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DEFINING RELATIONS FOR CERTAIN INTEGRALLY PARAMETERIZED CHEVALLEY GROUPS

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For each faithful finite dimensional irreducible representation R of a finite dimensional simple Lie algebra L over the complex field, this paper treats the integrally parameterized subgroup G_Z of the Chevalley Group G over the rational field Q. For L of type A_l, D_l, or E_l, Lie algebraic methods are used to extend a result of J. Nielson on $SL(3, Z)$ to obtain a finite set of defining relations for G_Z. Similar relations augmented by defining relations for $G_Z(B_2)$ are shown to define G_Z when L is of type B_l, C_l, or F_4. (The relations for $G_Z(B_2)$ are not listed here.)

Defining relations for the n-dimensional group of lattice transformations have been given by W. Magnus in [4]. His method easily yields relations for the group $SL(n, Z)$ respectively $PSL(n, Z)$ isomorphic to the universal respectively adjoint group G_Z for L of type A_{n-1}. H. Klingen [2] has proven the existence of a finite set of defining relations for $Sp(2n, Z)$, which is essentially the group G_Z for L of type C_n. Hence, the defining relations in § 2 extend Magnus' result to G_Z of types D_l and E_l and Klingen's result to G_Z of types B_l and F_4.

It might be helpful to the reader to note that a displayed equation is referred to by a symbol in parentheses, e.g., "(3.1)" or "(B)" and a theorem, lemma, or corollary is referred to by its title and a number without parentheses, e.g., "Lemma 3.1".

2. **Statement of results.** Let R be a faithful finite dimensional irreducible representation of a finite dimensional simple Lie algebra L over the complex field C, and let Σ be the set of nonzero roots of L with respect to some Cartan subalgebra. L has a *Chevalley basis* $\{X_\rho, H_\rho: \rho \in \Sigma\}$ as defined in [1, p. 24, Th. 1] or [9, p. 6, Th. 1]. The L module V associated with R contains a lattice M which is invariant under the action of the Chevalley basis. If M is properly chosen and K is an arbitrary field, the automorphism $x_\rho(t) = \exp tR(X_\rho)$ on $V_K = K \otimes_Z M$ can be defined for each ρ in Σ and t in K. The group G_K generated by all of these automorphisms is the Chevalley group over the field K of type L corresponding to the representation R. G_K is the *adjoint* respectively *universal* Chevalley group if R is the adjoint respectively universal representation of L. (See [9, pp. 42-45].)

We will be concerned with the *rational Chevalley group* G_Q (henceforth denoted by G) and its subgroup, the *integrally parameterized Chevalley group* G_Z generated by the $x_\rho(t)$ with ρ in Σ and t in Z.

The relations $x_\rho(s)x_\rho(t) = x_\rho(s+t)$ in G show the finite set $\{x_\rho(1): \rho \in \Sigma\}$ suffices to generate G_Z. Our goal is to find defining relations for G_Z in terms of these generators.

We will let P denote the set of positive roots and Π, the set of simple roots with respect to some (henceforth fixed) regular ordering (as defined in [1, p. 20] or [9, p. 266]) of Σ. Greek letters α and β will denote arbitrary simple roots, and γ, δ, ρ, σ, and τ will be generic symbols for any roots.

A set S of roots is called *closed* if $\rho, \sigma \in S$ and $\rho + \sigma \in \Sigma$ implies that $\rho + \sigma \in S$. A closed set S of roots is an *admissible system* if $-\rho \in S$ whenever $\rho \in S$. S is a *positive set of roots* if S is closed and $\rho \in S$ implies $-\rho \notin S$. If S is a positive set of roots, it is possible to find a regular ordering of Σ which makes all of the roots in S positive; such a regular ordering will be called a *relative ordering* corresponding to the positive set S, to distinguish it from the fixed regular ordering. Finally, we define L_R to be the additive group generated by the set of all weights of the representation R.

Consider the abstract generators $\{x_\rho: \rho \in \Sigma\}$ and define $w_\rho = x_\rho x_{-\rho}^{-1} x_\rho$ and $h_\rho = w_\rho^2$ for each $\rho \in \Sigma$. Designate the following relations:

$$\text{(A}'\text{)} \quad (x_\rho, x_\sigma) = \Pi x_{i\rho+j\sigma}^{C(i,j;\rho,\sigma)} \; (\rho, \sigma \in \Sigma,\ \rho + \sigma \neq 0)$$

where (x, y) denotes the commutator $xyx^{-1}y^{-1}$ and the product is over all positive integers i and j such that $i\rho + j\sigma \in \Sigma$, taken in increasing order of the roots $i\rho + j\sigma$. The $C(i, j; \rho, \sigma)$ are integers depending only on i, j, ρ, σ, the choice of Chevalley basis, and the structure of L. (See [1, p. 27] or [9, p. 22].)

$$\begin{array}{lll} \text{(A}''\text{)} & w_\rho x_\rho w_\rho^{-1} = x_{-\rho}^{-1} & (\rho \in \Sigma)\,. \\ \text{(B)} & h_\rho^2 = 1 & (\rho \in \Sigma) \\ \text{(C)} & \Pi h_\alpha = 1 \quad \text{if} & \Pi(-1)^{c(\mu,\alpha)} = 1 \text{ for all } \mu \in L_R\,, \end{array}$$

where $c(\beta, \alpha)$ is the Cartan integer $2(\beta, \alpha)/(\alpha, \alpha)$ and both products are over an increasing sequence (α) of (not necessarily all) simple roots. Let (A) denote the relations (A$'$) respectively (A$''$) when $rk\ \Sigma > 1$ respectively $rk\ \Sigma = 1$ (rk means *rank*). If L is of type A_l, D_l, E_l, or G_2, let (D) be the empty set of relations. If L is of type B_l, C_l, or F_4, let α and β be the simple roots forming a system of type B_2 with long root β, and let $G_Z(\alpha, \beta)$ be the subgroup of G_Z generated by $\{x_\alpha(1), x_{-\alpha}(1), x_\beta(1), x_{-\beta}(1)\}$. In this case, let (D) be the relations in $\{x_\alpha, x_{-\alpha}, x_\beta, x_{-\beta}\}$ obtained by replacing each $x_\rho(1)$ by x_ρ in a set of defining relations for $G_Z(\alpha, \beta)$. The principal result is

MAIN THEOREM. *Let L be a finite dimensional simple Lie algebra over C which is not of type G_2. Then the integrally parameterized*

Chevalley group G_Z *is isomorphic* (by the *canonical map* defined by $x_\rho \mapsto x_\rho(1)$) *to the abstract group* G' *generated by* $\{x_\rho: \rho \in \Sigma\}$ *subject to the relations* (A), (B), (C), *and* (D).

The relations (C) can be omitted if G is the universal Chevalley group. If G is the adjoint Chevalley group, then $L_R = Z\Pi$ and can be replaced by Π in the relations (C).

The main theorem is proved by showing (in §4) that a normal form for writing elements of G_Z (given in Theorem 3.3) can be duplicated in the abstract group G'.

In accomplishing the latter, it is shown that a set E of compatible relations (See §4.) containing (A), (B), and (C) suffices to define G_Z if it suffices to define the rank two subgroups of G_Z. This technique of parlaying defining relations for the rank two subgroups to defining relations for the whole group is reminiscent of Magnus' extension in [4] of Nielsen's result [6] and of Klingen's treatment in [2] of $Sp(2n, Z)$.

An explicit set of defining relations for $Sp(4, Z)$ would probably allow the relations (D) for L of type B_l, C_l, and F_4 to be explicitly stated, as suggested by R. Ree's identification $PSp(4, Q) \cong G_Z(C_2) \cong G_Z(B_2)$ in [7]. It seems likely that the relations (D) are in fact unnecessary.

3. A normal Form for G and G_Z. In this section we develop several notions, notations, and a normal form in the concrete group G_Z which we will use to study the abstract group in §4. Many of the results displayed in this section are known, and most of them appear in sources such as [1], [8], [9], and [10].

Let U be the subgroup of G generated by $\{x_\rho(r): \rho \in P, r \in Q\}$ and U_Z the subgroup of G_Z generated by $\{x_\rho(1): \rho \in P\}$. Corresponding to each root ρ we define the one parameter subgroups $\mathfrak{X}_\rho = \{x_\rho(r): r \in Q\}$ of G and $\mathfrak{X}_\rho(Z) = \{x_\rho(r): r \in Z\}$ of G_Z. More generally, for any $S \subseteqq \Sigma$, $\mathfrak{X}_S$ respectively $\mathfrak{X}_S(Z)$ is the subgroup of G respectively G_Z generated by $\{x_\rho(r): \rho \in S, r \in Q\}$ respectively $\{x_\rho(1): \rho \in S\}$. Then $G(\rho, \sigma, \cdots) = \mathfrak{X}_S$ and $G_Z(\rho, \sigma, \cdots) = \mathfrak{X}_S(Z)$, where S is the admissible system of roots generated by $\rho, \sigma, \cdots$.

Consider the homomorphism φ_ρ from $SL(2, Q)$ into G defined (See [1, pp. 33-37].) for each $\rho \in \Sigma$ by

$$(3.1) \qquad \varphi_\rho \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = x_{-\rho}(r), \varphi_\rho \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = x_\rho(r) \quad (r \in Q) .$$

φ_ρ maps $SL(2, Q)$ onto $G(\rho)$. Its restriction to $SL(2, Z)$, which we also denote by φ_ρ, is a homomorphism into G_Z. Since $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL(2, Z)$ (See [6, p. 8] or [3, vol. 2, Appendix B].), φ_ρ maps $SL(2, Z)$ onto the subgroup $G_Z(\rho)$ of G_Z generated by $x_{-\rho}(1)$ and $x_\rho(1)$. Now define

$$h_\rho(t) = \varphi_\rho\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix},\ w_\rho(t) = \varphi_\rho\begin{pmatrix} 0 & t \\ -t^{-1} & 0\end{pmatrix}, \tag{3.2}$$

for each $\rho \in \Sigma$ and $t \in Q^*$, where Q^* is the set of nonzero elements of Q. Then H respectively H_Z is the subgroup of G respectively G_Z generated by $\{h_\rho(t): \rho \in \Sigma, t \in Q^*\}$ respectively $\{h_\rho(-1): \rho \in \Sigma\}$, and N respectively N_Z is the subgroup of G respectively G_Z generated by $\{w_\rho(t): \rho \in \Sigma, t \in Q^*\}$ respectively $\{w_\rho(1): \rho \in \Sigma\}$.
The identities

$$w_\rho(t) = x_\rho(t)x_{-\rho}(-t^{-1})x_\rho(t) \qquad (\rho \in \Sigma,\ t \in Q^*)\ , \tag{3.3a}$$

$$h_\rho(t) = w_\rho(t)w_\rho(-1) \qquad (\rho \in \Sigma,\ t \in Q^*)\ , \tag{3.3b}$$

follow from corresponding identities in $SL(2, Q)$.

R. Steinberg showed in [9, p. 66 Th. 8 and p. 43 Lemma 28] (Also see [8].) that the group G is defined by the generators $\{x_\rho(r): \rho \in \Sigma, r \in Q\}$ subject to the relations

$$x_\rho(r)x_\rho(s) = x_\rho(r+s) \tag{3.4a}$$

$$(x_\rho(r), x_\sigma(s)) = \Pi x_{i\rho+j\sigma}(C(i, j; \rho, \sigma)r^i s^j) \qquad (\rho + \sigma \neq 0) \tag{3.4b}$$

$$w_\rho(t)x_\rho(r)w_\rho(t)^{-1} = x_{-\rho}(-t^2 r) \tag{3.4b'}$$

$$h_\rho(t)h_\rho(t') = h_\rho(tt') \tag{3.4c}$$

$$\Pi h_\alpha(t_\alpha) = 1 \text{ if } \Pi t_\alpha^{c(\mu,\alpha)} = 1 \text{ for every } \mu \in L_R\ , \tag{3.4d}$$

where $w_\rho(t)$ and $h_\rho(t)$ are defined by (3.3ab), $\rho, \sigma \in \Sigma$, $r, s \in Q$, $t, t', t_\alpha \in Q^*$. The product (3.4b) is over all positive integers i, j such that $i\rho + j\sigma$ is a root, taken in increasing root order, and the product in (3.4d) is taken over all $\alpha \in \Pi$ in increasing root order. The relations (3.4b) respectively (3.4b′) can be omitted if $rk\ \Sigma = 1$ respectively $rk\ \Sigma > 1$, and the relations (3.4d) can be omitted if G is the universal Chevalley group. If R is the adjoint representation, it suffices to have $\Pi t_\alpha^{c(\mu,\alpha)} = 1$ for all $\mu \in \Pi$ in (3.4d).

Let W denote the Weyl group of Σ and let ω_ρ denote the reflection in the hyperplane perpendicular to the root ρ. For $\rho, \sigma \in \Sigma$, $\sigma' = \omega_\rho(\sigma)$, $c = c(\sigma', \rho)$, $d = c(\sigma, \rho)$, $t, s \in Q^*$, and $n = n(\rho, \sigma) = \pm 1$, the relations

$$w_\rho(t)x_\sigma(s)w_\rho(-t) = x_{\sigma'}(nt^c s)\ , \tag{3.5a}$$

$$w_\rho(t)w_\sigma(s)w_\rho(-t) = w_{\sigma'}(nt^c s)\ , \tag{3.5b}$$

$$w_\rho(t)h_\sigma(s)w_\rho(-t) = h_{\sigma'}(nt^c s)h_{\sigma'}(nt^c)^{-1}\ , \tag{3.5c}$$

$$h_\rho(t)x_\sigma(s)h_\rho(t)^{-1} = x_\sigma(t^d s)\ , \tag{3.5d}$$

$$h_\rho(t)w_\sigma(s)h_\rho(t)^{-1} = w_\sigma(t^d s)\ , \tag{3.5e}$$

$$h_\rho(t)h_\sigma(s)h_\rho(t)^{-1} = h_\sigma(t^d s)h_\sigma(t^d)^{-1}\ , \tag{3.5f}$$

hold in G. (See [1], [8, p. 119], or [9, p. 67] (3.5c) corrects a misprint in [8].) The last five relations are immediate from the first and the properties

$$(3.6)\qquad n(\rho, \sigma) = n(\rho, -\sigma) = (-1)^{c(\sigma,\rho)} n(\rho, \sigma'),\ n(\rho, \rho) = -1\ ,$$

of the function $n\colon \Sigma \times \Sigma \to \{-1, 1\}$, using (3.3ab).

Now it is clear from (3.5d) that $B = HU$ respectively $B_Z = H_Z U_Z$ is a group containing U respectively U_Z as a normal subgroup, and, by [1, p. 42], $U \cap H = U_Z \cap H_Z = \{1\}$. An element b of B can be uniquely represented in the form

$$(3.7)\qquad b = hu$$

with $h \in H$ and $u \in U$ expressed as

$$(3.7a)\qquad h = \Pi h_\alpha(t_\alpha) \qquad (t_\alpha \in Q^*)\ ,$$

$$(3.7b)\qquad u = \Pi x_\rho(r_\rho) \qquad (r_\rho \in Q)\ ,$$

where the product in (3.7a) is over all $\alpha \in \Pi$ in increasing root order and the product in (3.7b) is over all $\rho \in P$ in increasing root order. The expression (3.7b) is unique, and (3.7a) is unique modulo the relations (3.4d). (See [8, p. 122] and [1, p. 39, Lemme 6].) Moreover, by [9, p. 114, Lemma 49, and p. 115 Cor. 3], $B_Z = B \cap G_Z$, $H_Z = H \cap G_Z$, $U_Z = U \cap G_Z$, and an element $b = hu$ in the form (3.7ab) is in B_Z if and only if each $t_\alpha = \pm 1$ and each $r_\rho \in Z$.

We will have use for the following easily proved result.

LEMMA 3.1. *Let the group $\mathfrak{X}$ be a product $\mathfrak{X} = \mathfrak{X}_1 \cdots \mathfrak{X}_n$ of subgroups $\mathfrak{X}_1, \cdots, \mathfrak{X}_n$ such that $(\mathfrak{X}_i, \mathfrak{X}_j) \subseteqq \prod_{k \geqq j} \mathfrak{X}_k$ and uniqueness holds for the representation $x = x_1 \cdots x_n$ $(x_i \in \mathfrak{X}_i)$, and let p be any permutation of the numbers $1, 2, \cdots, n$. Then $\mathfrak{X} = \mathfrak{X}_{p(1)} \cdots \mathfrak{X}_{p(n)}$ with uniqueness of representation.*

Now if S is a positive set of roots it follows from (3.4ab) that

$$\mathfrak{X}_S = \Pi \mathfrak{X}_\rho\ ,$$

where the product is over all $\rho \in S$ taken in increasing relative order. Since $\mathfrak{X}_S \subseteqq U' = \mathfrak{X}_{P'}$ where P' is a positive system of roots containing S, the representation (3.7b) (for U' instead of U) of $u \in \mathfrak{X}_S$ is unique, so (3.4b) and Lemma 3.1 show that any element $u \in \mathfrak{X}_S$ can be uniquely expressed in the form $u = \Pi x_\rho(r_\rho)$, with the product taken over all $\rho \in S$ in any fixed order.

There is a unique homomorphism ζ of N onto the Weyl group W of L, with kernel H, such that $w \cdot X_\rho \in Q \cdot X_{\omega(\rho)}$ when $\zeta(w) = \omega$. (Recall that X_ρ is an element of the Chevalley basis of L, and that elements of G act as automorphisms on L. See [1, p. 37, Lemma 3] or [9, pp. 29–31, Lemmas 20 and 22].) Thus $\psi\colon N/H \to W\colon Hw \mapsto \zeta(w)$ is an isomorphism onto W. Moreover, for any $t \in Q^*$, $Hw_\rho(t) = Hw_\rho(1)$ and $\psi(Hw_\rho(t)) = \zeta(w_\rho(1)) = \omega_\rho$. It is clear that H_Z is the kernel of the restriction of ζ to N_Z and $\zeta(N_Z) = W$. Thus $\psi_Z\colon N_Z/H_Z \to W\colon H_Z w \mapsto \zeta(w)$ is an isomorphism onto W.

A set N^* of representatives of N modulo H (as well as N_Z modulo H_Z) can be chosen in N_Z so that $w_\rho(1) \in N^*$ for each $\rho \in P$. For each $\omega \in W$ there is a unique representative $w(\omega) \in N^*$ such that $\zeta(w(\omega)) = \omega$. Henceforth, the elements $\omega \in W$ are frequently identified with the representatives $w(\omega)$, and both are denoted by w. We will also denote the reflection ω_ρ by w_ρ.

The Chevalley group G has Bruhat decomposition

$$G = \bigcup_{w \in W} BwB \tag{3.8}$$

into disjoint double cosets $BwB = BwU_w$, where U_w is the group generated by the x_ρ such that $\rho > 0$ and $w(\rho) < 0$. This provides the normal form

$$x = bwu \quad (b \in B,\ w \in N^*,\ u \in U_w) \tag{3.9}$$

for uniquely expressing any element $x \in G$. Since (3.9) is invalid in G_Z (since we might have $x = bwu \in G_Z$ with $b, u \notin G_Z$), we must modify this normal form to a normal form for G which applies to G_Z as well.

A reflection w_α in W corresponding to a simple root α is called a *simple reflection*. It is well known that the simple reflections generate the Weyl group W. For each root ρ, let

$$Y_\rho = \{\varphi_\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z),\ 0 \leqq a < c\}\,,$$

where $\varphi_\rho\colon SL(2, Q) \to G(\rho)$ is the canonical homomorphism described above (3.1). Then Y_ρ is a system of representatives for $B\backslash Bw_\rho B$, and we have

LEMMA 3.2. *For every $w \in W$ choose a minimal expression $w = w_\alpha w_\beta \cdots w_\delta$ as a product of simple reflections. Then*

$$BwB = BY_\alpha Y_\beta \cdots Y_\delta \tag{3.10}$$

with uniqueness of expression on the right.

Lemma 3.2 is a special case of [9, pp. 99-100, Theorem 15 and Lemma 43]. (A more detailed proof for the special case was given in [10].)

