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INVERSE SEMIGROUPS OF PARTIAL TRANSFORMATIONS AND θ-CLASSES

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If S is an inverse semigroup and θ is the relation on the lattice $\Lambda(S)$ of congruences on S defined by saying that two congruences ρ_1,ρ_2 are θ-equivalent if and only if they induce the same partition of the idempotents then θ is a congruence on $\Lambda(S)$ and each θ-class is a complete modular sublattice of $\Lambda(S)$. If X is a partially ordered set then J_X denotes the inverse semigroup of one-to-one partial transformations of X which are order isomorphisms of ideals of X onto ideals of X, while if X is a semilattice, T_X denotes the inverse subsemigroup of J_X consisting of those elements α whose domain $\Delta(\alpha)$ and range $\nabla(\alpha)$ are principal ideals. It is shown that any inverse semigroup is isomorphic to an inverse subsemigroup of J_X for some semilattice X.

For an inverse subsemigroup of J_X, $\theta(S) = \Lambda(S)/\theta$ is related to certain equivalence relations on X. The weakest of these is a convex congruence which is an equivalence relation on X, convex in the partial ordering and compatible with the operation in S. It is shown that there is a natural order preserving mapping α of $\theta(S)$ into the lattice $\Gamma(X)$ of convex congruences. If X is a semilattice, the set of those convex congruences which are also semilattice congruences on X is denoted by $\Gamma_2(X)$. If S contains the idempotents of T_X, that is, if S is full in J_X, then α is a semilattice homomorphism of $\theta(S)$ onto $\Gamma_2(X)$. If S is full in T_X then α is a lattice isomorphism of $\theta(S)$ onto $\Gamma_2(X)$. Conversely, there exists an order preserving mapping β of $\Gamma_2(X)$ into $\theta(S)$. If S is full in J_X, then β is an order isomorphism into $\theta(S)$: if S is full in T_X, then β is a lattice isomorphism onto $\theta(S)$ and $\beta = \alpha^{-1}$.

We adopt the notation and terminology of (2). In particular, a semigroup S is called an *inverse semigroup* if $a \in aSa$, for all $a \in S$, and the idempotents of S commute. Then there is a unique element x such that $a = axa$ and $a = xax$. We call x the *inverse* of a and write $x = a^{-1}$. For any inverse semigroup S, we denote by E_S the subsemigroup of idempotents of S. If we define a partial ordering on E_S by saying that $e \leqq f$ if $ef = e$ then S is a semilattice where, by a *semilattice*, we mean a partially ordered set in which any two elements have a greatest lower bound. For the basic results on inverse semigroups the reader is referred to (2). All semigroups considered in this paper will be inverse semigroups.

Denote by $\Lambda(S)$ the lattice of congruences on the inverse semigroup S; that is, the lattice of equivalence relations ρ such that, for $a, b, c \in S$, $(a, b) \in \rho$ implies that $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$. Define the relation θ (cf. 9) on $\Lambda(S)$ by

$$(\rho_1, \rho_2) \in \theta \quad \text{if and only if} \quad \rho_1 | E_S = \rho_2 | E_S$$

where $\rho_i | E_S$ denotes the restriction of the congruence ρ_i to E_S. Then

LEMMA 1.1. ((9) *Theorem* 5.1). *Let S be an inverse semigroup and the relation θ be defined as above.*
Then

(i) *θ is a congruence on $\Lambda(S)$;*

(ii) *each θ-class is a complete modular sublattice of $\Lambda(S)$ (with a greatest and least element).*

We shall denote the lattice of θ-classes of an inverse semigroup S by $\Theta(S)$.

Now each congruence on an inverse semigroup S determines a *normal partition* of E_S; that is a partition $P = \{E_\alpha : \alpha \in J\}$ such that

E(i) $\alpha, \beta \in J$ implies that there exists a $\gamma \in J$ such that $E_\alpha E_\beta \subseteqq E_\gamma$;

E(ii) $\alpha \in J$ and $a \in S$ implies that there exists a $\beta \in J$ such that $aE_\alpha a^{-1} \subseteqq F_\beta$.

Likewise we call an equivalence relation ρ on E_S a *normal* equivalence if its classes constitute a normal partition of E_S.

Conversely, if P is a normal partition of E_S then P is induced by some congruence on S. Thus the lattice of normal partitions of E_S is, clearly, just (isomorphic to) $\Theta(S)$.

