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EXTREME MARKOV OPERATORS AND THE ORBITS OF RYFF

RAY C. SHIFLETT

Let X be the unit interval with the Lebesgue structure and let m be Lebesgue measure. A Markov operator with invariant measure m is an operator T on $L_\infty(X, m)$ such that $T1 = 1$ and $\int Tfdm = \int fdm$ for all f in $L_\infty(X, m)$. If θ is a measure-preserving transformation on X, then $\hat{\theta}f = f\circ\theta$ defines a Markov operator. Each such $\hat{\theta}$ is an extreme point in the convex set of Markov operators.

Let $\Omega(f)$ be the set of all $g \in L_1(X, m)$ such that $Tf = g$ for some Markov operator T. This convex set is called the orbit of f. The extreme points of $\Omega(f)$ are equimeasurable to f and arise from Markov operators of the form $\hat{\theta}\hat{\sigma}^*$. This paper shows the connection between extreme points of the set of Markov operators and the extreme points of $\Omega(f)$. The set of Markov operators which carry f to a given extreme point of $\Omega(f)$ is shown to contain an extreme Markov operator. The Markov operators of the from $\hat{\theta}\hat{\sigma}^*$ are shown to be extreme when θ is invertible. It is also shown that not all extreme operators factor into $\hat{\theta}\hat{\sigma}^*$ and that there are θ and σ such that $\hat{\theta}\hat{\sigma}^*$ is not extreme.

This paper deals with the problem of extreme points in the convex set of Markov operators and how they relate to the extreme points of orbits of elements from L_1 as defined by J. V. Ryff. The author would like to express his gratitude to Professor J. V. Ryff for discussing this work with the author and to the referee for his helpful comments.

A Markov operator, with Lebesgue measure invariant, is an operator T defined on $L_\infty(X, m)$ ($X = [0, 1]$ and m is Lebesgue measure) which satisfies:

(1) T is a positive operator

(2) $T1 = 1$

(3) $\int_X fdm = \int_X Tfdm$.

The norm of T, in the L_∞ norm, is one. T may be extended to L_1 such that $\|T\|_1 = 1$ and, using the Riesz convexity theorem, T may be extended uniquely to L_p as a contraction mapping for each p, $1 < p < \infty$. Therefore, T is defined on the Hilbert space L_2 and the adjoint, T^*, is well defined.

If $\theta: X \to X$ is a measure-preserving transformation, that is, if θ is measurable and $m(\theta^{-1}A) = m(A)$ for every measurable A, then the operator defined by $\hat{\theta}f = f\circ\theta$ is an extreme point in the set of Markov operators. If T is extreme, then T^* is extreme. James R. Brown proved that the set of operators induced by invertible measure-preserving transformations was dense in the set of Markov operators in the weak operator topology on L_p, $1 < p < \infty$. He also proved that the set of Markov operators is the closed convex hull of the set of operators induced by invertible measure-preserving transformations in the strong operator topology [1].

There are examples of self-adjoint, extreme Markov operators which are not induced by a measure-preserving transformation. R. G. Douglas [3] and J. Lindenstrauss [5] gave, independently, the only known characterization of the extreme points of the set of Markov operators.

J. V. Ryff gave the following definition.

DEFINITION. The orbit of $f \in L_1$, $\Omega(f)$, is the set of $g \in L_1$ where $g = Tf$ for some Markov operator T.

Ryff's work, [6, 7, 8], with these orbits suggests a possible connection between the extreme points of the set of Markov operators and the extreme points of $\Omega(f)$. The first theorem makes this connection explicit. Theorems 2, 3, and 4 give further clarification of this relationship. Theorems 5 and 6 show the limitation of this approach.

THEOREM 1. *If M_{fg} is the set of Markov operators which map f to g and if g is an extreme point of $\Omega(f)$, then M_{fg} contains an extreme point of the convex set of Markov operators.*

Proof. Let $tT_1 + (1 - t)T_2$ be in M_{fg} where T_1 and T_2 are Markov operators and $0 < t < 1$. Then $tT_1f + (1 - t)T_2f = g$. Therefore, $T_1f = T_2f = g$ since g is given as extreme. Thus M_{fg} is an extremal subset of the set of Markov operators.

