

FIXED POINT AND COINCIDENCE SETS OF BICONNECTED MULTIFUNCTIONS ON TREES

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Various properties of fixed point sets of monotone single-valued mappings on dendrites have been studied by L. E. Ward, Jr. and the author. Some of them are extended here to coincidences of two mappings between two trees, and to biconnected multifunctions. Examples are given which show that others are no longer true in these cases.

1. Introduction. We know several properties of fixed point sets of monotone mappings onto dendrites. A theorem by G. E. Schweigert [5] and L. E. Ward, Jr. [6] states that every monotone mapping of a dendrite onto itself which leaves one end point fixed also leaves at least one other point fixed. Further results are contained in [3], where e.g., monotone mappings which leave all end points or all but finitely many end points fixed are investigated. It is shown in [3] that the fixed point set contains all branch points of order $\geq n$ if it contains all but n of the end points.

The theorem by Schweigert and Ward has been extended in [4] in two directions: from fixed points to coincidences of two mappings, and from monotone single-valued mappings to biconnected multifunctions. The purpose of the present paper is an attempt to extend results from [3] in a similar way. But this is not entirely possible. In the Main Theorem (see §3) we only prove that points of order $\geq n + 3$ rather than order $\geq n$ are invariant, and examples in §4 show that this statement cannot be sharpened to include points of order $\leq n + 2$. Neither can another result from [3], namely the non-existence of a fixed point set consisting of two points of order two for monotone surjections of dendrites, be extended to coincidences or to biconnected multifunctions, as can be seen from two examples given at the end.

A dendrite is a metric tree. The conclusions of this paper are valid for trees, as a metric is not needed in the proofs.

2. Trees and multifunctions. A tree is a continuum (i.e., compact connected Hausdorff space) in which every pair of points is separated by a third one. We use the partial order structure of trees which was developed by L. E. Ward, Jr. [6, 7]. It is obtained by selecting an arbitrary point $r \in T$ as root, and defining $x \leq y$ if $x = r$, $x = y$, or x separates r and y . Let

$$L(x) = \{y \in T \mid y \leq x\},$$

$$M(x) = \{y \in T \mid x \leq y\}.$$

Then $L(x)$ and $M(x)$ are closed for every $x \in T$, and $M(x) \setminus \{x\}$ is open. If $x < y$, then the (linearly ordered) set $[x, y] = M(x) \cap L(y)$ is connected [6]. A connected subset A of T which contains the points x and y , where $x < y$, also contains $[x, y]$, for $M(z) \setminus \{z\}$ and $T \setminus M(z)$ would induce a separation of A if $z \in [x, y]$ but $z \notin A$.

A point $m \in A$ is called a maximum of the subset A of T if $m \prec x$ for every $x \in A$. A root $r \in A$ is a point such that $r \leq x$ for every $x \in A$. Every nonempty closed subset of T has a maximum [6, Theorem 1], and every nonempty closed connected subset has a root [1, Lemma 2]. We define the order of a point $a \in T$ as the number of components of $T \setminus \{a\}$. An end point is a point of order one, a branch point is a point of order ≥ 3 .

The following lemma will be needed in the proof of the Main Theorem.

LEMMA. *If $a \in T$, then every component of $T \setminus \{a\}$ contains either the root or an end point.*

Proof. This lemma was proved in [3, Lemma 5.2] for dendrites. It is easy to adapt the proof given there to the nonmetric case.