The least and greatest congruence in the θ-class corresponding to the normal partition P can be characterized as follows:

LEMMA 1.2. ((9) *Theorem* 4.2) *Let $P = \{E_\alpha : a \in J)$ be a normal partition of the semilattice of idempotents of S. Let $\sigma = \{(a, b) \in S \times S$: there exists an $\alpha \in J$ with $aa^{-1}, bb^{-1} \in E_\alpha$ and, for some $e \in E_\alpha$, $ea = eb\}$ and $\rho = \{(a, b) \in S \times S$: $\alpha \in J$ implies that, for some $\beta \in J$, $a\ E_\alpha a^{-1}$, $b\ E_\alpha b^{-1} \subseteqq E_\beta\}$. Then σ and ρ are, respectively, the smallest and largest congruences on S in the θ-class corresponding to the normal partition P.*

By a *one-to-one partial transformation* of a set X we mean a one-to-one mapping α of a subset Y of X onto a subset $Y' = Y\alpha$ of X. We call Y the *domain* of α, Y' the *range* of α and write $\Delta(\alpha) = Y$, $\nabla(\alpha) = Y'$. If we denote by I_X the set of all one-to-one partial transformations of X then, with respect to the natural multiplication of mappings, I_X is an inverse semigroup called the *symmetric inverse* *semigroup* on X (2).

Let X be a partially ordered set. By an *ideal* of X we mean a subset Y of X such that $x \leqq y \in Y$ implies that $x \in Y$. If X is trivially ordered, that is, if no two distinct elements are comparable, then any subset of X will be an ideal. We consider the empty set $\varnothing$ as being an ideal of X. By a principal ideal we mean an ideal of the form $\{x: x \leqq y\}$ for some fixed element y. Then we call $\{x: x \leqq y\}$ the *(principal) ideal generated by* y and denote it by $< y >$. For an arbitrary subset A of X we write $< A > = \{x \in X: x \leqq a,$ for some $a \in A\}$.

If X is a partially ordered set, let J_X denote the set of all $\alpha \in I_X$ such that

(i) $\Delta(\alpha)$ and $\nabla(\alpha)$ are ideals of X;

(ii) α is an order isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$; that is, a one-to-one mapping of $\Delta(\alpha)$ onto $\nabla(\alpha)$ such that, for $x, y \in \Delta(\alpha)$, $x \leqq y$ if and only if $x\alpha \leqq y\alpha$.

It is straightforward to verify that J_X is an inverse subsemigroup of I_X. If X is trivially ordered then, of course $J_X = I_X$.

By the following theorem, any inverse semigroup S can be embedded in I_S.

THEOREM 1.3. ((2) *Theorem* 1.20) *Let S be an inverse semigroup and for each $a \in S$ define the element α_a of I_S by*

(i) $\Delta(\alpha_a) = Sa^{-1}$;

(ii) *for* $x \in \Delta(\alpha_a)$, $x\alpha_a = xa$.

Then the mapping $\alpha: a \rightarrow \alpha_a$ is an isomorphism of S into I_S.

Considering S as a trivially ordered set we then have that S can be embedded in J_S. However, on any inverse semigroup S there exists a partial ordering, called the *natural partial ordering* which can be defined as follows: for any $a, b \in S$,

$$a \leqq b \quad \text{if and only if} \quad a^{-1}b = a^{-1}a \,.$$

For several equivalent definitions of this partial ordering see §7.1 of (2). The natural partial ordering is compatible with the multiplication of S.

Suppose that $y \in Sa^{-1}$ and that $x \leqq y$. Then $y = sa^{-1}$, for some $s \in S$ and $x^{-1}y = x^{-1}x$. Hence $x = xx^{-1}x = xx^{-1}y = xx^{-1}as^{-1} \in Sa^{-1}$. Thus $\Delta(\alpha_a)$ is an ideal in the partially ordered set S. Moreover, for any $x \leqq y$, with $x, y \in \Delta(\alpha_a)$, $x\alpha_a = xa \leqq ya = y\alpha_a$, since the natural partial ordering is compatible with the multiplication. Conversely, if $x\alpha_a \leqq y\alpha_a$, for $x, y \in \Delta(\alpha_a)$ then $xa \leqq ya$ and $xaa^{-1} \leqq yaa^{-1}$. