] on $K_g \times K_g$ which is induced from the Lie algebra structure of g. Let $\varphi, \psi \in K_g$, then

$$[\varphi, \psi] = \begin{cases} 0 & \text{if } [e_\varphi, e_\psi] = 0 \\ \varphi + \psi & \text{if } [e_\varphi, e_\psi] = e_{\varphi+\psi} . \end{cases}$$

Note that $\alpha[\varphi, \psi] = [\alpha(\varphi), \alpha(\psi)] \forall \alpha \in W$. Denote by (φ, ψ) the standard inner product on h^*, the dual space to h, given by the Cartan-Killing form.

Now let p be reductive, i.e., p is given uniquely as the direct sum of simple algebras and its center. Thus there exists non-isomorphic simple Lie subalgebras of g: $X_1, X_2, \cdots, X_r$ such that $p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T$ where T is the center of p and where $X_i^{k_i}$ denotes the direct sum of all ideals of p isomorphic to X_i, and k_i is the number of such ideals. Note that since $X_i = (h \cap X_i) \oplus \sum_\varphi e_\varphi$, where the φ are unique, K_{X_i} and W_{X_i} make sense.

LEMMA 1.2. *Let* $q_i = X_i^{k_i}$. *Then* $W_p \subset W_{q_i}$.

Proof. Let $\alpha \in W_p$. Let f be an inner automorphism of g representing α. Then $f(p) = p$. Also $f(X_i)$ is an ideal of p isomorphic to X_i. Thus $f(X_i) \subset q_i$ since all ideals of p isomorphic to X_i are in q_i. Hence $f(q_i) = q_i$ and $\alpha \in W_{q_i}$.

Let z be a subalgebra of g with Cartan subalgebra $h_z \subset h$. Assume that z is regular (in the sense of Dynkin [2]), i.e., let $K_z = \{\varphi \in K_g \mid e_\varphi \subset z\}$, then $z = h_z \oplus \sum_{\varphi \in K_z} e_\varphi$. Denote by $K_z^\perp$ all of the roots in K_g orthogonal to the set K_z. Let $h_z^\perp$ be the subspace of h orthogonal to h_z.

Let $z^\perp = h_z^\perp \oplus \sum_{\varphi \in K_z^\perp} e_\varphi$. Note that if φ, ψ are roots of g such that $[\varphi, \psi] = 0$, then $(\varphi, \psi) = 0$. We then leave it to the reader to show that

LEMMA 1.3. $z^\perp$ *is a subalgebra of* g, *and* $K_{z^\perp} = K_z^\perp$.

THEOREM 1.4. *Let p be a maximal rank, reductive subalgebra. Let $p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T$ (as described above). Let $q_i = X_i^{k_i}$, and let $Y = q_1^\perp$. If p is primitive, then either $Y = q_2 \oplus \cdots \oplus q_r$ or the subalgebra, l, generated by the vector subspace $q_1 + Y + h$ is g.*

Proof. Since elements of the Weyl group act as isometries, $W_{q_1} = W_Y$. Now $[q_2 \oplus \cdots \oplus q_r, q_1] = 0$, thus $(q_2 \oplus \cdots \oplus q_r, q_1) = 0$. Hence $X_2^{k_2} \oplus \cdots \oplus X_r^{k_r} \subseteqq Y$, and $p \subseteqq l$.

Now if $\alpha \in W_p$ then (a) $\alpha(K_{q_1}) \subset K_l$ and (b) $\alpha(K_Y) \subset K_l$. (a) follows from Lemma 1.2 and (b) from the above remark that $X_{q_1} = W_Y$. Now $K_{q_1} \cup K_Y$ forms a set of generators for K_l (under $[\,,]$). Thus $\alpha(K_l) \subset K_l$ (since α acts as a "homomorphism" relative to $[\,,]$). So we have shown that $W_p \subset W_l$. By Proposition 1.2 and by the primitivity of p, either $l = p$ or $l = g$. If $l = p$, then clearly $Y = X_2^{k_2} \oplus \cdots \oplus X_r^{k_r}$. Otherwise $l = g$.

Let φ, ψ be roots in a simple Lie algebra where all of the roots have the same length. If $(\varphi, \psi) = 0$, then $[\varphi, \psi] = 0$. Otherwise $[\varphi, \psi] = \varphi + \psi$ and the length of $\varphi + \psi$ is greater than the length of φ or ψ since $(\varphi, \psi) = 0$. Hence the following.

COROLLARY 1.5. *Let g be a simple Lie algebra all of whose roots have the same length (in particular E_6, E_7 and E_8). Then if p is a primitive subalgebra, $Y = X_2^{k_2} \oplus \cdots \oplus X_r^{k_r}$.*

Proof. In such an algebra $X_1^{k_1} + X + h = l$.

If $l = g$, then g can be decomposed into to direct sum of two ideals, which contradicts the fact that g is simple.

As an immediate application of this corollary we get:

COROLLARY 1.6. *Let g be a simple Lie algebra whose roots all have the same length. Let p be a subalgebra of g. If $K_p^\perp \neq \emptyset$, then p is not primitive.*

Note: This corollary is true even if the roots of g do not have the same length.

2. E_6, E_7 and E_8. We first describe the roots of the algebras E_6, E_7 and E_8 (see [1]). Let $z_1, z_2, \cdots, z_8$ be the standard orthonormal basis for the dual space to a fixed Cartan subalgebra of E_8. With respect to this basis the roots of E_8 are given by

$$(I)_8 = \{\pm z_i \pm z_j \mid 1 \leqq i < j \leqq 8\}$$

and

$$(II)_8 = \{\pm 1/2(z_1 \pm z_2 \pm \cdots \pm z_8) \mid \text{the number of minus signs is even.}\}$$

We shall refer to these as type I roots and type II roots respectively. The roots of type I will be denoted by $\pm i \pm j$ when no confusion arises; e.g., $-z_2 + z_3$ is denoted by $-2 + 3$, etc. The roots of Type II will be denoted by the corresponding sequences of signs, e.g., $1/2(z_1 + z_2 - z_3 + z_4 - z_5 - z_6 - z_7 + z_8) = (+ + - + - - - +)$.

We take E_6 and E_7 to be regular subalgebras of E_8. The roots of E_7 are all of those roots of E_8 orthogonal to $7 + 8$. Thus

$$(I)_7 = (I)_8 \cap K_{E_7} = \{\pm i \pm j \mid 1 \leqq i < j \leqq 6\} \cup \{7 - 8\}$$

$$(II)_7 = (II)_8 \cap K_{E_7} = \{\pm(* * * * * * + -) \mid \text{an odd number of the } * \text{ are } - \text{ and the others are } +\}\,.$$

The roots of $E_6 \subseteqq E_7$ are as follows:

$$(I)_6 = (I)_8 \cap K_{E_6} = \{\pm i \pm j \mid 1 \leqq j < j \leqq 5\}$$
$$(II)_6 = (II)_8 \cap K_{E_6} = \{\pm(* * * * * + + -) \mid \text{and odd number of the } * \text{ are } - \text{ and the others are } +\} .$$

When no confusion arises, we shall write merely (I) or (II). Also, we note that $(I)_s$ generates a maximal rank subalgebra, which we also denote by $(I)_s$ (or sometimes by (I)). The algebra $(I)_s$ is maximal in E_s.

We note that all roots of E_8 have the same length, $\sqrt{2}$. Thus the Weyl group acts transitively in each of these algebras. As observed in Corollary 3.3. of [3], the Cartan subalgebras in E_s are primitive.

In order to classify the primitive, reductive, maximal rank subalgebras we need some information about their Weyl groups. We note first that the reflections about roots of type I are just determined by signed permutations of $z_1, \cdots, z_8$. For if α and β are roots, then

$$S_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha = \beta - (\alpha, \beta)\alpha, \text{ as } (\alpha, \alpha) = 2 .$$

$$\text{If } \alpha = i - j, \text{ then } S_\alpha(z_k) = \begin{cases} z_i & \text{if } k = j \\ z_j & \text{if } k = i \, . \\ z_k & \text{otherwise} \end{cases}$$

$$\text{If } \alpha = i + j, \text{ then } S_\alpha(z_k) = \begin{cases} -z_i & \text{if } k = j \\ -z_j & \text{if } k = i \\ z_k & \text{otherwise} \, . \end{cases}$$

We can thus, for example, identify S_α with the transposition $\begin{pmatrix} 1\,2\,3 \cdots 8 \\ 2\,1\,3 \cdots 8 \end{pmatrix}$ if $\alpha = 1 - 2$, and with $\begin{pmatrix} 1 & 2\,3 \cdots 8 \\ -2 & -1\,3 \cdots 8 \end{pmatrix}$ if $\alpha = 1 + 2$. These reflections, then, generate signed permutations with an even number of sign changes.

LEMMA 2.1. (a) *Any two sets of three mutually orthogonal roots in E_s are conjugate by an inner automorphism* (i.e., *by a Weyl group element*).

(b) *Any set of mutually orthogonal roots is conjugate by a Weyl group element to a set of roots (mutually orthogonal) of type I.*

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be mutually orthogonal roots in E_s. Since the Weyl group acts transitively on the roots, we may assume $\alpha_1 = 1 - 2$.

Suppose α_2 is a root of type II. Since $(\alpha_2, \alpha_1) = 0$, $\alpha_2 = \pm(+ + * * * * * *)$, say $(+ + * * * * * *)$. Let $\beta = (- - * * * * * *)$, where β is taken to agree with α in the last six signs. Then $S_\beta(1-2) = 1-2$, $S_\beta(\alpha) = 1+2$. Since no root of type II is orthogonal to both of these, (b) holds. In particular, $S_\beta(\alpha_3)$ is of type I, say $i+j$, i, $j \neq 1$ or 2. Then there is some signed permutation f fixing 1 and 2, and taking $i+j$ into $3-4$. The permutation can be taken from W_{E_6} or W_{E_7} if $i-j$ is in K_{E_6} or K_{E_7}, respectively. Thus fS_β takes $\alpha_1, \alpha_2, \alpha_3$ into 1 ± 2, $3-4$, and (a) holds.

On the other hand, if α_2 is a root of type I, and if $\alpha_2 \neq 1+2$, then we can find a signed permutation f, as above, such that f fixes 1 and 2 and $f(\alpha_2) = 3-4$. Then let $\beta = (+++-+++-)$. $S_\beta(1-2) = 1-2$, $S_\beta(3-4) = (--+-----+)$. Thus we have reduced the problem to the case where α_2 is of type II, and the lemma is proved.

Let N_s be the subgroup of the Weyl group of E_s consisting of the signed permutations. We have the following normal forms for elements of the Weyl group:

THEOREM 2.2. *Let g be a Weyl group element. Then one of the following three cases holds*:

(a) $g \in N_s$.

(b) $\exists f \in N_s,\ \alpha \in (II)_s \ni g = fS_\alpha$.

(c) $\exists f \in N_s,\ \alpha, \beta \in (II)_s \ni (\alpha, \beta) = 0$ *and* $g = fS_\beta S_\alpha$.

Proof. Any Weyl group element g has the form $g = S_{\alpha_n} \cdots S_{\alpha_1}$, where the α_i are roots of the algebra. Recalling that $S_\alpha S_\beta = S_\beta S_{S_\beta(\alpha)}$, we may move any S_{α_i}, where α_i is of type I, to the left past all roots of type II. Thus we may assume $g = S_{\beta_1} \cdots S_{\beta_k} \ S_{\gamma_1} \cdots S_{\gamma_l}$ $(k+l=n)$, where the β_i are of type I, and the γ_j of type II. Let this be a representation of g so that l is minimal. Then all the γ_j are mutually orthogonal. For if $(\alpha, \beta) = 0$, then $S_\alpha S_\beta = S_\beta S_\alpha$, and thus if there are any γ_i and γ_j not orthogonal, we may assume that we have $(\gamma_1, \gamma_2) = \pm 1$ or ± 2. If $(\gamma_1, \gamma_2) = \pm 2$, then $\gamma_1 = \pm\alpha$, and $S_{\gamma_1} S_{\gamma_2} = 1$, the identity. If $(\gamma_1, \gamma_2) = \pm 1$, then $S_{\gamma_1} S_{\gamma_2} = S_{\gamma_2} S_{S_{\gamma_2}(\gamma_1)}$, where $S_{\gamma_2}(\gamma_1)$ is a root of type I. In either case, we can decrease l, a contradiction. Thus we may write $g = fS_{\gamma_1} \cdots S_{\gamma_\gamma}$ where all the γ_j are mutually orthogonal type II roots, and f is a signed permutation.

Two roots of type II are orthogonal if and only if they agree in four signs, and disagree in four signs, e.g., $(++++++++)$ and $(++++----)$. Thus in E_6 it is easy to see that there can be at most two mutually orthogonal roots of type II. Thus, for E_6 the theorem is proved.

Suppose that for E_7 or E_8 we have γ_1, γ_2, and γ_3 mutually orthogonal. Then for an appropriate signed permutation h we have

$$h(\gamma_1) = (+ - + - + - + -)$$
$$h(\gamma_2) = (- + - + + - + -)$$
$$h(\gamma_3) = (- + + - - + + -) .$$

Thus $S_{\gamma_1}S_{\gamma_2}S_{\gamma_3} = h^{-1}hS_{\gamma_1}h^{-1}hS_{\gamma_2}h^{-1}hS_{\gamma_3}h^{-1}h = h^{-1}S_{h(\gamma_1)}S_{h(\gamma_2)}S_{h(\gamma_3)}h$. Let $\beta = - + + - + - - +$. Then $S_{h(\gamma_1)}S_{h(\gamma_2)}S_{h(\gamma_3)}S_\beta = (-1)\cdot(12)(34)(56)(78) = a \in N_s$. Thus $S_{\gamma_1}S_{\gamma_2}S_{\gamma_3} = h^{-1}S_{h(\gamma_1)}S_{h(\gamma_2)}S_{h(\gamma_3)}S_\beta S_\beta h = h^{-1}aS_\beta h$. Thus $g = fh^{-1}aS_\beta hS_{\gamma_4}\cdots S_{\gamma_l}$. By the arguments above, this reduces l by two, a contradiction. Hence $l \leqq 2$. This proves the theorem.

We now proceed with the classification of the primitive, maximal rank, reductive subalgebras, p, of E_s.

The general plan of attack in classifying the possible algebras p will be to assume that X_1 is some particular algebra, say A_3 or D_4, etc., and then to conjugate X_1 (by Weyl group elements) into a form suitable for deciding whether or not the various maximal rank, reductive subalgebras with X_1 as an ideal are primitive. Theorem 2.1 is used for the latter. The number of cases to consider is kept small this way since there are only a few choices for X_1, and, of course, the rank of X_1 is limited by 8. In particular, X_1 cannot be B_n, C_n, G_2 or F_4 since the roots of E_s have only one length. Because the arguments are essentially the same, we treat E_6, E_7 and E_8 together.

We consider the following cases for X_1:

$$A_1, A_2, A_3, A_4D_l(l > 4, A_l(l > 4))E_7, E_6 .$$

Case 1. $X_1 = A_1$.

(a) $k_1 = 1$. (See § 1 for the definition of k_1).

By Corollary 1.5 and the primitivity of p, $p = A_1 \oplus Y$, where $Y = A_1^\perp$.

(i) In E_8 let $K_{A_1} = \{\pm(7+8)\}$. This can be done without loss of generality since the Weyl group acts transitively. Then $A_1^\perp = E_7$. As is easily checked, $A_1 \oplus E_7$ is maximal, and hence primitive.

(ii) In E_7, let $K_{A_1} = \{\pm(7-8)\}$. Since the roots of Y must be orthogonal both to $7-8$ (A_1) and to $7+8$ (since they are roots of E_7), Y must have only roots of type I. Then

$$K_Y = \{\pm\, i \pm j \mid 1 \leqq i < j \leqq 6\} ,$$

or $Y = D_6$. $A_1 \oplus D_6$ is the algebra $(I)_7$, which is maximal, and thus primitive.

(iii) In E_6, let $K_{A_1} = \{\pm(1-2)\}$. Then $Y = A_1^\perp$, and $K_Y = \{\pm(1+2)$, $\pm i \pm j$, $3 \leqq i < j \leqq 5$, and all type II roots of the form $\pm(+ + * * * + + -)$ and $\pm(+ + * * * - - +)\}$.

We see, then, that $Y = A_5$. In this case $p = A_1 \oplus A_5$ is maximal, and thus primitive.

(b) $k_1 > 1$

Here we let $p = A_1^{k_1} \oplus Y$.

(i) In E_8, since each A_1 summand has a root orthogonal to the others, we can assume that for the first two summands $A_1 \oplus A_1 = A_1^2$, $K_{A_1^2} = \{\pm 1 \pm 2\}$, by Lemma 2.1. Let $\bar{Y} = A^{k_1-2} \oplus Y$. Then $\bar{Y} \subseteqq (A_1^2)^\perp$, and thus $K_{\bar{Y}} \subseteqq (I)$.

If $k_1 = 2$, then $K_Y = K_{(A_1^2)^\perp} = D_6$.

Here we digress to prove a lemma about D_l factors of reductive subalgebras.

LEMMA 2.3. *$D_l \subseteqq E_s$ is always conjugate to the algebra with roots $\{\pm i \pm j \mid 1 \leqq i < j \leqq l\}$. If D_l has only type I roots, and if $l > 4$, then $W_{D_l} \subset W_{(I_s)}$.*

Proof. Let $D_l \subset E_s$. Let h be the Cartan subalgebra of E_s. Let $\bar{h}$ be the Cartan subalgebra of D_l. Note that $\dim \bar{h} = l$. We can choose an orthonormal basis $w_1, w_2, \cdots, w_l$ of $\bar{h}$ so that the roots of D_l are just $\pm w_i \pm w_j$, $1 \leqq i < j \leqq l$.

Now $w_1 \pm w_2$ are orthogonal. Thus by Lemma 2.1 we can assume $\{\pm w_1 \pm w_2\} = \{\pm 1 \pm 2\}$ in E_s. $w_2 - w_3$ is not orthogonal to $w_1 + w_2$ nor to $w_1 - w_2$. But any type *II* root is orthogonal either to $1 - 2$ or to $1 + 2$. Thus $w_2 - w_3$ is of type *I*. By use of a signed permutation fixing the set $\{\pm 1 + 2\}$, we can take $w_2 - w_3$ to be $2 - 3$. Clearly $w_2 + w_3$ then becomes $2 + 3$ under this same signed permutation. Thus we may assume $\pm 1 \pm 2$, $\pm 2 \pm 3$ and therefore $\pm 1 \pm 3$ are roots of D_l. Continuing in this way, we can assume $\pm i \pm j$, $1 \leqq i < j \leqq l$, are roots of D_l. These are all the roots of D_l, and hence the first part of the lemma is proved.

Now in fact what we saw was that if D_l has only type *I* roots, then up to conjugacy by a signed permutation the roots of D_l are $\pm i \pm j$, $1 \leqq i < j \leqq l$.

Thus it is sufficient for establishing the last part of the lemma to assume that the roots of D_l are $\pm i \pm j$, $1 \leqq i < j \leqq l$.

Let $g \in W_{D_l}$. Suppose, contrary to the assertion, that g is not a signed permutation. Then by Theorem 2.1, either $g = fS_\alpha$, or $g = fS_\beta S_\alpha$, f a signed permutation, and α, β roots of type *II* with $(\alpha, \beta) = 0$. In the first case, either $S_\alpha(1 + 2)$ or $S_\alpha(1 - 2)$ is of type *II*, and thus $fS_\alpha(1 + 2)$ or $fS_\alpha(1 - 2)$ is of type *II*, and hence not in K_{D_l}, contradicting the assumption that $g \in W_{D_l}$.

For the second case, let $g = fS_\alpha S_\beta$, $(\alpha, \beta) = 0$, and let $i \pm j \in K_{D_l}$. As above, $g(i \pm j) \in K_{D_l} \subseteqq (I)$. Thus $S_\alpha S_\beta(i \pm j) \subseteqq (I)$. Now if α and β, disagree in sign in one of the i or j positions, and agree in the other, then $S_\alpha S_\beta(i \pm j)$ are both in (II). (We see this by observing $S_\alpha S_\beta = S_\beta S_\alpha$, and if $(\gamma, \alpha) = 0$, then $S_\alpha(\gamma) = \gamma$. $i - j$ is orthogonal to one of α and β, and $i + j$ to the other). Thus α and β must agree in both the i and j positions, or both disagree. Changing i and j, we see that α and β must totally agree or totally disagree on positions $1, 2, \cdots, l$. If $l > 4$, this contradicts $(\alpha, \beta) = 0$. This completes the proof of the lemma.

We now return to the case at hand, namely $A_1^2 \oplus D_6$ in E_8. By Lemma 1.2 we have $W_p \subseteqq W_{D_6}$ and by Lemma 2.3, $W_{D_6} \subseteqq W_{(I)}$. But $A_1^2 \oplus D_6 \subsetneqq (I)$. Hence $A_1^2 \oplus D_6$ is not primitive, by Prop. 1.1.

This means that we may assume $k_1 > 2$. By Lemma 2.1 we may assume that $K_{A_1^3} = \{\pm(1 \pm 2), \pm(3 - 4)\}$, the roots for the first three factors A_1. Now $3 + 4$ is in $K^{\perp}_{(A_1^3)}$. If $3 + 4 \notin K_p$, then p is not primitive, by Corollary 1.6. If $3 + 4 \in K_Y$, then since Y has no summands of type A_1 in it, $K_Y \subseteqq (I)$ contains another root α such that $[3 + 4, \alpha] \neq 0$. But then either $[-3 + 4, \alpha] \neq 0$ or $[3 - 4, \alpha] \neq 0$, contradicting the fact that $\pm(3 - 4)$ are the roots for a direct summand A_1 of p.

This all implies that k_1 can't just be 3. In fact, the same kind of reasoning implies that k_1 must be even if $k_1 > 1$. $k_1 = 6$ is not possible, since, by Lemma 2.1, we can assume $K_{A_1^6}$ consists of six of the following eight roots, together with their negatives: 1 ± 2, 3 ± 4, 5 ± 6, 7 ± 8. But then $K^{\perp}_{A_1^6}$ consists of the other two, and we have two more summands of A_1. Thus $k_1 = 4$ and $k_1 = 8$ are the only possibilities. By Corollary 1.5, then, the only remaining possibilities are $A_1^4 \oplus D_4$ and A_1^8, where $K_{A_1^4 \oplus D_4} = \{\pm(1 \pm 2), \pm(3 \pm 4), \pm i \pm j, 5 \leqq i < j \leqq 8\}$ and $K_{A_1^8} = \{\pm(1 \pm 2), \pm(3 \pm 4), \pm(5 \pm 6) \pm (7 \pm 8)\}$.

We first show that $A_1^4 \oplus D_4$ is not primitive by showing $W_{A_1^4 \oplus D_4} = W_{D_4 \oplus D_4}$, where $K_{D_4 \oplus D_4} = \{\pm i \pm j \mid i, j \leqq 4 \text{ or } i, j \geqq 5\}$. Let $g \in W_{A_4 \oplus D_4}$. If g is a signed permutation, then $g \in W_{D_4 \oplus D_4}$. If $g = fS_\alpha$, f a signed permutation, $\alpha \in (II)$, then fS_α must take one of $1 + 2$ and $1 - 2$ into (II), contradicting $g \in W_{A_1^4 \oplus D_4}$.

If $g = fS_\beta S_\alpha$, $(\alpha, \beta) = 0$ then, just as in the proof of Lemma 2.3, α and β must totally agree or totally disagree in sign in positions 5, 6, 7, 8, say they agree. Then since $(\alpha, \beta) = 0$, they must totally disagree in positions 1, 2, 3, 4. Thus $S_\beta S_\alpha$ are in $W_{D_4 \oplus D_4}$ (by computation). Since $g \in W_{A_4 \oplus D_4}$, $f = gS_\beta S_\alpha$ is in $W_{A_4 \oplus D_4}$, and thus in $W_{D_4 \oplus D_4}$ by the observations above. Hence $g \in W_{D_4 \oplus D_4}$. This completes the proof that $W_{A_4 \oplus D_4} \subseteqq W_{D_4 \oplus D_4}$. Hence $A_4 \oplus D_4$ is not primitive.

A_1^8, the last case, is primitive. We show this by noting that the following elements are in $W_{A_1^8}$: (c) (13)(24), (d) (35)(46), (e) (57)(68), (f) arbitrary sign changes (even number), and (g) $S_\beta S_\alpha$, where $\alpha =$

$(+ + + + + + + +)$ and $\beta = (+ + + + - - - -)$. Let q be an algebra strictly containing A_1^8, and invariant under $W_{A_1^8}$. If K_q contains a root of (I) not in K_p, then using (c), (d) and (e) above we see that K_q must contain all of (I). But $S_\beta S_\alpha(4 + 5)$ is in (II), and thus K_q also has a root of type II. But then K_q has all roots of type II since (I) is a maximal subalgera, and $q = E_8$. If K_q contains a root of type II, say α_1 then we can assume $\alpha_1 = \alpha$, using (f). Similarly, using (f) we see that $(+ - - + + + + +) \in K_g$. But then $S_\beta S_\alpha(+ - - + + + + +) = -2 - 3 \in K_g$, and $q = E_8$ by the case above, as this is a new root of type I. Thus the only algebra larger than A_1^8 invariant under $W_{A_1^8}$ is E_8, and A_1^8 is primitive. This completes the case $X_1 = A_1$ for E_8. (ii) In E_7 we may assume that the first copy of A_1 is given by $K_{A_1} = \{\pm(7 - 8)\}$, and the second by $\{\pm(1 - 2)\}$, using Lemma 2.1. Just as k_1 had to be even in the E_8 case above, the same argument shows that k_1 must be odd for E_7, and thus $k_1 = 3,5$ or 7, as $k_1 > 1$. But if $k_1 = 5$, then $K_{A_1^5} = \{\pm 1 \pm 2,\ \pm 3 \pm 4,\ \pm(7 - 8)\}$ up to signed permutations, and thus $K_{A_1^5}^{\perp} = \{\pm 5 \pm 6\}$, and $(A_1^5)^{\perp} = A_1^2$, contradicting $k_1 = 5$. Hence this case is impossible. The only remaining possibilities are $A_1^3 \oplus D_4$ and A_1^7, by rank considerations and Corollary 1.5.

$$K_{A_1^3 \oplus D_4} = \{\pm 1 \pm 2,\ \pm i \pm j,\ 3 \leqq i < j \leqq 6,\ \pm(7 - 8)\}\ .$$

We show that $A_1^3 \oplus D_4$ is primitive. Let $\alpha = (+ - + + + + + -)$, $\beta = (- + + + + + - +)$. Then $S_\beta S_\alpha$ is in $W_{A_1^3 \oplus D_4}$. Let q be a subalgebra of E_7 properly containing $A_1^3 \oplus D_4$ and invariant under $W_{A_1^3 \oplus D_4}$. Suppose K_q contains a root of type I not in $K_{A_1^3 \oplus D_4}$. Then K_q contains all of $(I)_7$, using the algebra multiplication $[\,,]$ of q. Now $S_\beta S_\alpha(2 - 3) \in K_q$ a root of type II. Thus, since $(I)_7$ is a maximal subalgebra of E_7, $q = E_7$. On the other hand, if K_q contains a root, γ, of type (II), then, since $f = \begin{pmatrix} 1 & 2 & 3\,4 \cdots 8 \\ 1 & -2 & -3\,4 \cdots 8 \end{pmatrix}$ is in $W_{A_1^3 \oplus D_4}$, $f(\gamma)$ must be in K_q. Thus $[\gamma, f(\gamma)] \in K_q$. But this is one of the four roots $\pm 2 \pm 3$ which is not in $K_{A_1^3 \oplus D_4}$. Thus $q = E_7$ by the previous argument, and $A_1^3 \oplus D_4$ is primitive.

Now consider A_1^7. We claim that this is primitive also,

$$K_{A_7} = \{\pm 1 \pm 2,\ \pm 3 \pm 4,\ \pm 5 \pm 6,\ \pm(7 - 8)\}\ .$$

Let q be a subalgebra properly containing A_1^7 and invariant under $W_{A_1^7}$, and in particular under (c) (13)(24), (d) (35)(46), (e) $\begin{pmatrix} 1 & 2 & 3\,4 \cdots 8 \\ 1 & -2 & -3\,4 \cdots 8 \end{pmatrix} = f$ and $S_\beta S_\alpha$, where $\alpha = (+ - + + + + + -)$ and $\beta = (- + + + + + - +)$. If K_q contains a root of type I not in $K_{A_1^7}$, then using (c), (d) and the algebra multiplication we see K_q contains all of $(I)_7$. Further, K_q contains $S_\beta S_\alpha(2 - 3)$, which is a root of type II. Thus, since $(I)_7$

is maximal, q $= E_7$. On the other hand, if K_q has a root γ, of type II, then $f(\gamma) \in K_q$, $[\gamma, f(\gamma)] \in K_q$ and $[\gamma, f(\gamma)]$ is a root of type I not in $K_{A_1^7}$. Then the previous argument yields $q = E_7$. Thus A_1^7 is primitive.

Note. As can be seen in the last few arguments, the crucial thing in demonstrating that a given subalgebra p is primitive is knowing which elements of W_p to use in showing that larger algebra invariant under W_p is E_s. The arguments thereafter are just like the ones above. Thus, from now on, rather than show in detail why a given algebra is primitive, we shall simply indicate which elements of W_p one uses

(iii) In E_6 we can assume $K_{A_1^2} = \{\pm 1 \pm 2\}$, by Corollary 1.5. Then $K_{A_1^2}^{\perp} \subseteqq (I)_6$. In this case, as in E_8, we must have k_1 even, by the same arguments. $k_1 = 6$ is impossible since $(I)_6$ doesn't contain $K_{A_1^6}$. Thus only $A_1^2 \oplus A_3$ and A_1^4 remain. We claim that neither of these is primitive.

$K_{A_1^2 \oplus A_3} = \{\pm 1 \pm 2,\ \pm 3 \pm 4,\ \pm 4 \pm 5,\ \pm 3 \pm 5\}$. We show that $W_{A_1^2 \oplus A_3} \subseteqq W_{(I)_6}$. Let $g \in W_{A_1^2 \oplus A_3} \cdot g$ cannot be of the form fS_α, f a signed permutation and $\alpha \in (II)_6$, for either $fS_\alpha(1-2)$ or $fS_\alpha(1+2)$ would be in $(II)_6$, and thus not in $K_{A_1^2 \oplus A_3}$. Suppose $g = fS_\beta S_\alpha$, $(\alpha, \beta) = 0, f$ a signed permutation, $\alpha, \beta \in (II)_6$. As we saw in the proof of Lemma 2.3 (since $A_3 = D_3$), β and α must agree in each of positions 3, 4 and 5, or disagree in these three position. Similarly they must agree or disagree in both positions 1 and 2. Since they are both in $(II)_6$, they must totally agree or totally disagree in positions 6, 7, 8. It is impossible, then, for α and β to agree in exactly four positions, contradicting $(\alpha, \beta) = 0$. Thus g must be in $W_{(I)_6}$, and $W_{A_1^2 \oplus A_3} \subset W_{(I)_6}$. So $A_1^2 \oplus A_3$ is not primitive.

We now consider the last case, A_1^4. $K_{A_1^4} = \{\pm 1 \pm 2, \pm 3 \pm 4\}$. Here we show that $W_{A_1^4} \subseteqq W_{D_4}$, where $K_{D_4} = \{\pm i \pm j \mid 1 \leqq i < j \leqq 4\}$, and thus that A_1^4 is not primitive. Let $g \in W_{A_1^4}$. As above, $g = fS_\alpha$ is not possible, as $fS_\alpha(K_{A_1^4}) \nsubseteqq (I)_6$ If $g = fS_\beta S_\alpha$, as above, α and β must agree in both positions 1 and 2 or disagree in both, and similarly for positions 3 and 4. Also, as $\alpha, \beta \in (II)_6$, they totally agree or totally disagree in positions 6, 7 and 8. Since $(\alpha, \beta) = 0$, α and β must agree in exactly four positions, and these must be 1, 2, 3 and 4 or 5, 6, 7 and 8. In either case, $S_\alpha S_\beta \in W_{A_1^4}$ and $S_\alpha S_\beta \in W_{D_4}$ (by computation). Then $f = gS_\beta S_\alpha \in W_{A_1^4}$. But a signed permutation in $W_{A_1^4}$ is clearly also in W_{D_4}. Thus $f \in W_{D_4}$, and $g = fS_\beta S_\alpha \in W_{D_4}$, and A_1^4 is not primitive.

Before beginning Case 2, we prove a lemma about subalgebras $A_l \subseteqq E_s$.

LEMMA 2.4. *Let* $A_l \subset E_s$.

(a) *If* $l \leqq 6$ *for* $s = 8$, $l \leqq 4$ *for* $s = 7$, *or* $l \leqq 3$ *for* $s = 6$, *then all* A_l *are conjugate, and in particular conjugate to* A_l *with* $K_{A_l} = \{\pm(i - j) \mid 1 \leqq i < j \leqq l + 1\}$.

(b) *If* $l = 7$ *for* E_8, $l = 5$ *for* E_7, *or* $l = 4$ *for* E_6, *then there are two conjugacy classes of* A_l *whose simple roots are respectively* $\pm(i - (i + 1))$, $1 \leqq i \leqq l$, *and* $\pm(i - (i + 1))$, $1 \leqq i \leqq l - 1$, *together with* $\pm(l + (l + 1))$. (Note: where we write $(i + 1)$ and $(l + 1)$ above we mean numerical addition, of course).

Proof. Let h be the fixed Cartan subalgebra of E_s. Let $h_l \subseteqq h$ be the Cartan subalgebra of A_l. We can choose vectors $w_1, \cdots, w_{l+1}$ in h such that h_l is the hyperplane in the span of $\{w_1, \cdots, w_{l+1}\}$ determined by the $w_i - w_{i+1}$, $1 \leqq i \leqq l$. We can assume that the $w_i - w_{i+1}$ are the simple roots of A_l, since $l < s$ in all cases. We can assume $w_1 - w_2$ is $1 - 2$, by conjugating with a Weyl group element. If $w_2 - w_3$ is of type I, then a signed permutation can conjugate $\{w_1 - w_2, w_2 - w_3\}$ to $\{1 - 2, 2 - 3\}$. If $w_2 - w_3$ is of type II, it must have the form $\pm(+ - * * * * * *)$. Using a signed permutation, we can assume that $w_2 - w_3$ is $(+ - + + * * * *)$. Let $\beta = (+ + - + * * * *)$, where β and $w_2 - w_3$ agree in the last four positions. Then $S_\beta(1 - 2) = 1 - 2$, $S_\beta(+ - + + * * * *) = -2 + 3$. Hence we can assume $w_2 - w_3$ is a root of type I, and $(w_1 - w_2, w_2 - w_3\} = \{1 - 2, 2 - 3\}$. Next consider $w_3 - w_4$. If it is in $(I)_s$, then using a signed permutation we can assume that $w_3 - w_4 = 3 - 4$. If $w_3 - w_4 \in (II)_s$ then since it is orthogonal to $1 - 2$ and not to $2 - 3$, using a signed permutation we can assume it is $(+ + - + + * * *.)$ Letting $\beta = (+ + + - + * * *)$, and using S_β, we can assume $w_3 - w_4 \in (I)_s$. Continuing in this way yields (a).

For (b) we do the E_8 case, E_6 and E_7 being entirely similar. By (a) we can assume $w_i - w_{i+1} = i - (i + 1)$ for $1 \leqq i \leqq 6$. Now if $w_7 - w_8$ is a root of type I, then since $(w_7 - w_8, i - (i + 1)) = 0$ for $1 \leqq i \leqq 5$, and $(w_7 - w_8, 6 - 7) \neq 0$, we can assume $w_7 - w_8$ is $7 - 8$ or $7 + 8$, and if $w_7 - w_8$ is of type II, then it must be $\pm(+ + + + + + - -)$ for the same reasons. Let $\alpha = (+ + + + + + + +)$. Then S_α fixes $i - (i + 1)$, $i \leqq 6$, and takes $w_7 - w_8$ into a root of type I. Thus it is reduced to the previous case. The two forms (the one with $7 - 8$ and the one with $7 + 8$) obtained here are not conjugate, since in the first case all the roots are orthogonal to $(+ + + + + + + +)$, whereas in the second case no root is orthogonal to all of them. This completes the proof of (b).

Case 2. $X_1 = A_2$

(a) $k_1 = 1$, $Y = A_2^\perp$

(i) In E_8, we can take $K_{A_2} = \{\pm(6-7),\ \pm(7+8),\ \pm(6+8)\}$, since by Lemma 2.4 all A_2 in E_8 are conjugate. Then $A_2^\perp = E_6$. Now $A_2 \oplus E_6$ is maximal and thus primitive.