For any rational number r, define

$$x(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},\ y(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix},\ \text{and } \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$ with $0 \leqq a < c$. If $a = 0$, then $c = 1$, $b = -1$, and $A = y(-d)\Omega^{-1}$. If $a > 0$, there is a positive integer r_1 such that $c = r_1 a + c_1$ with $a > c_1 \geqq 0$ and $d = r_1 b + d_1$. If $c_1 = 0$, then $a = d_1 = 1$ and $A = y(r_1)x(b)$; if $c_1 > 0$, there is a positive integer s_1 such that $a = s_1 c_1 + a_1$ with $c_1 > a_1 \geqq 0$, $b = s_1 d_1 + b_1$, and $A = y(r_1)x(s_1)A_1$ with $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in SL(2, Z)$ with $c > c_1 > a_1 \geqq 0$. By induction on c, repeated application of the division algorithm will yield

$$A = y(r_1)x(s_1) \cdots y(r_{n-1})x(s_{n-1})y(r_n)x(k)$$

or

$$A = y(r_1)x(s_1) \cdots x(s_{n-1})y(r_n)x(s_n)y(k)\Omega^{-1} ,$$

where $n \geqq 0$, r_i and s_i are positive integers which do not appear if $n = 0$, and k is an integer. Clearly, the integers n, r_i, s_i, and k are uniquely determined by A. In view of (3.1) and (3.2), transforming the above result by the homomorphism φ_ρ shows that every element g_ρ of Y_ρ can be expressed in exactly one of the forms

$$g_\rho = x_{-\rho}(r_1)x_\rho(s_1) \cdots x_{-\rho}(r_{n-1})x_\rho(s_{n-1})x_{-\rho}(r_n)x_\rho(k) \tag{3.11a}$$

or

$$g_\rho = x_{-\rho}(r_1)x_\rho(s_1) \cdots x_\rho(s_{n-1})x_{-\rho}(r_n)x_\rho(s_n)x_{-\rho}(k)w_\rho(-1) , \tag{3.11b}$$

where the integers $n \geqq 0$, the positive integers r_i and s_i (which do not occur if $n = 0$) and the integer k are uniquely determined by g_ρ. Thus we have

THEOREM 3.3. *For every $g \in G$, there is a unique $w \in W$ such that $g \in BwB$. Thus, for any minimal decomposition $w = w_\alpha w_\beta \cdots w_\gamma$ of w as a product of simple reflections w_ρ, g can be expressed as a product of generators $x_\rho(r)$, $h_\delta(t)$, and $w_\delta(1)$ ($\rho \in \Sigma$, $\delta \in \Pi$) by writing*

$$g = bg_\alpha g_\beta \cdots g_\gamma \tag{3.12}$$

with b in the form (3.7ab), *and each $g_\delta \in Y_\delta$ in one of the forms* (3.11). *Moreover, $g \in G_Z$ if and only if every parameter r_ρ is an integer in* (3.7b) *and every parameter $t_i = \pm 1$ in* (3.7a). *Thus,* (3.12) *provides a normal form for G_Z.*

4. **The abstract group.** In this section, we will consider several abstract groups generated by the symbols x_ρ ($\rho \in \Sigma$) and defined by different sets of relations. A set E of relations among the generators x_ρ ($\rho \in \Sigma$) is called *compatible* if the corresponding relations in G_Z, obtained by replacing each x_ρ by $x_\rho(1)$, are valid. Henceforth, E will

denote a compatible set of relations which contains the relations (A) of §2, and G' will denote the abstract group generated by the symbols x_ρ ($\rho \in \Sigma$) and defined by the relations E. Thus the mapping $x_\rho \mapsto x_\rho(1)$ extends to an epimorphism π: $G' \to G_Z$, which we call the *canonical projection* of G' onto G_Z.

For each $S = \{\sigma_1, \sigma_2, \cdots\} \subseteqq \Sigma$, let $\mathfrak{X}'(S) = \mathfrak{X}'(\sigma_1, \sigma_2, \cdots)$ be the subgroup of G' generated by $\{x_\sigma \in G' \colon \sigma \in S\}$, and let $G'(\sigma_1, \sigma_2, \cdots) = \mathfrak{X}'(\bar{S})$, where $\bar{S}$ is the admissible system generated by S.

LEMMA 4.1. *Let S be any positive set of roots. Then*

$$\mathfrak{X}'(S) = \Pi \mathfrak{X}'(\sigma) , \tag{4.1a}$$

and any element $x \in \mathfrak{X}'(S)$ can be expressed in the form

$$x = \Pi\, x_\sigma^{k(\sigma)} , \tag{4.1b}$$

where both products are taken over all roots σ in S in any (fixed) *order, and $k(\sigma)$ are uniquely determined by x and the order in which the product is taken.*

Proof. First, induction on the cardinality m of S is used to show that $\mathfrak{X}'(S) = \Pi \mathfrak{X}'(\sigma)$ with uniqueness of expression, where the product is taken in increasing order (by some fixed relative ordering associated with S) of $\sigma \in S$. If $m = 1$, this is clear. Suppose $m > 1$. Let γ be the smallest and δ the largest root of S with respect to the relative ordering. Denote $S' = S - \{\gamma\}$, $\mathfrak{X}'(S) = \mathfrak{X}'$, $\mathfrak{X}'(S') = \mathfrak{X}''$. S' is a positive set of roots with cardinality $m - 1$, so the lemma holds for $\mathfrak{X}''$. Now, $\gamma + \delta > \delta$, so $\gamma + \delta \notin S$, and hence $\gamma + \delta \notin \Sigma$, since S is closed. Since S contains at least two linearly independent roots, $rk\ \Sigma > 1$ and the relations (A') hold in G'. Thus $(x_\gamma, x_\delta) = 1$ and $x_\gamma^{-1} x_\delta x_\gamma = x_\delta \in \mathfrak{X}''$. Suppose that for some $\sigma \in S'$, $x_\gamma^{-1} x_\rho x_\gamma \in \mathfrak{X}''$ for every $\rho \in S'$ with $\rho > \sigma$. Then $(x_\gamma, x_\sigma) = \Pi x_\rho^{n(\rho)}$ $(\rho > \sigma, \rho \in S')$ is in $\mathfrak{X}''$, so $(x_\sigma, x_\gamma^{-1}) = x_\gamma^{-1}(x_\gamma, x_\sigma)x_\gamma = \Pi\, (x_\gamma^{-1} x_\rho x_\gamma)^{n(\rho)}$ $(\rho > \sigma, \rho \in S')$ is in $\mathfrak{X}''$. Hence $x_\gamma x_\sigma x_\gamma^{-1} = (x_\gamma, x_\sigma) x_\sigma$ and $x_\gamma^{-1} x_\sigma x_\gamma = (x_\sigma, x_\gamma^{-1})^{-1} x_\sigma$ are in $\mathfrak{X}''$ for every $\sigma \in S'$. Thus the elements of $\mathfrak{X}'(\gamma)$ can be commuted to the left in any product of elements of $\mathfrak{X}'$, leaving an element of $\mathfrak{X}''$ to the right, and the factors of this element can be taken in increasing root order by the induction hypothesis. Hence, the product $x = \Pi\, x_\sigma^{k(\sigma)}$ can be taken in increasing root order. Since the $k(\sigma)$ are uniquely determined in the image $\pi(x) = \Pi x_\sigma(k(\sigma))$ of x under the canonical projection π, the $k(\sigma)$ are uniquely determined by x. (Note that this shows π is biunique when restricted to $\mathfrak{X}'(S)$.) The result for arbitrary root order now follows from Lemma 3.1.

COROLLARY 4.2. *Let S be a positive set of roots. Then the subgroup $\mathfrak{X}'(S)$ of G' is isomorphic by the canonical projection π to the corresponding subgroup $\pi(\mathfrak{X}'(S)) = \mathfrak{X}_S(Z)$ of G_Z.*

COROLLARY 4.3. *The relations*

$$(4.2) \qquad (x_\rho^r, x_\sigma^s) = \Pi x_{i\rho+j\sigma}^{C_{i,j,\rho,\sigma}r^i s^j} \qquad (\rho, \sigma, \in \Sigma, \rho + \sigma \neq 0, r, s, \in Z),$$

where the product is taken over all positive integers i and j such that $i\rho + j\sigma \in \Sigma$ in increasing order of the roots $i\rho + j\sigma$, hold in G'.

Proof. $S = \{i\rho + j\sigma \in \Sigma$: i and j are nonnegative integers$\}$ is a positive set of roots, so the result follows from the isomorphism π and the corresponding relations in $\mathfrak{X}_S(Z)$.

We now investigate various other relations in G'. First, it will be useful to show the equivalence of three formulas (4.3 abc). The definition $w_\rho = x_\rho x_{-\rho}^{-1} x_\rho$ $(\rho \in \Sigma)$ gives the identity $w_\rho w_{-\rho} = w_\rho x_\rho w_\rho^{-1} x_{-\rho}$, and $h_\rho = w_\rho^2$ by definition, so the relation

$$(4.3a) \qquad w_\rho x_\rho w_\rho^{-1} = x_{-\rho}^{-1}$$

is equivalent to

$$(4.3b) \qquad w_\rho w_{-\rho} = h_\rho h_{-\rho} = 1 .$$

The latter yields $1 = w_{-\rho} w_\rho = x_{-\rho} w_\rho^{-1} x_\rho w_\rho$, and so $w_\rho^{-1} x_\rho w_\rho = x_{-\rho}^{-1}$. Conjugating this last relation or (4.3a) with appropriate powers of w_ρ and using (4.3b) gives

$$(4.3c) \qquad w_\rho^t x_\sigma w_\rho^{-t} = x_{-\sigma}^{-1} \qquad (\sigma = \pm\rho, t = \pm 1) .$$

Clearly, for any $\rho \in \Sigma$, the relations (4.3abc) are equivalent. Moreover, the relation (4.3a) and the definition $h_\sigma = w_\sigma^2$ implies

$$(4.4) \qquad (w_\sigma x_\sigma)^3 = h_\sigma^2 \qquad (\sigma = \pm \rho) .$$

We will now imitate the methods of [8] to discover more of the structure of G'.

LEMMA 4.4. *Let $\rho, \sigma \in \Sigma$, $\sigma' = \omega_\rho \sigma$, $c = c(\sigma', \rho)$, $t = \pm 1$, and $s \in Z$. Then*

$$(4.5) \qquad w_\rho^t x_\sigma^s w_\rho^{-t} = x_{\sigma'}^{n(\rho,\sigma)t^c s}$$

in G', where $n(\rho, \sigma) = n(\rho, -\sigma) = \pm 1$ depends only on ρ and σ.

The proof of Lemma 4.4 is quite similar to the proof of Lemma 7.2 in [8, p. 118], and is hence omitted. It is perhaps helpful to note that when considering the case $\rho = \pm \sigma$ for the integrally parameterized group, one uses the existence of roots γ and δ such that $(x_\gamma, x_\delta) = x_\rho^v \Pi'$ *with* $v = \pm 1$ occurs among the relations (A').

Note that for any $\rho \in \Sigma$, (4.5) specializes to (4.3a), which implies that (4.3abc) and (4.4) hold in G'.

Utilizing Lemma 4.4 and the definitions of w_ρ and h_ρ, we get the next corollary.

COROLLARY 4.5. *Let the notation be as in Lemma* 4.4, *and let* $d = c(\sigma, \rho)$. *Then the following relations hold in* G':

$$w_\rho^t w_\sigma^s w_\rho^{-t} = w_{\sigma'}^{n(\rho,\sigma)t^c s}, \tag{4.6}$$

$$w_\rho^t h_\sigma^s w_\rho^{-t} = h_{\sigma'}^{n(\rho,\sigma)t^c s}, \tag{4.7}$$

$$h_\rho^t x_\sigma^s h_\rho^{-t} = x_\sigma^{(-1)^d s}, \tag{4.8}$$

$$h_\rho^t w_\sigma^s h_\rho^{-t} = w_\sigma^{(-1)^d s}, \tag{4.9}$$

$$h_\rho^t h_\sigma^s h_\rho^{-t} = h_\sigma^{(-1)^d s}. \tag{4.10}$$

The next corollary is immediate from the relations (B), (4.3b), (4.4), (4.7) and (4.10).

COROLLARY 4.6. *Let the notation be as in Lemma* 4.4. *If the defining relations E of G' contain the relations* (B), *then the following relations hold in* G':

$$h_\rho = h_\rho^{-1} = h_{-\rho}, \tag{4.11}$$

$$(w_\rho x_\rho)^3 = 1, \tag{4.12}$$

$$w_\rho^t h_\sigma^s w_\rho^{-t} = h_{\sigma'}^s, \tag{4.13}$$

$$h_\rho h_\sigma = h_\sigma h_\rho. \tag{4.14}$$

We define N' (respectively H') to be the subgroup of G' generated by the elements w_ρ (respectively h_ρ) for all roots $\rho \in \Sigma$. For each root ρ, $H'(\rho)$ is the subgroup of H' generated by h_ρ. It is easy to show that H' is a normal subgroup of N' and that the mapping $H'w_\rho \mapsto \omega_\rho$ $(\rho \in \Sigma)$ extends to an isomorphism of N'/H' onto the Weyl group W, but we will not need this fact here.

LEMMA 4.7. *For any $\rho \in \Sigma$, $H'(\rho)$ is a normal subgroup of H', and $H' = \Pi H'(\alpha)$, where the product is taken over all simple roots α. If E contains the relations* (B), *then H' is abelian.*

The proof of Lemma 4.7 is omitted because of its similarity to the proof of [8, 7.7, p. 120].

Now, let $U' = \mathfrak{X}'(P)$ be the subgroup of G' generated by $\{x_\rho \colon \rho \in P\}$

and let B' be the subgroup of G' generated by $H' \cup U'$.

LEMMA 4.8. *$U'H' = H'U' = B'$ and $U' \cap N' = U' \cap H' = \{1\}$.*

Proof. $U'H' = H'U' = B'$ by (4.8). Since $\pi(U' \cap N') = U_Z \cap N_Z = \{1\}$ in G_Z and π is an isomorphism on U' by Corollary 4.2, $U' \cap N' = \{1\}$ in G'.

COROLLARY 4.9. *Any element b in B' can be uniquely decomposed in the form*

$$b = hu \qquad (h \in H', u \in U') . \tag{4.15a}$$

Any $h \in H'$ can be written in the form

$$h = \Pi h_\alpha^{t(\alpha)} \qquad (t(\alpha) \in Z) , \tag{4.15b}$$

where the product is over all $\alpha \in \Pi$ in increasing root order. Any $u \in U'$ can be written in the form

$$u = \Pi x_\rho^{k(\rho)} \qquad (k(\rho) \in Z) , \tag{4.15c}$$

where the product is over all $\rho \in P$ in any order. The $k(\rho)$ in (4.15c) *are uniquely determined by u and the order in which the product is taken. Furthermore, the integers $t(\alpha)$ in* (4.15b) *can be chosen in* $\{0, 1\}$ *if E contains the relations (B).*

Proof. (4.15a), (4.15b), and (4.15c) are immediate from Lemmas 4.8, 4.7, and 4.1, respectively.

COROLLARY 4.10. *An element b in B' is in the kernel of π if and only if $b = \Pi h_\alpha^{t(\alpha)}$ with $\Pi(-1)^{t(\alpha)c(\mu,\alpha)} = 1$ for every $\mu \in L_R$, where both products are taken over all $\alpha \in \Pi$* (in any order). *If E contains the relations (B) and (C), then π restricted to B' is an isomorphism of B' onto B_Z.*

Proof. Expressing b in the form (4.15a), we have $\pi(b) = \pi(h)\pi(u) = 1$, so $\pi(u)$ lies in $U_Z \cap H_Z = \{1\}$. Thus $\pi(u) = 1$ implies $u = 1$ by Corollary 4.2, and $b = h$ can be written in the form (4.15b). Now, $\pi(h_\rho) = \pi(w_\rho^2) = w_\rho(1)^2 = w_\rho(-1)w_\rho(-1) = h_\rho(-1)$, so $\pi(h) = \Pi h_\alpha((-1)^{t(\alpha)}) = 1$ if and only if $\Pi(-1)^{t(\alpha)c(\mu,\alpha)} = 1$ for every $\mu \in L_R$, by (3.4d) and [9, p. 43, Lemma 28(c)]. If E contains the relations (B) and (C), the relations (B) "reduce" the t_α to 0 or 1, and then the condition for b to be in the kernel of π implies $b = 1$ by the relations (C), so π restricted to B' is an isomorphism.

Corollary 4.9 establishes a normal form for elements of B' which corresponds to the normal form in B_Z under the canonical projection π. Following the next technical lemma, we will seek to extend this normal form to all of G'.

LEMMA 4.11. *Let $\rho, \sigma \in \Sigma$ be simple roots relative to some ordering of Σ. Then every element of $G'(\rho)$ commutes with every element of $G'(\sigma)$ if and only if $(\rho, \sigma) = 0$.*

Proof. Since ρ, σ are simple roots in the relative ordering, $\rho - \sigma$ and $\sigma - \rho$ are not roots, and $\rho + \sigma$ is not a root if and only if $-q = c(\rho, \sigma) = 0$. Thus, $(\rho, \sigma) = 0$ implies there are no roots $i\rho + j\sigma \in \Sigma$ $(i, j \in Z)$ except $\pm\rho$ and $\pm\sigma$, so $(x_\rho, x_\sigma) = (x_\rho, x_{-\sigma}) = (x_{-\rho}, x_\sigma) = (x_{-\rho}, x_{-\sigma}) = 1$ by (A'). If $(\rho, \sigma) \neq 0$, $(x_\rho, x_\sigma) \neq 1$, since $\rho + \sigma$ is a root and the right hand product in (A') has the factor $x_{\rho+\sigma}^k$ with $k = C(1,1; \rho, \sigma) = \pm 1$, and hence cannot be 1 because of the uniqueness in (4.15c).

We now try to duplicate the normal form of Theorem 3.3 for G_Z in the abstractly defined group G'. An element x in G' such that $\pi(x) \in BwB$ is called *completely decomposable* (*c.d.*) if, for every minimal representation $w = w_1 w_2 \cdots w_n$ of w in terms of simple reflections w_i, x can be written in the normal form

$$x = hug_1 g_2 \cdots g_n\,, \tag{4.16a}$$

where h and u are in the forms (4.15b) and (4.15c), respectively, and for each $i = 1, 2, \cdots, n$, either

$$g_i = x_{-\alpha}^{r(1)} x_\alpha^{s(1)} \cdots x_{-\alpha}^{r(m)} x_\alpha^k \qquad (m > 0) \tag{4.16b}$$

or

$$g_i = x_{-\alpha}^{r(1)} x_\alpha^{s(1)} \cdots x_\alpha^{s(m)} x_{-\alpha}^k w_\alpha^{-1} \qquad (m \geqq 0)\,, \tag{4.16c}$$

where $r(j)$ and $s(j)$ are positive integers, k is an integer, and α is the simple root such that $w_i = w_\alpha$. A subset of G' is *completely decomposable* (*c.d.*) if every one of its elements is c.d. We denote by Y'_α the set of all elements g_i in $G'(\alpha)$ which can be written in the form (4.16b) or (4.16c).

The expressions (4.16) are more conveniently treated in terms of the following generators of G'.

LEMMA 4.12. *G' is generated by $\{x_\alpha, w_\alpha \in G' : \alpha \in \Pi\}$.*

Proof. Let G^* be the subgroup of G' generated by $\{x_\alpha, w_\alpha \in G' : \alpha \in \Pi\}$. For any $\rho \in P$, we show by induction on $ht(\rho)$ that x_ρ and $x_{-\rho}$ are in G^*. If $ht(\rho) = 1$, $\rho \in \Pi$ implies x_ρ and $x_{-\rho} = w_\rho x_\rho^{-1} w_\rho^{-1}$ are in G^*. If $ht(\rho) > 1$, there is a root $\alpha \in \Pi$ such that $\rho' = w_\alpha(\rho)$ is a positive root with $ht(\rho') < ht(\rho)$. (See [9, p. 267 (10), (11)].) Thus, $x_{\rho'}, x_{-\rho'} \in G^*$ implies that $x_\rho = w_\alpha x_{\rho'}^{n(\alpha,\rho')} w_\alpha^{-1}$, $x_{-\rho} = w_\alpha x_{-\rho'}^{n(\alpha,\rho')} w_\alpha^{-1} \in G^*$ by

(4.5). Since the x_ρ $(\rho \in \Sigma)$ generate G', $G^* = G'$.

Now, the identity of G' is trivially c.d., and it follows inductively that G' is c.d. if it can be shown for each $x \in G'$ that x c.d. implies $x_\alpha^t x$ and $w_\alpha^t x$ $(\alpha \in \Pi,\ t = \pm 1)$ are c.d. We use this to prove

LEMMA 4.13. *G' is c.d. if $G'(\alpha, \beta)$ is c.d. for every $\alpha, \beta \in \Pi$.*

Proof. Let $x \in \pi^{-1}(BwB)$ be c.d., let α be a simple root, and let $t = \pm 1$. For every minimal representation $w = w_1 \cdots w_n$, x can be put in the corresponding normal form (4.16) $x = hu\, g_1 \cdots g_n$. Writing $x_\alpha^t hu = h'u' \in B'$ in the form (4.15), $x_\alpha^t x = h'u'g_1 \cdots g_n$ is in the form (4.16) corresponding to the minimal representation $w = w_1 \cdots w_n$ of w, and $\pi(x_\alpha^t x) \in BwB$. Thus, $x_\alpha^t x$ is c.d.

Now, consider $w_\alpha^t x$. The normality of H' in W' implies $w_\alpha^t H' = H'w_\alpha^t = H'w_\alpha^{-1}$, where we use $w_\alpha = h_\alpha w_\alpha^{-1}$ if $t = 1$. By Lemma 4.1, any element $u \in U'$ can be written in the form $u = (\Pi x_\rho^{k(\rho)})x_\alpha^k$, where the product $\Pi x_\rho^{k(\rho)}$ is taken over all $\rho \in P - \{\alpha\}$. Since $w_\alpha(P - \{\alpha\}) = P - \{\alpha\}$ (See [9, p. 267, (11)].), the relations (4.5) imply $u' = w_\alpha^{-1}(\Pi x_\rho^{k(\rho)})w_\alpha$ is in U'. Thus, $w_\alpha^t x = h'u'w_\alpha^{-1}x_\alpha^k g = h'u'x_{-\alpha}^{-k}w_\alpha^{-1}g$, with h' and u' in the forms (4.15b) and (4.15c), respectively, and g in the form $g = g_1 \cdots g_n$ is obtainable for every form $x = hug_1 \cdots g_n$ for x. Noting that $g_\alpha = x_{-\alpha}^{-k}w_\alpha^{-1}$ is in the form (4.16c), we see that $w_\alpha^t x = h'u'g_\alpha g_1 \cdots g_n$ is in the normal form (4.16) if $w_\alpha w_1 \cdots w_n$ is a minimal representation for $w_\alpha w$. If not, then there is a minimal representation for w which begins with w_α. (See [9, p. 270, (21)].) Since x is c.d., we can assume that $x = hug_1 \cdots g_n$ with $g_1 \in Y_\alpha'$. Since $G'(\alpha) = G'(\alpha, \alpha)$ is c.d. by hypothesis, we can write $g_\alpha g_1 = h_1 u_1 g_1'$ in the normal form (4.16) with $g_1' = 1$ or in the form (4.16b) or (4.16c). In either case,

$$w_\alpha^t x = h'u'h_1u_1g_1'g_2 \cdots g_n = h''u''g_1'g_2 \cdots g_n$$

can be put in the normal form (4.16), since both $w_1 \cdots w_n$ and $w_2 \cdots w_n$ are minimal representations. The proof that $w_\alpha^t x$ is c.d. is completed in the following lemma, which shows that the decomposition (4.16) can be obtained corresponding to every minimal representation of the related Weyl group element.

LEMMA 4.14. *Let $G'(\alpha, \beta)$ be c.d. for every $\alpha, \beta \in \Pi$. Suppose that $w = w_1 \cdots w_n = w_1' \cdots w_n'$ are two minimal representations of $w \in W$, and that $g_1, \cdots, g_n$ are elements of G' such that $g_i \in Y_\alpha'$ when $w_i = w_\alpha$. Then there exist $h \in H'$, $u \in U'$, and g_i' such that $g_i' \in Y_\alpha'$ when $w_i' = w_\alpha$ and $g_1 \cdots g_n = hug_1' \cdots g_n'$.*

Proof. By [5], $w_1' \cdots w_n'$ can be obtained from $w_1 \cdots w_n$ by successive substitutions of terms $w_\alpha w_\beta w_\alpha \cdots$ of k factors by terms $w_\beta w_\alpha w_\beta \cdots$ of k factors, where k is the order of $w_\alpha w_\beta$, so it suffices to prove the lemma in the case

$$w_1 \cdots w_j w_\alpha w_\beta w_\alpha \cdots w_{j+k+1} \cdots w_n = w_1 \cdots w_j w_\beta w_\alpha w_\beta \cdots w_{j+k+1} \cdots w_n .$$

Since $G'(\alpha, \beta)$ is c.d., we can write

$$g_{j+1} \cdots g_{j+k} = h'u'g'_{j+1} \cdots g'_{j+k} ,$$

with $g_{j+i}, g'_{j+i+1} \in Y'_\alpha$ and $g'_{j+i}, g_{j+i+1} \in Y'_\beta$ for odd integers i between 1 and k.

Now we complete the proof by showing that the factor $h'u'$ can be "commuted" to the left to get

$$g_1 \cdots g_n = hu\, g'_1 \cdots g'_n ,$$

by showing that for any $\gamma \in \Pi$, $g_\gamma \in Y'_\gamma$, $h' \in H'$, and $u' \in U'$, $g_\gamma h'u' = h''u''g'_\gamma$ with $h'' \in H'$, $u'' \in U'$, and $g'_\gamma \in Y'_\gamma$. First, since g_γ is a product of powers of x_γ and $x_{-\gamma}$, and h' is a product of h_ρ $(\rho \in \Sigma)$, we see by (4.8) that $g_\gamma h' = h'g$, where $g \in G'(\gamma)$ can be written $g = h_1 u_1 g''_\gamma$ with $g''_\gamma \in Y'_\gamma$, since $G'(\gamma)$ is c.d. and $\pi(g) \in Bw_\gamma B$.

Consider $u' = x^s_\gamma \Pi_{\rho \neq \gamma} x_\rho^{k(\rho)}$. Since $G'(\gamma)$ is c.d., we can write $g''_\gamma x^s_\gamma = b_1 g'_\gamma$ with $b_1 \in B'$ and $g'_\gamma \in Y'_\gamma$. Let $\rho \in S = P - \{\gamma\}$. Then $p\rho \pm q\gamma \in S$ for any positive integers p and q such that $p\rho \pm q\gamma \in \Sigma$. Hence, $x = \Pi_{\rho \in S} x_\rho^{k(\rho)} \in \mathfrak{X}'(S)$ implies that $x^r_{\pm\gamma}\, x\, x^{-r}_{\pm\gamma} = \Pi_{\rho \in S}(x^r_{\pm\gamma}, x_\rho^{k(\rho)}) x_\rho^{k(\rho)}$ is in $\mathfrak{X}'(S)$ by (4.2). Since g'_γ is a product of powers of x_γ and $x_{-\gamma}$, it follows that $x' = g'_\gamma x g'^{-1}_\gamma \in \mathfrak{X}'(S) \subseteqq U'$.

Combining the above results and applying Corollary 4.9 to $h'h_1u_1b_1x' \in B'$, we obtain

$$g_\gamma h'u' = h'h_1u_1b_1x'g'_\gamma = h''u''g'_\gamma$$

with $h'' \in H'$, $u'' \in U'$, and $g'_\gamma \in Y'_\gamma$. This completes the proof.

Now, if G' is c.d., π gives a one-to-one correspondence between the normal form (4.16) in G' and the normal form of Theorem 3.3 in G_Z, modulo the subgroup B' of G'. Thus the kernel of π is the subgroup of H' described in Lemma 4.10, and, if E contains the relations (B) and (C), then π is an isomorphism of G' onto G_Z. We utilize this notion and Lemma 4.13 to complete the proof of the main theorem, by showing that each $G'(\alpha, \beta)$ $(\alpha, \beta \in \Pi)$ is c.d. when E contains the relations (B), (C), and (D).

Case 1. $\alpha = \beta$ and $G'(\alpha, \beta) = G'(\alpha)$. By [6, p. 8] the group $SL(2, Z)$ is generated by $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ subject to the relations

$$yx^{-1}yxy^{-1}x = (xy^{-1}x)^4 = 1 .$$

Substituting x_α for x and $x_{-\alpha}$ for y, $w_\rho = x_\rho x_{-\rho}^{-1} x_\rho$, and $h_\rho = w_\rho^2$, these relations are

$$w_{-\alpha} w_\alpha = h_\alpha^2 = 1 ,$$

which hold in $G'(\alpha)$ by (4.3b) and (B). Since $SL(2, Z)$ is c.d. (by proof of Theorem 3.3) in terms of its generators x and y, $G'(\alpha)$ is also c.d.

Case 2. $\alpha, \beta \in \Pi$, $(\alpha, \beta) = 0$. In this case, every element of $G'(\alpha)$ commutes with every element of $G'(\beta)$ by Lemma 4.11. Thus, $G'(\alpha, \beta) = G'(\alpha)G'(\beta)$ and this case follows from Case 1 and the above commutativity property.

Case 3. α, β form a system of type A_2. The admissible set of roots generated by α and β is $\Sigma_0 = \{-\alpha - \beta, -\beta, -\alpha, \alpha, \beta, \alpha + \beta\}$, and $G'(\alpha, \beta)$ is generated by $\{x_\rho : \rho \in \Sigma_0\}$. For this case, it can be shown that the structural constants satisfy $C(i, j; \rho, \sigma) = 0$ if $(i, j) \neq (1, 1)$, $C(1,1; \rho, \sigma) = N_{\rho,\sigma}$, $N_{\rho,\sigma} = n(\rho, \sigma)$ if $\rho + \sigma \in \Sigma_0$, and $N_{\rho,\sigma} = 0$ if $\rho + \sigma \notin \Sigma_0$, for every $\rho, \sigma \in \Sigma_0$. Moreover, $n(\alpha, \beta) = -n(\beta, \alpha) = n(\beta, \alpha + \beta) = -n(\alpha, \alpha + \beta) = n(\alpha + \beta, \beta) = -n(\alpha + \beta, \alpha)$, $n(\rho, \rho) = n(\rho, -\rho) = -1$, and $n(\rho, \sigma) = n(\rho, -\sigma) = -n(-\rho, \sigma)$ for any $\rho, \sigma \in \Sigma_0$ with $\rho \neq \pm\sigma$. Thus, all of the constants are determined by the value of $n(\alpha, \beta)$. We assume $n(\alpha, \beta) = 1$, renaming α and β, if necessary.

To parallel the notation of [6, §2], we let the roots ρ in Σ_0 correspond to subscripts ij as follows:

$$\alpha \longleftrightarrow 12 , \qquad \beta \longleftrightarrow 23 , \qquad \alpha + \beta \longleftrightarrow 13 ,$$
$$-\alpha \longleftrightarrow 21 , \qquad -\beta \longleftrightarrow 32 , \qquad -\alpha - \beta \longleftrightarrow 31 .$$

For $\rho \leftrightarrow ij$, we write $x_\rho = x_{ij}$, $w_\rho = w_{ij}$, and $h_\rho = h_{ij}$. Different letters will always denote different subscripts. In writing down the following relations, which are numbered to match corresponding relations from [6], we will put a reference to the relations implying them to the right. With this notation, the following relations hold in $G'(\alpha, \beta)$:

(A1)	$w_{ij}^2 = h_{ij} = h_{ji}$	$w_\rho^2 = h_\rho$ and (4.11) ,
(A2)	$w_{ij} w_{ji} = 1$	(4.3b) ,
(A3)	$w_{ij} w_{jk} = w_{jk} w_{ik}$	(4.6) ,
(A4)	$h_{ij}^2 = 1$	(4.11) ,
(A5)	$h_{ij} h_{jk} = h_{jk} h_{ij}$	(4.14) ,
(A6)	$h_{12} h_{23} h_{13} = 1$	(A1), (4.9), (4.7), (A4) ,
(B7)	$w_{ij}^{-1} x_{ij} w_{ij} = x_{ji}^{-1}$	(4.5) ,
(B8)	$w_{ik}^{-1} x_{ij} w_{ik} = x_{kj}$	(4.5) ,

(B9)	$w_{jk}^{-1}x_{ij}w_{jk} = x_{ik}$	(4.5) ,
(B10)	$h_{ij}x_{ij} = x_{ij}h_{ij}$	(4.8) ,
(B11)	$h_{ik}^{-1}x_{ij}h_{ik} = x_{ij}^{-1}$	(4.8) ,
(B12)	$h_{jk}^{-1}x_{ij}h_{jk} = x_{ij}^{-1}$	(4.8) ,
(C13)	$w_{ij}^{-1}x_{ij}x_{ji}^{-1}x_{ij} = 1$	$w_\rho = x_\rho x_{-\rho}^{-1}x_\rho$,
(C14)	$(x_{ij}, x_{ik}) = 1$	(A′) ,
(C15)	$(x_{ij}, x_{kj}) = 1$	(A′) ,
(C16)	$(x_{ij}, x_{jk}) = x_{ik}$	(A′) .

The proof that these relations define $PSL(3, Z) = SL(3, Z)$ proceeds almost exactly as in [6, §2]. R. Ree has shown $PSL(3, Z) \cong G_Z(A_2)$ in [7]. Since one can obtain a "canonical" isomorphism of $G'(\alpha, \beta)$ onto $G_Z(A_2)$, the group $G'(\alpha, \beta)$ is c.d.

Case 4. α, β form a system of type B_2 with long root β. Then the relations (D) show that the "canonical" projection π': $G'(\alpha, \beta) \rightarrow G_Z(B_2)$ defined by $x_\rho \mapsto x_\rho(1)$ (in $G_Z(B_2)$) is an isomorphism onto the c.d. group $G_Z(B_2)$, and hence, $G'(\alpha, \beta)$ is c.d.

This completes the proof of the main theorem.

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