The set of Markov operators is compact in the weak operator topology [1]. Now let $< T_\alpha >$ be a net in M_{fg} which converges to T in the strong operator topogy; that is, $T_\alpha f$ converges to Tf for every f in L_1. Thus $Tf = g$ since $T_\alpha f = g$ for every α. Therefore, $T \in M_{fg}$. This proves that M_{fg} is closed in the strong operator topology. A convex set has the same closure in the weak operator topology as in the strong operator topology. Thus M_{fg} is a closed, compact, convex, extremal subset of the set of Markov operators, which is a convex subset of a locally convex topological vector space. Thus M_{fg} contains an extreme Markov operator, see page 67; [9].

Ryff characterized the extreme points of $\Omega(f)$ as those elements which are equimeasurable to f [6]. These arise from Markov operators which may be written as $(\hat{\theta}) \circ (\hat{\sigma})^*$ where θ and σ are measure-preserving transformations. Let I_A be the characteristic function of the set A.

THEOREM 2. *If θ and σ are measure-preserving transformations with θ invertible then $T = \hat{\theta}\hat{\sigma}^*$ is extreme.*

Proof. θ is an invertible measure-preserving transformation if and only if $\hat{\theta}$ is unitary [1]. $\hat{\theta}$ is unitary if and only if $\hat{\theta}$ and $\hat{\theta}^*$ are isometries. Thus $\hat{\theta}^{-1} = \hat{\theta}^*$. Therefore, $\hat{\theta}\hat{\sigma}^* = (\hat{\sigma}\hat{\theta}^{-1})^*$. Since $\hat{\sigma}\hat{\theta}^{-1}I_B = I_B(\theta^{-1}\sigma)$ and $m((\theta^{-1}\sigma)^{-1}B) = m(B)$, the operator $\hat{\theta}\hat{\sigma}^*$ is the adjoint of that operator induced by the measure-preserving transformation $\theta^{-1}\sigma$ and is therefore extreme.

THEOREM 3. *If θ and σ are measure-preserving transformations, then there is an $f \in L_1$ such that $\hat{\theta}\hat{\sigma}^*f$ is extreme in $\Omega(f)$.*

Proof. Let A be given with $m(A) > 0$ and lef f be the characteristic function $\sigma^{-1}A$. Any g which is extreme in $\Omega(f)$ must be equimeasurable to f and is, therefore, essentially a characterictic function of a set B with $m(B) = m(A)$. Let C be a measurable set, then $(I_C, \hat{\sigma}^*f) \equiv (\hat{\sigma}I_C, f) = (\hat{\sigma}I_C, \hat{\sigma}I_A) = (I_C, I_A)$. Thus $\hat{\sigma}^*f = I_A$. Therefore, $\hat{\theta}\hat{\sigma}^*f = \hat{\theta}I_A = I_A \circ \theta$, which is the characteristic function of $\theta^{-1}A$ with $m(\theta^{-1}A) = m(A)$.

It is well known that any Markov operator which carries characteristic functions to characteristic functions is extreme. The next theorem is a partial result to a conjecture suggested by the above fact and Theorem 3. The conjecture is that given an extreme Markov operator there is some measurable set B such that $0 < m(B) < 1$ and such that TI_B is extreme in $\Omega(I_B)$. This would say that $TI_B = I_A$ for some A with $m(A) = m(B)$.

THEOREM 4. *If T is extreme then there is a set B such that $0 < m(B) < 1$ and such that $TI_B = I_A + F$ where $m(A) > 0$ and $F = 0$ on A.*

Proof. Suppose for every B, with $0 < m(B) < 1$, it is true that $0 \leqq TI_B < 1$ $[m]$-almost everywhere. Then $1 \geqq TI_{-B} > 0$ $[m]$-almost everywhere. But $-B$ is measurable with $0 < m(-B) < 1$ so that $0 < TI_{-B} < 1$ $[m]$-almost everywhere. Thus, for every B, $0 < TI_B < 1$ $[m]$-almost everywhere. This implies that $(I_A, TI_B) > 0$ for every A and B with $m(A) > 0$ and $m(B) > 0$. By Theorem 2, [2], T is not extreme.

After the discovery that the extreme points of $\Omega(f)$ were given by $\hat{\theta}\hat{\sigma}^*f$, the conjecture was made that the extreme points of the set of Markov operators are characterized by those operators which factor into $\hat{\theta}\hat{\sigma}^*$. The last two results of this paper answer this conjecture completely.

J. R. Brown proved that $\mu(A \times B) = (I_A, TI_B)$ gives a one to one correspondence between the set of doubly stochastic measures and Markov operators [1]. The notation μ_θ will be used to denote the measure associated with $\hat{\theta}$. The measure for $\hat{\sigma}^*$ is denoted by μ_σ^* and for $\hat{\theta}\hat{\sigma}^*$ by μ. The next technical result is needed for the last two theorems.

PROPOSITION. *$\mu_\theta(A \times B) = \mu(A \times \sigma^{-1}B)$ and $\mu_\sigma^*(A \times B) = \mu(\theta^{-1}A \times B)$.*

Proof.

$$\begin{aligned}\mu_\theta(A \times B) &= (I_A, \hat{\theta}I_B) = (\hat{\sigma}\hat{\theta}^*I_A, \hat{\sigma}I_B) = (\hat{\theta}^*I_A, \hat{\sigma}^*I_B\circ\sigma)\\ &= (I_A, \hat{\theta}\hat{\sigma}^*I_B\circ\sigma) = \mu(A \times \sigma^{-1}B)\ .\end{aligned}$$

The other equality is established in a similar manner.

THEOREM 5. *There are extreme Markov operators which are not of the form $\hat{\theta}\hat{\sigma}^*$.*

Proof. Let T be defined by

$$\begin{aligned}TI_B &= (1/2)I_{2B+(1/3)} \text{ if } B \subset [0, 1/3] \text{ and}\\ TI_B &= I_{(1/2)(B-(1/3))} + (1/2)I_B \text{ if } B \subset [1/3, 1]\ .\end{aligned}$$

Let μ be the associated doubly stochastic measure. It is easily seen that 1/3 of the mass of μ is uniformly distributed over the sets

$$\begin{aligned}&\{(x, y)\colon y = (1/2)(x - 1/3)\} \cap ([1/3, 1] \times [0, 1/3])\ ,\\ &\{(x, y)\colon y = x\} \cap ([1/3, 1] \times [1/3, 1]) \text{ and}\\ &\{(x, y)\colon y = 2x + 1/3\} \cap ([0, 1/3] \times [1/3, 1])\ .\end{aligned}$$

By Theorem 2, [2], μ is extreme. It can be shown that $T = T^*$ [4].

Suppose $T = \hat{\theta}\hat{\sigma}^*$. By the proposition, $\mu_\theta(B \times A) = (I_B, TI_A\circ\sigma) = (TI_B, I_A\circ\sigma) = ((1/2)I_{2B+(1/3)}, I_A\circ\sigma) = (1/2)\mu_\sigma([2B + (1/3)] \times A)$ for every $B \subset [0, 1/3]$ and A measurable. Also $\mu_\theta(B \times A) = m(B \cap \theta^{-1}A)$ and therefore,

$$m(B \cap \theta^{-1}A) = (1/2)m([2B + (1/3)] \cap (\sigma^{-1}A)\ . \tag{1}$$

By the proposition, $\mu_\sigma^*(A \times B) = \mu(\theta^{-1}A \times B)$. Similar manipulations as those yielding (1) will yield

$$m(B \cap \sigma^{-1}A) = (1/2)m(\theta^{-1}A \cap [2B + (1/3)]) \tag{2}$$

for $B \subset [0, 1/3]$ and A measurable.

Let $C = 2B + (1/3) \subset [1/3, 1]$. Then $\mu_\sigma^*(A \times C) = \mu[\theta^{-1}A \times C] = (I_A \circ \theta, TI_C) = (I_A \circ \theta, I_{(1/2)(C-(1/3))}) + (1/2(I_A \circ \theta, I_C) = m((1/2)[C-(1/3)] \cap \theta^{-1}A) + (1/2)m(C \cap {}^{-1}A)$. Therefore,

$$(3) \qquad m([2B + (1/3)] \cap \sigma^{-1}A) = m(B \cap \theta^{-1}A) + (1/2)m([2B + (1/3)] \subset \theta^{-1}A) .$$

Equations (1) and (3) yield

$$(4) \qquad m([2B + (1/3)] \cap \sigma^{-1}A) = m([2B + (1/3)] \cap \theta^{-1}A)$$

for all measurable $B \subset [0, 1/3]$ and all measurable A. Equations (2) and (4) yield

$$(5) \qquad (1/2)m([2B + (1/3)] \cap \sigma^{-1}A) = m(B \cap \sigma^{-1}A)$$

for $B \subset [0, 1/3]$ and all A. Then (5) and (1) give

$$m(B \cap \sigma^{-1}A) = m(B \cap \theta^{-1}A)$$

for all measurable $B \subset [0, 1/3]$ and all measurable A. Since every $C \subset [1/3, 1]$ is the image of some $B \subset [0, 1/3]$ under $2B + (1/3)$, for any measurable A and C, $\mu_\theta(A \times C) = \mu_\sigma(A \times C)$. Thus $\mu_\theta = \mu_\sigma$ and $\theta = \sigma$.

Thus, if $T = \hat{\theta}\hat{\sigma}^*$, it must be that $T = \hat{\theta}\hat{\theta}^*$. However,

$$(I_{[0,1/3]}, TI_{[0,1/3]}) = (1/2)(I_{[0,1/3]}, I_{[1/3,1]}) = 0 .$$

Then $(I_{[0,1/3]}, \hat{\theta}\hat{\theta}^* I_{[0,1/3]}) = (\hat{\theta}^* I_{[0,1/3]}, \hat{\theta}^* I_{[0,1/3]}) = \int_X (\hat{\theta}^* I_{[0,1/3]})^2 dm = 0$. Then $\hat{\theta}^* I_{[0,1/3]} = 0$ $[m]$-almost everywhere. This says that $(1, \hat{\theta}^* I_{[0,1/3]}) = 0$, which is a contradiction. Thus T is not of the form $\hat{\theta}\hat{\sigma}^*$.

THEOREM 6. *There are operators $T = \hat{\theta}\hat{\sigma}^*$ which are not extreme.*

Proof. Let $\sigma(x) = 2x (\mathrm{mod}\ 1)$ and $\theta(x) = 3x$ if $x \in [0, 1/3]$ and $\theta(x) = (1/2)(3x - 1)$ if $x \in [1/3, 1]$. For any

$$f \in L_2,\ (\hat{\sigma}^* f)(x) = (1/2)f(x/2) + (1/2)f((x + 1)/2) .$$

Thus

$$\begin{aligned} \mu(A \times B) = (I_A, TI_B) &= (I_A, \hat{\theta}[(1/2)I_{2B} + (1/2)I_{2B-1}]) \\ &= (1/2)\int_{[0,1/3]} I_A(x) I_{2B}(3x) m(dx) \\ &\quad + (1/2)\int_{[1/3\ 1]} I_A(x) I_{2B}((3x - 1)/2) m(dx) \\ &\quad + (1/2)\int_{[0,1/3]} I_A(x) I_{2B-1}(3x) m(dx) \\ &\quad + (1/2)\int_{[1/3,1]} I_A(x) I_{2B-1}((3x - 1)/2) m(dx) . \end{aligned}$$

If $B \subset [0, 1/2]$, this equality will yield

$$\mu(A \times B) = (1/2)m(A \cap (1/3)[2B \cup (4B + 1)]) .$$

If $B \subset [1/2, 1]$, $\mu(A \times B) = (1/2)m(A \cap (1/3)[(2B - 1) \cup (4B - 1)])$. So, for $A \subset [0, 1/3]$ and $B \subset [0, 1/2]$, $\mu(A \times B) = (1/2)m[A \cap (2B)/3]$. Then 1/6 of the mass of μ is distributed uniformly on $y = (3/2)x$ in $[0, 1/3] \times [0, 1/2]$.

Similar manipulations show that the 1/3 of the mass is on $y = (3/4)(x - (1/4))$ in $[1/3, 1] \times [0, 1/2]$ and 1/6 on $y = (3/2)x + (1/2)$ in $[0, 1/3] \times [1/2, 1]$ and 1/3 on $y = (3/4)x + (1/4)$ in $[1/3, 1] \times [1/2, 1]$. By Theorem 1, [2], this T is not extreme.

Theorem 5 does not answer the more general conjecture, which the author made, that every extreme Markov operator factors into a product of operators induced by measure preserving transformations and the adjoints of such operators. The author has not been able to answer this question. It is easy to show that this conjecture could be stated as every extreme Markov operator T may be written as $T = T_1T_2 \cdots T_n$ where $T_i = \hat{\theta}_i\hat{\sigma}_{i*}$ (θ_1 or σ_n may be the identity transformation). Theorem 6 shows this property can not characterize the extreme points.

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