(ii) In E_7, take $K_{A_2} = \{\pm(7-8),\ \pm(+\,+\,+\,+\,+\,-\,+\,-),\ \pm(+\,+\,+\,+\,+\,-\,-\,+)\}$ (by Lemma 2.4). Then $A_2^\perp = A_5$, where $K_{A_5} = \{\pm(i-j),\ \pm(i+6) \mid i \neq j,\ i, j = 1, \cdots, 5\}$. This is maximal, and thus primitive.

(iii) In E_6, by Lemma 2.4, we can take

$$K_{A_2} = \{\pm(1-2),\ \pm(2-3),\ \pm(1-3)\}\,.$$

Then $K_{A_2}^\perp = \{\pm 4 \pm 5,\ \pm(+\,+\,+\,\varepsilon_1\varepsilon_2\,+\,+\,-),\ \pm(+\,+\,+\,\varepsilon_3\varepsilon_4\,-\,-\,+) \mid \varepsilon_1 \neq \varepsilon_2,\ \varepsilon_3 = \varepsilon_4\}$. Thus $A_2^\perp = A_2 \oplus A_2$. Either we have a summand of A_1, and Case 1 applies, another summand of A_2, violating $k_1 = 1$, or neither and hence a root orthogonal to p, contradicting primitivity by Corollary 1.6.

(b) $k_1 > 1$

(i) In E_8, let one copy of A_2 have $K_{A_2} = \{\pm\alpha, \pm\beta, \pm\delta\}$ where $\delta = \alpha + \beta$, and let γ be a root in another copy of A_2. By Lemma 2.1 we can assume $\alpha = 1 + 2$, $\gamma = 1 - 2$. Now $(\beta, 1+2) \neq 0$, and $(\beta, 1-2) = 0$. Thus β must be of type *II*, and using a signed permutation we may assume $\beta = (+\,+\,+\,+\,+\,+\,+\,+)$. Since $[\alpha, \beta] = \delta$, we have $\delta = (-\,-\,+\,+\,+\,+\,+\,+)$.

The roots in the second copy of A_2 are orthogonal to δ and β, but not to $1-2$. Thus they are $\pm(1-2)$, $\pm\eta$, $\pm\tau$, where η and τ are of type *II*. Using a signed permutation, then, which fixes z_1 and z_2, we can assume $\eta = (+\,-\,+\,+\,+\,-\,-\,-)$ and $\tau = (-\,+\,+\,+\,+\,-\,-\,-)$.

Then $K_{A_2^2}^\perp = \{\pm(3-4),\ \pm(4-5),\ \pm(3-5),\ \pm(6-7),\ \pm(7-8),\ \pm(6-8)\}$, and $A_2^{2\perp} = A_2^2$ (two more copies). Thus $Y \subseteqq A_2^2$. If $Y \neq A_2^2$, then either $Y = A_1 \oplus Z$, which is already covered by Case 1, or $Y = A_2$ or A_2^2. If $Y = A_2$, then there is a root orthogonal to p in E_8, which can't happen by Corollary 1.6. Thus $Y = A_2^2$ and $p = A_2^4$. This is primitive, as can be seen using the following elements of $W_{A_2^4}$: $(46)(57)(38)$ and $\begin{pmatrix} 1 & 2\,3\,4\,5 & 6 & 7 & 8 \\ 1 & -2\,3\,4\,5 & -6 & -7 & -8 \end{pmatrix}$.

(ii) In E_7 we can assume that for one copy of A_2, $K_{A_2} = \{\pm(7-8),\ \pm(+\,+\,+\,+\,+\,-\,+\,-),\ \pm(+\,+\,+\,+\,+\,-\,-\,+)\}$. $A_2^\perp \subseteqq (I)_7$, since any root of $A_2^\perp \subseteqq E_7$ must be orthogonal to $7+8$ and, in this case, to $7-8$. Thus, by using signed permutations, we can assume that for the second copy of A_2 we have $K_{A_2} = \{\pm(1-2),\ \pm(2-3),\ \pm(1-3)\}$. Then $K_{A_2^2}^\perp = \{\pm(4-5),\ \pm(5+6),\ \pm(4+6)\}$. Hence A_2^3 is the only possibility. A_2^3 is primitive, which can be seen using the elements of $W_{A_2^3}$: $\begin{pmatrix} 1\,2 & 3\,4\,5 & 6\,7\,8 \\ 4\,5 & -6\,1\,2 & -3\,7\,8 \end{pmatrix}$ and $fS_\beta S_\alpha$, where $f = \begin{pmatrix} 1\,2 & 3 & 4 & 5 & 6\,7\,8 \\ 1\,2 & -3 & -4 & -5 & -6\,7\,8 \end{pmatrix}$ $\alpha = (+\,-\,-\,+\,+\,-\,+\,-)$, $\beta = (+\,-\,+\,-\,-\,+\,+\,-)$.

(iii) In E_6 we can assume that the first copy of A_2 has $K_{A_2} = \{\pm(1-2), \pm(2-3), \pm(1-3)\}$, by Lemma 2.4. Then $A_2^\perp = A_2 \oplus A_2$. As in subcase (a) (iii) of case 2, $p = A_2^3$ is the only possibility. This is maximal in E_6 and hence primitive.

Case 3. $X_1 = A_3$

(a) $k_1 = 1$.

In E_8 we can assume by Lemma 2.4, that

$$K_{A_3} = \{\pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3\} ,$$

(i) In E_8, then $A_3^\perp = D_5$, with $K_{D_5} = \{\pm i \pm j \mid 4 \leqq i < j \leqq 8\}$. By Lemma 2.3, we have $W_{A_3 \oplus D_5} \subsetneqq W_{D_5} \subsetneqq W_{(I)_8}$. Thus $A_3 \oplus D_5$ is not primitive.

(ii) In E_7, $A_3^\perp = A_3 \oplus A_1$, where $K_{A_3}^\perp = \{\pm 4 \pm 5, \pm 5 \pm 6, \pm(7-8)\}$. Either this violates $k_1 = 1$ or it has been included in a former case.

(iii) In E_6, $A_3^\perp = A_1^2$ with $K_{A_3}^\perp = \{\pm 4 \pm 5\}$. This was treated in Case 1.

(b) $k_1 > 1$. Then $k_1 = 2$ since the rank of A_3^3 is greater than 8.

(i) In E_8 there are two non-conjugate ways to imbed A_3^2 in E_8. We see this as follows. By Lemma 2.4 we can assume that the first copy of A_3 has $K_{A_3} = \{\pm(i-j) \mid 1 \leqq i < j \leqq 4\}$. Let $\bar{h}$ be the Cartan subalgebra of the second copy of A_3. We can assume the coordinates are chosen so that the roots are given by

$$\pm(w_i - w_j)(1 \leqq i < j \leqq 4) .$$

Now $w_i - w_j$ is orthogonal to the roots of the first copy of A_3. So if $w_1 - w_2$ is a root of type I, by use of a signed permutation we can assume that $w_1 - w_2$ is $5 - 6$. If $w_1 - w_2$ is of type II, it must be of the form $\pm(+ + + + * * * *)$ in order to be orthogonal to the first copy of A_3. But then conjugating by an element β which agrees with $w_1 - w_2$ in all positions except 5 and 6 gives us a root of type I while keeping fixed the first copy of A_3. Thus we may assume $w_1 - w_2 = 5 - 6$.

Now if $w_2 - w_3$ is of type II, then it must be $\pm(+ + + + \varepsilon_5 \varepsilon_6 * *)$, where ε_5 and ε_6 disagree, in order that it not be orthogonal to $5 - 6$. Now conjugating by a root β which agrees with $w_2 - w_3$ everywhere except in positions 6 and 7 we fix the first copy of A_3 and $5 - 6$ as well. $w_2 - w_3$ becomes a root of type I, which can be assumed to be $6 - 7$, using a signed permutation. Thus, we can assume $w_2 - w_3 = 6 - 7$.

Now $w_3 - w_4$ must be orthogonal to $5 - 6$ but not to $6 - 7$, and, of course, orthogonal to the first copy of A_3. If $w_3 - w_4$ is of type II, then it must be of the form $\pm(+ + + + + + - -)$ or

$\pm(+ + + + - - + +)$. Conjugating by $(+ + + + + + + +)$ or $(+ + + + - - - -)$, respectively, we fix $5-6$, $6-7$ and the first copy of A_3, while conjugating $w_3 - w_4$ into a root of type I, in particular $\pm(7+8)$. Hence we may assume $w_3 - w_4$ is of type I. There are two cases: $w_3 - w_4 = 7+8$ or $7-8$.

Thus we have the two non-conjugate imbeddings of A_3^2 in E_8, namely

$$K_{A_3^2} = \{\pm(i-j) \mid 1 \leqq i < j \leqq 4,\ 5 \leqq i < j \leqq 8\}$$

and

$$K_{A_3^2} = \{\pm(i-j) \mid 1 \leqq i < j \leqq 4, 5 \leqq i < j \leqq 7\} \cup \{\pm(i+8) \mid 5 \leqq i \leqq 7\}\ .$$

These are not conjugate since in the first case there is a root, $(+ + + + + + + +)$, orthogonal to $K_{A_3^2}$, whereas in the second case there is no such root.

But then in the first case, $(A_3^2)^\perp \neq \emptyset$, and thus p has a factor of lower rank, which has already been covered in Cases 1 and 2. Thus only the second case need be considered. Here we claim that A_3^2 is not primitive, and in fact that $W_{A_3^2} \subsetneqq W_{D_4^2}$, where

$$K_{D_4^2} = \{\pm i \pm j \mid 1 \leqq i < j \leqq 4, 5 \leqq i < j \leqq 8\}\ .$$

Let $g \in W_{A_3^2}$. If g is a signed permutation, then clearly $g \in W_{D_4^2}$. Suppose $g = fS_\alpha$, where f is a signed permutation and $\alpha \in (II)_8$. Then S_α must take all roots of A_3^2 into roots of type I, and $\alpha = \pm(+ + + + + + + -)$ or $\pm(+ + + + - - - +)$ and neither of these is a root of E_8. Thus $g \neq fS_\alpha$. Finally, then, let $g = fS_\beta S_\alpha$, $(\beta, \alpha) = 0$, f a signed permutation. But β and α must agree completely or disagree completely on the first four positions, by the same reasons as were used in the proof of Lemma 2.3. Thus, as $(\alpha, \beta) = 0$, α and β must disagree completely or agree completely, respectively, on positions 5, 6, 7 and 8. But then $S_\beta S_\alpha \in W_{A_3^2}$, and therefore $f = gS_\alpha S_\beta$ is in $W_{A_3^2}$. We already saw that this implies $f \in W_{D_4^2}$ since α and β agree or disagree completely on 1, 2, 3 and 4. Thus $g \in W_{D_4^2}$, and A_3^2 is not primitive.

(ii) In E_7 we can assume for the first copy of A_3 that $K_{A_3} = \{\pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3\}$. Then as in Case 2, (a), (ii) above, $A_3^\perp = A_3 \oplus A_1$. Then either $p = A_3 \oplus A_3 \oplus A_1$, which was covered in Case 1, or $p = A_3 \oplus A_3$ and there is a root, namely the root of A_1, orthogonal to p, and hence p isn't primitive (Corollary 1.6).

(iii) In E_6 $A_3^\perp = A_1^2$, so A_3^2 isn't a subalgebra of E_6.

Case 4. $X_1 = A_4$.

(i) In E_8, by Lemma 2.4, we can assume

$$K_{A_4} = \{\pm(i - j) \mid 1 \leqq i < j \leqq 5\} .$$

Then $K_{A_4}^{\perp} = \{\pm 6 \pm 7, \pm 7 \pm 8, \pm 6 \pm 8$; all roots of type II having the same sign in positions 1 through 5$\}$. Thus $A_4^{\perp} = A_4$, and A_4^2 is the only possibility in this case. This subalgebra is maximal, and thus primitive.

(ii) In E_7, by Lemma 2.4, we can assume

$$K_{A_4} = \{\pm(i - j) \mid 1 \leqq i < j \leqq 5\} .$$

Then $K_{A_4}^{\perp} \neq \emptyset$, and there must be another summand. This has rank less than 4, and hence is covered by previous cases.

(iii) In E_6 Lemma 2.4 implies that there are two possible imbeddings of A_4, up to conjugacy. In both cases $K_{A_4}^{\perp} \neq \emptyset$, and thus we obtain only cases previously considered.

Case 5. $X_1 = D_4$.

We can assume by Lemma 2.3 that $K_{D_4} = \{\pm i \pm j \mid 1 \leqq i < j \leqq 4\}$.

(i) In E_8 $K_{D_4}^{\perp} = \{\pm i \pm j \mid 5 \leqq i < j \leqq 8\}$. Thus $D_4^{\perp} = D_4$, and the only case not previously considered is $D_4 \oplus D_4$. This algebra is primitive in E_8. To see this we use $S_\beta S_\alpha \in W_{D_4 \oplus D_4}$, where $\alpha = (+ + + + - + + -)$, $\beta = (+ + + + + - - +)$, and sign changes, all of which are in $W_{D_4 \oplus D_4}$.

(ii) In E_7, $K_{D_4}^{\perp} \neq \emptyset$. The only possibilities here were covered by earlier cases.

(iii) In E_6, $K_{D_4}^{\perp} = \emptyset$. In fact, D_4 is primitive in E_6, which can be seen by using $S_\beta S_\alpha$ from (i) above, together with sign changes on the first five coordinates.

Case 6. D_l, $l > 4$.

By Lemma 2.3 we can assume that $K_{D_l} = \{\pm i \pm j \mid 1 \leqq i < j \leqq l\}$. Let p be a primitive subalgebra of E_s with a D_l summand. Since $K_{D_l}^{\perp} \subset (I)_s$, we can apply Lemma 2.3 to get

$$W_p \subset W_{D_l} \subset W_{(I)_s} .$$

Since p is primitive and $p \subset (I)_s$, we have that $p = (I)_s$. Thus the only possibilities are D_8 in E_8, D_5 in E_6, (both are maximal and hence primitive) and $D_6 \oplus A_1$ in E_7 (which is covered by a previous case).

Case 7. $X_1 = A_l$, $l > 4$.

(i) In E_8, suppose $l \leqq 7$ and suppose

$$K_{A_l} = \{\pm(i - j) \mid 1 \leqq i < j \leqq l + 1\} .$$

Then $K_{A_l}^{\perp} \neq \emptyset$, as it contains $(+ + + + + + + +)$, and since $l > 4$, the rank of $A_l^\perp$ is less than 4, and these possibilities have been covered by previous cases. Thus, by Lemma 2.4, we have one other possibility, namely $l = 7$ and

$$K_{A_7} = \{\pm(i - j) \mid 1 \leqq i < j \leqq 7\} \cup \{\pm(i + 8) \mid 1 \leqq i \leqq 7\} ,$$

$K_{A_7}^\perp = \varnothing$. We claim that A_7 is not primitive. In particular, we show $W_{A_7} \subset W_{D_8}$.

To see this, let $g \in W_{A_7}$. Then if g is a signed permutation, $g \in W_{D_8}$. Let $g = fS\alpha$, f a signed permutation, $\alpha \in (II)_8$. Then S_α must not take any root of K_{A_7} into a root of type II. The only possibilities are $\alpha = \pm(+ + + + + + + -)$, which is not a root. Finally, suppose $g = fS_\beta S_\alpha$, $(\beta, \alpha) = 0$, f a signed permutation. But β and α must either agree completely or disagree completely in all positions, or else some root of K_{A_7} would be taken by $S_\beta S_\alpha$ into a root of type II, and $fS_\beta S_\alpha \notin W_{A_7}$. This violates $(\alpha, \beta) = 0$. Hence if $g \in W_{A_7}$, then g is a signed permutation, and $W_{A_7} \subsetneqq W_{D_8}$.

The only other case of $A_l \subsetneqq E_8$ is $l = 8$. Since $A_6 \subset A_7 \subset A_8$, using Lemma 2.4, we can assume that K_{A_8} contains

$$\{\pm(i - j) \mid 1 \leqq i < j \leqq 7\} .$$

Then it is easy to see that this can only happen when

$$\begin{aligned} K_{A_8} = &\{\pm(i - j) \mid 1 \leqq i < j \leqq 7\} \cup \{i + 8 \mid 1 \leqq i \leqq 7\} \\ &\cup \{\pm(+ + + + + + + +), \pm(+ + + + + + - -)\} . \end{aligned}$$

Then A_8 is maximal, and hence primitive.

(ii) In E_7, $A_5 \subsetneqq A_l$, and we can assume by Lemma 2.4 that $K_{A_5} \subset \{\pm i \pm j \mid 1 \leqq i < j \leqq 6\}$. Thus $K_{A_5}^\perp \neq \varnothing$, as it contains $7 - 8$. Since $K_{A_5}^\perp$ has rank less than 4, we have already covered these possibilities by previous cases.

Let $\alpha_1, \alpha_2, \cdots, \alpha_6$ be the simple roots of A_6. By Lemma 2.4 we can assume that for $i \leqq 4$, $\alpha_i = i - (i + 1)$, and α_5 is either $5 - 6$ or $5 + 6$. We have $(\alpha_6, \alpha_i) = 0$ for $i \leqq 4$, and $(\alpha_6, \alpha_5) \neq 0$. Thus α_6 must be a root of type II of the form $\pm(+ + + + + - + -)$ or $\pm(+ + + + + - - +)$ (since we are in E_7), and thus α_5 must in fact be $5 - 6$.

Now $A_7 \supset A_6$, and hence we can assume $K_{A_7} = K_{A_6} \cup \{\pm(7 - 8)\}$. A_7 is maximal, and thus primitive.