A multifunction $\varphi: X \rightarrow X'$ from a space X into a space X' is a correspondence which assigns to each point of X a nonempty subset of X' . We say that φ is point-closed if $\varphi(x)$ is closed for every $x \in X$. It is called connected if $\varphi(A) = \bigcup (\varphi(x) \mid x \in A)$ is a connected subset of X' whenever A is a connected subset of X , inverse-connected if $\varphi^{-1}(A') = \{x \in X \mid \varphi(x) \cap A' \neq \emptyset\}$ is a connected subset of X whenever A' is a connected subset of X' , and biconnected if it is both connected and inverse-connected. If $\varphi: T \rightarrow T'$ is a point-closed and biconnected multifunction from a tree T into a tree T' , then $\varphi(x)$ is a nonempty, closed and connected subset of T' for every $x \in T$ and hence has a root, which we denote by $f(x)$. The function $f: T \rightarrow T'$ is single-valued, but in general not continuous. If f transforms the root of T onto the root of T' , then f is isotone (i.e., $x \leq y$ in T implies $f(x) \leq f(y)$ in T'), as the proof of Lemma 2 in [4] still holds for point-closed instead of upper semicontinuous (use) multi-functions.

3. Results. We shall now state and prove the theorem which is the main result of this paper. It concerns coincidences of multifunctions, and specializes to fixed points.

DEFINITION. A fixed point of a multifunction $\varphi: X \rightarrow X$ is a point $x \in X$ with $x \in \varphi(x)$. A coincidence of two multifunctions $\varphi, \psi: X \rightarrow X'$ is a point $x \in X$ with $\varphi(x) \cap \psi(x) \neq \emptyset$.

MAIN THEOREM. Let $\varphi, \psi: T \rightarrow T'$ be two point-closed and biconnected multifunctions from a tree T onto a tree T' . If all but n of the end points are coincidences, then all points of order $\geq n + 3$ are coincidences.

Proof. If $b \in T$ is a branch point of order $\geq n + 3$, then $T \setminus \{b\}$ consists of at least $n + 3$ components. Therefore we see from the Lemma in §2 that T has at least $n + 2$ end points. By assumption all but n of these are coincidences, hence we can select a point e_0 as root for T which is both an end point and a coincidence. Select also a point $e'_0 \in \varphi(e_0) \cap \psi(e_0)$ as root of T' .

The Lemma permits us now to choose in each component of $T \setminus \{b\}$ which does not contain e_0 an end point. Thus we obtain at least $n + 2$ end points, and according to the assumptions of the Main Theorem at least two of them must be coincidences. Call them e_1 and e_2 .

Then $b < e_1$, as otherwise the maximum $m = \max [L(b) \cap L(e_1)] < b$, so that the connected set $[e_0, m] \cup [m, e_1]$ would contain both e_0 and e_1 but not b . As e_0 and e_1 have been chosen in different components of $T \setminus \{b\}$, this is impossible. Similarly we have $b < e_2$.

Now select points $e'_i \in \varphi(e_i) \cap \psi(e_i)$ for $i = 1, 2$, and let the maximum of $L(e'_i) \cap L(e'_2)$ be b' . As f is an isotone function, we have

$$f(e_0) = e'_0 \leq f(b) \leq f(e_i) \leq e'_i \quad (i = 1, 2),$$

hence $f(b) \leq b' \leq e'_i$. Therefore the connected set $\varphi([b, e_i])$, which contains both $f(b)$ and e'_i , must also contain $[f(b), e'_i]$ and hence b' . From this we see that there exist points $x_i \in [b, e_i]$ for $i = 1, 2$ with $b' \in \varphi(x_i)$.

We now repeat the argument with the points e_0 and e_1 interchanged (i.e., with e_1 as root of T), and see that there also exists a point x_0 which separates e_0 and b , and for which $b' \in \varphi(x_0)$.

Now take again e_0 as root. As $x_j \in \varphi^{-1}(b')$ for $j = 0, 1$, it follows that $[x_0, x_1] \in \varphi^{-1}(b')$. But $b \in [x_0, x_1]$, hence $b \in \varphi^{-1}(b')$ or $b' \in \varphi(b)$. In the same way it follows that $b' \in \psi(b)$, so that b is a coincidence of φ and ψ .

The following corollary is obtained from the Main Theorem by taking $n = 0$.

COROLLARY 1. If all end points are coincidences of two point-closed and biconnected multifunctions from one tree onto another, then

all branch points are coincidences.

A further special case of interest is the fixed point case, i.e., the case where $T = T'$ and ψ is the identity mapping $\psi(x) = x$.

COROLLARY 2. *Let $\varphi: T \rightarrow T$ be a point-closed and biconnected multifunction of a tree T onto itself. If the fixed point set of φ contains all but n of the end points, then it contains all branch points of order $\geq n + 3$.*

4. Remarks. The following theorem was proved in [3, Theorem 5.1]: If $f: D \rightarrow D$ is a monotone mapping of a dendrite D (i.e., a metric tree) onto itself, and if at most n end points of D are not fixed, then every point of order n (where $n \geq 3$) is fixed. In view of this result it might be expected that the Main Theorem can be sharpened to include branch points of order n instead of $n + 3$. But this is not the case, not even for fixed points of point-closed and biconnected multifunctions or for coincidences of monotone single-valued mappings, as the next two examples show.

EXAMPLE 1. Let T be the tree (triad) in the xy -plane with end points $e_1 = (0, -1)$, $e_2 = (0, 1)$, and $e_3 = (1, 0)$, and branch point $b = (0, 0)$ of order three. Let further $a = (0, -(1/2))$ and $c = (0, 1/2)$. Define a single-valued and linear mapping $f: T \rightarrow T$ by $f(e_1) = e_1$, $f(e_2) = e_2$, $f(e_3) = a$, $f(b) = a$, $f(c) = b$; and let $\varphi: T \rightarrow T$ be defined by $\varphi(x) = f(x)$ if $x \in T \setminus \{c\}$, $\varphi(c) = [b, e_3]$. Then φ is point-closed and biconnected, leaves only the end point e_3 not fixed, but leaves also the branch point b of order 3 not fixed. Hence Corollary 2 does not hold if $n + 3$ is replaced by $n + 2$.

EXAMPLE 2. Let T be the triad of Example 1 with vertices $\{e_1, e_2, e_3, b\}$, and let T' be the segment $[e_1, e_2]$. Take $f: T \rightarrow T'$ as the linear mapping determined by $f(e_1) = f(e_3) = f(b) = e_1$, $f(e_2) = e_2$, and $g: T \rightarrow T'$ as the linear mapping determined by $g(e_1) = e_1$, $g(e_2) = g(e_3) = g(b) = e_2$. Then the end points e_1 and e_2 are coincidences, but e_3 and b are not coincidences. Hence the Main Theorem cannot even in the single-valued case be strengthened to include points of order $n + 2$.

The theorem by Schweigert and Ward mentioned in the introduction states that the fixed point set of a monotone mapping of a tree onto itself cannot consist of one point of order one. This result was extended in [3, Theorem 4.1(i)], where it was proved that the fixed point set also cannot consist of two points of order two. We conclude by giving two examples which show that this extension is not valid for fixed points of point-closed and biconnected multifunctions, or for

coincidences of monotone mappings.

EXAMPLE 3. Let T be the segment $[e_1, e_2]$ and define a and c as in Example 1. Define a multifunction $\varphi: T \rightarrow T$ by $\varphi(x) = a$ for $x \in [e_1, a)$, $\varphi(a) = [e_1, a]$, $\varphi(x) = 1/4 + x - x^2$ for $x \in (a, c)$, $\varphi(c) = [c, e_2]$, $\varphi(x) = c$ for $x \in (b, e_2]$. Then φ is point-closed and biconnected, and its fixed point set consists of the two points a and c which are both of order two.

EXAMPLE 4. Let again T have the vertices $\{e_1, e_2, e_3, a, b, c\}$, and let T' be the subtree with vertices $\{e_1, e_2, a, c\}$. Let $f, g: T \rightarrow T'$ be the linear mappings determined by $f(e_1) = e_1$, $f(e_2) = f(b) = f(c) = c$, $f(e_3) = e_2$, $f(a) = a$, and $g(e_1) = g(a) = g(b) = a$, $g(e_2) = e_2$, $g(e_3) = e_1$, $g(c) = c$. Then the coincidence set of f and g consists of the two points a and c of order two.

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