A_6 itself is not primitive, and in fact $W_{A_6} \subsetneqq W_{A_7}$. For let $g \in W_{A_6}$. If g is a signed permutation, then $g \in W_{A_7}$ also. Suppose $g = fS_\alpha$, f a signed permutation, $\alpha \in (II)_7$. S_α must take one of $1 - 2, 2 - 3, \cdots,$ or $5 - 6$ into a root of type II, for not all of the first six positions of α can have the same sign (in $(II)_7$). Thus we can assume $S_\alpha(1-2)$ is in $(II)_7$. Now if $S_\alpha(3-4)$, $S_\alpha(4-5)$ or $S_\alpha(5-6)$ is a root of type II, say $fS_\alpha(3-4)$, then $fS_\alpha(3-4)$ and $fS_\alpha(1-2)$ would be two orthogonal roots of type II in A_6. Such a pair of roots does not exist. Thus S_α must not take $3-4, 4-5, 5-6$ into roots of type II. α has the same sign on positions 3, 4, 5, 6, and must have in positions 1 and 2 different signs. There are four possibilities:

$$\begin{aligned}
&\text{(a)} \quad \pm(+\ -\ +\ +\ +\ +\ +\ -)\\
&\text{(b)} \quad \pm(+\ -\ -\ -\ -\ -\ +\ -)\\
&\text{(c)} \quad \pm(-\ +\ +\ +\ +\ +\ +\ -)\\
&\text{(d)} \quad \pm(-\ +\ -\ -\ -\ -\ +\ -)\,.
\end{aligned}$$

We note that in all four cases, $\alpha \in K_{A_7}$. In cases (a) and (c), $\alpha \in K_{A_6}$, hence $S_\alpha \in W_{A_6}$, and $f \in W_{A_6}$. By the remark above, $f \in W_{A_7}$. Since $K_{A_6} \subset K_{A_7}$, $S_\alpha \in W_{A_7}$. Thus $g \in W_{A_7}$.

In case (b), for $i = 2, 3, 4, 5$, $fS_\alpha(i-(i+1)) = f(i-(i+1))$, and these must not be $\pm(7-8)$, as $\pm(7-8)$ is orthogonal to all other roots of type I. Thus f fixes $\pm(7-8)$. Now $f = f''f'$, where f' is a permutation, and f'' changes some signs. Clearly, if f'' changes any signs on $f'(2), \cdots, f'(6)$, then it changes them all, or else some $f(i-(i+1)) \notin K_{A_6}$. Since f'' must change an even number of signs, and f' must fix $\{7, 8\}$, then if f'' changes any signs on $f'(2), \cdots, f'(6)$, it must also change sign on either $f'(1)$ or 7, 8. In either case, f'' changes sign on $\{f'(1), , \cdots, f'(6)\} = \{1, \cdots, 6\}$. Thus f'' either changes all signs or no signs on $1, 2, \cdots, 6$. Then $f \in W_{A_6}$, and therefore $f \in W_{A_7}$. But $\alpha \in K_{A_7}$, and thus S_α and fS_α are in W_{A_7}.

In case (d), the argument is the same with 1 and 2 interchanged.

Finally, let $g = fS_\alpha S_\beta$, $(\alpha, \beta) = 0$. Then since α and β are orthogonal, they can't totally agree or totally disagree in sign on 1, 2, 3, 4, 5 and 6. Hence there is some root $i-j$ which $S_\beta S_\alpha$ takes into a type II root. We can assume $S_\beta S_\alpha(1-2) \in (II)_7$. Now since all of $i-j$, $3 \leqq i < j \leqq 6$ are orthogonal to $1-2$, none of them are taken into roots of type II, as there aren't two orthogonal type II roots in K_{A_6}, as noted above. Thus α and β must totally agree or totally disagree in sign on 3, 4, 5 and 6. Since $S_\beta S_\alpha(1-2) \in (II)_7$, β and α must agree on exactly one of positions 1 and 2. Thus α and β totally agree or totally disagree on five positions, contradicting $(\alpha, \beta) = 0$. Thus g cannot be of the form $fS_\beta S_\alpha$.

(iii) In E_6 let $A_5 \subset E_6$. We have $A_4 \subset A_5$. Now by Lemma 2.4, K_{A_4} is either $\{\pm(i-j) \mid 1 \leqq i < j \leqq 5\}$ or $\{\pm(i-j) \mid 1 \leqq i < j \leqq 4\} \cup \{(i+5) \mid 1 \leqq i \leqq 4\}$.

In the first case, there is a simple root in A_5 orthogonal to $1-2$, $2-3$, $3-4$ and not to $4-5$. This root must be of type II, so it is $\pm(+ + + + - + + -)$, and A_5 is the subalgebra generated by $1-2$, $2-3$, $3-4$, $4-5$ and $(+ + + + - + + -)$. But then $K_{A_5}^{\perp}$ contains $(+ + + + + - - +)$, and thus we have considered this in previous cases.

In the second case, the argument is the same.

Finally, A_6 cannot be imbedded in E_6. For $A_5 \subset A_6$, and we can assume that K_{A_5} is one of the two possibilities described above. Then there would be a simple root α in K_{A_6} orthogonal to $1-2$, $2-3$, $3-4$, and $4-5$ (resp. $4+5$), and not orthogonal to $(+ + + + - + + -)$ (resp. $(+ + + + + - - +)$). This is impossible in E_6.

Case 8. $X_1 = E_7$.

We need only consider $E_7 \subset E_8$. There are seven mutually orthogonal roots of E_7. By Lemma 2.1 we can assume that these roots are all of type *I*. There is a root of type *I* orthogonal to these seven. This root is in $K_{E_7}^{\perp}$, and thus we are done by previous cases.

Case 9. $X_1 = E_6$.

Let $E_6 \subset E_s$, $s = 7, 8$. $D_5 \subset E_6$. By Lemma 2.3 we can assume that $K_{D_5} = \{\pm i \pm j \mid 1 < i < j \leqq 5\}$. There is no further root of type *I* in E_6 or else either we obtain $D_6 \subset E_6$ or we obtain five mutually orthogonal roots in E_6, both impossible. Thus all other roots of E_6 are of type *II*. We can assume that $(+ + + + + - - +)$ is in K_{E_6}. But then $K_{E_6}^{\perp} \neq \varnothing$ in E_8, and hence we are done by previous cases. In E_7, E_6 is maximal and hence primitive.

3. F_4. The roots of F_4 are described as follows. Let z_1, z_2, z_3, z_4 be the standard orthonormal basis for the dual space to a fixed Cartan subalgebra h of F_4. With respect to this basis the roots of F_4 are given by

$$\begin{aligned}
(\mathrm{I}) &= \{\pm z_i \mid i = 1, 2, 3, 4\} \\
(\mathrm{II}) &= \{\pm z_i \pm z_j \mid 1 \leqq i, j \leqq 4\} \\
(\mathrm{III}) &= \left\{\frac{1}{2}(\pm z_1 \pm z_2 \pm z_3 \pm z_4)\right\} .
\end{aligned}$$

As in the case of the E_s, we denote the roots of type *III* by the corresponding sequence of $+$ and $-$ signs, the roots of type *II* by the corresponding $i - j$, and the roots of type *I* merely by the corresponding $\pm j$. Thus $1/2(z_1 + z_2 + z_3 - z_4)$ is denoted $(+ + + -)$, $z_3 - z_2$ is denoted $3 - 2$, z_4 is denoted 4, and so on. Also, we use $(I) \cup (II)$, (II), etc. to denote the subalgebras determined by these roots, when no confusion results.

We note that the roots have two lengths. Roots in (I) and (III) have length 1, and roots in (II) have length $\sqrt{2}$. The Weyl group W of F_4 acts transitively on the roots of each length.

As in the case of E_8 above, using Theorem 1.4 we see that if p is a maximal rank, reductive, primitive subalgebra, and if $p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r}$, then either

$$(X_1^{k_1})^\perp = X_2^{k_2} \oplus \cdots \oplus X_r^{k_r}, \quad \text{or} \quad (X_1^{k_1} \oplus (X_1^{k_1})^\perp) + h$$

generates (as a subalgebra) F_4

Case 1. $X_1 = A_1$.

(a) $k_1 = 1$.

By transitivity of the Weyl group on roots of each length, we may assume that $K_{A_1} = \{\pm 1\}$ or $K_{A_1} = \{\pm(1 - 2)\}$. In the first case $K_{A_1}^\perp = \{\pm 1,\ (i \pm j) \mid 2 \leqq i,\ j \leqq 4\}$. Thus $A_1^\perp = B_3$. But $K_{A_1} \cup K_{B_3}$ generates $(I) \cup (II)$, which is not $K_{A_1} \cup K_{B_3}$ nor K_{F_4}. Thus $A_1 \oplus Z$ can't be primitive, by Theorem 1.4. In the second case,

$$\begin{aligned} K_{A_1}^\perp = \{&\pm(1 + 2), \pm 3, \pm 4, \pm 3 \pm 4, \pm(+ + + +), \\ &\pm(+ + + -), \pm(+ + - +), \pm(+ + - -)\}\,. \end{aligned}$$

Then $A_1^\perp = C_3$, and $A_1 \oplus C_3$ is maximal, and hence primitive.

(b) $k_1 = 2$.

First we observe that if $K_{A_1} = \{\pm\alpha\}$ and the other $K_{A_1} = \{\pm\beta\}$, then not both α and β can be shorter roots, or else $[A_1, A_1] \neq 0$ contradicting fact that the two copies of A_1 were ideals in p. Thus it is sufficient to consider only three cases for α, β (up to conjugacy):

(1) $1,\ 2 - 3$

(2) $1 + 2,\ 1 - 2$

(3) $1 - 2,\ 3 - 4$.

In the first case, $K_{A_1^2}^\perp = \{\pm 4\}$, and the Lie algebra generated by $A_1^2 \oplus (A_1^2)^\perp = A_1^2 \oplus A_1$ is more than $A_1^2 \oplus A_1$ (e.g., $1 + 4$), but not all of F_4 (e.g., $(+ + + +)$). Thus $A_1^2 \oplus Z$ can't be primitive.

In the second case $K_{A_1^2}^\perp = \{\pm 3, \pm 4, \pm 3 \pm 4\}$, and $(A_1^2)^\perp = B_2$. In the third case, $K_{A_1^2}^\perp = \{\pm(1 + 2), \pm(3 + 4), \pm(+ + + +), \pm(+ + - -)\}$, and again $(A_1^2)^\perp = B_2$. Since the Weyl group is transitive on the shorter roots, there is some element w such that $w(+ + + +) = 1$. Then $w(+ + - -)$ is a shorter root orthogonal to 1, and hence, using a signed permutation if necessary, we may assume $w(+ + - -) = 2$. Then since $w(1 + 2)$ and $w(3 + 4)$ are orthogonal to neither 1 nor 2, and they are orthogonal to each other, we must have $w(\{\pm(3 + 4), \pm(1 + 2)\}) = \{\pm 1 \pm 2\}$. Hence w takes the B_2 from case (3) into the B_2 of case (2). Thus w takes $B_2^\perp$ into $B_2^\perp$, which is just A_1^2 in each case. Hence cases (2) and (3) are conjugate, and we only need to consider case (2).

In this case we claim $A_1^2 \oplus B_2$ is not primitive. In particular $A_1^2 \oplus B_2$ is a subalgebra of the algebra $(I) \cup (II)$, and $W_{A_1^2 \oplus B_2} \subset W_{(I) \cup (II)}$. For let $w \in W_{A_1^2 \oplus B2}$. Then w must preserve each summand, and within each summand it must preserve the roots of each length. Hence w takes $\{\pm 3, \pm 4\}$ onto itself. Also w preserves $\{\pm 1 \pm 2\}$.

If $w(1) \in (III)$ (and thus $w(2) \in (III)$, as $w(1)$ and $w(2)$ are orthogonal), then we may assume $w(1) = (++++)$, $w(2) = (++--)$, or $w(2) = (+--+)$. If $w(2) = (++--)$, then $w(1-2) = 3+4$, contradicting the fact that w fixes $\{\pm 3, \pm 4\}$. If $w(2) = (+--+)$, then $w(1-2) = 2+3$, which is not orthogonal to $w(3-4) \in \{\pm 3 \pm 4\}$. Thus neither case can occur, and $w(1)$ and $w(2)$ are of type I. Hence w is a signed permutation, therefore fixing $(I) \cup (II)$ as well. Thus $w_{A_1^2 \oplus B_2} \subset W_{(I) \cup (II)}$, and $A_1^2 \oplus B_2$ is not primitive.

(c) $k_1 = 3$.

Let the three copies of K_{A_1} be $\{\pm\alpha\}$, $\{\pm\beta\}$, $\{\pm\gamma\}$. Then at most one of α, β, γ can be a shorter root, or else two copies of A_1 would have $[A_1, A_1] \neq 0$, as above in the $k_1 = 2$ case. Thus there are two cases to consider (up to conjugacy) for α, β, γ:

(1) $1,\ 2-3,\ 2+3$

(2) $1-2,\ 1+2,\ 3-4$,

In the first case, $K_{A_1^3}^{\perp} = \{\pm 4\}$. Then $(A_1^3)^{\perp} = A_1$, and $A_1^3 \oplus A_1$ generates as a Lie algebra $B_2 \oplus A_2$. Thus there is no primitive subalgebra in this case.

In the second case, $K_{A_1^3}^{\perp} = \{\pm(3+4)\}$, and $(A_1^3)^{\perp} = A_1$. Here we get A_1^4 which is treated below.

(d) $k_1 = 4$.

A_1^4 contains A_1^3 as a summand, and by the $k_1 = 3$ case we can assume $K_{A_1^4} = \{\pm 1 \pm 2, \pm 3 \pm 4\}$. But this is not primitive, since $A_1^4 \subsetneqq (II)$, and $W_{II} = W_{F_4} \supset W_{A_1^4}$.

Case 2. $X_1 = A_2$.

If $K_{A_2} \subsetneqq (I) \cup (III)$, then we may assume $K_{A_2} = \{\pm 1, \pm(++++), \pm(-+++)\}$. Then $K_{A_2}^{\perp} = \{\pm(2-3), \pm(3-4), \pm(2-4)\}$. Thus we get $A_2 \oplus A_2$. Similarly, if $K_{A_2} \subsetneqq (II)$, then we may assume $K_{A_2} = \{\pm(2-3), \pm(3-4), \pm(2-4)\}$, and then

$$K_{A_2}^{\perp} = \{\pm 1, \pm(++++), \pm(-+++)\}\,.$$

Hence we get in either case $A_2 \oplus A_2$ with one $K_{A_2} \subset (I) \cup (III)$ and the other $K_{A_2} \subset (II)$. This algebra is maximal and thus primitive.

Case 3. $X_1 = B_2$.

As we observed in Case 1 above, all $B_2 \subset F_4$ are conjugate. Thus we can assume $K_{B_2} = \{\pm 1, \pm 2, \pm 1 \pm 2\}$. Then $K_{B_2}^{\perp} = \{\pm 3, \pm 4, \pm 3 \pm 4\}$, and we have $B_2 \oplus B_2$. But this generates $(I) + (II) = B_4$. Thus there are no primitive subalgebras in this case.

Case 4. $X_1 = A_3$.

If $K_{A_3} \subset (II)$, then we may assume $K_{A_3} = \{\pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3\}$. Then $K_{A_3}^{\perp} = \{\pm 4\}$, and we have $A_1 \oplus A_3$. This is covered by Case 1. On the other hand, K_{A_3} can have at most one root (and its negative) in (I), or else it would also have one in (II), contradicting the fact that all roots of A_3 have the same length. Thus, if $K_{A_3} \subset (I) \cup (III)$, then all but at most one root (and its negative) are in (III). This is impossible.

Case 5. $X_1 = A_4$.

This is not contained in F_4 as a maximal rank subalgebra.

Case 6. $X_1 = B_3$.

Just as in Case 3, there is only one way, up to conjugacy, to have $B_3 \subset F_4$, namely $K_{B_3} = \{\pm i, \pm i \pm j \mid 2 \leqq i, j \leqq 4\}$. Then $K_{B_3}^{\perp} = \{\pm 1\}$. Thus $B_3^{\perp} = A_1$. $A_1 + B_3$ generates $(I) + (II) = B_4$. Thus there is no primitive subalgebra in this case.

Case 7. $X_1 = B_4$.

As above, all $B_4 \subset F_4$ are conjugate, and we may assume $K_{B_4} = (I) \cup (II)$. This is maximal, and thus primitive.

Case 8. $X = D_4$.

$K_{D_4} = (II)$ is the only possibility. This is primitive, since $W_{D_4} = W_{F_4}$, and W_{F_4} is transitive on shorter roots. However D_4 is not maximal, since $D_4 \subset B_4 = (I) \cup (II)$.

Case 9. $X_1 = C_3$.

All C_3 are conjugate $K_{C_3}^{\perp} = K_{A_1}$ as in Case 1 (a).

Case 10. $X_1 = C_4$.

This is not a maximal rank subalgebra of F_4.

4. $\mathbf{G}_2$. Let α_1, α_2 be simple roots of G_2. Then $K_{G_2} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}$

Case 1. $p = A_2$.

The roots of A_2 are all of the same length.

The only way to imbed A_2, then, is as roots of longer length, i.e., $K_{A_2} = \{\pm\alpha_2, \pm(3\alpha_1 + \alpha_2) \pm (3\alpha_1 + 2\alpha_2)\}$. This is maximal and hence primitive.

Case 2. $p = A_1$.

There are two possibilities up to conjugacy for $K_{A_1} = \{\pm\beta\}$, namely, $\beta = \alpha_1$ and $\beta = \alpha_2$, since the Weyl group acts transitively on roots of each length, and there are two lengths of roots in K_{G_2}, that of α_1 and that of α_2.

If $\beta = \alpha_2$, this is not primitive, since $W_{A_1} \subsetneqq W_{A_2} = W_{G_2}$. If $\beta = \alpha_1$, then the only root γ with $[\gamma, \alpha_1] = [\gamma, -\alpha_1] = 0$ is $\gamma = \pm(3\alpha_1 + 2\alpha_2)$. Thus we have A_1^2 with $K_{A_1^2} = \{\pm\alpha_1, \pm(3\alpha_1 + 2\alpha_2)\}$.

Further, if $f \in W_{A_1}$, i.e., $f(\pm\alpha_1) = \pm\alpha_1$, then

$$\begin{aligned} 0 = f(0) = f[\pm\alpha_1, 3\alpha_1 + 2\alpha_2] &= [f(\pm\alpha_1), f(3\alpha_1 + 2\alpha_2)] \\ &= [\pm\alpha_1, f(3\alpha_1 + 2\alpha_2)] \, . \end{aligned}$$

Thus $f(\pm(3\alpha_1 + 2\alpha_2) = \pm(3\alpha_1 + 2\alpha_2)$. Hence $f \in W_{A_1^2}$. Thus A_1 is not primitive.

Case 3. $p = A_1^2$.

Since $K_{A_1^2}$ is not contained in the roots of one length, we have, up to conjugacy, $K_{A_1^2} = \{\pm\alpha_1, \pm(3\alpha_1 + 2\alpha_2)\}$. This is maximal, and thus primitive.

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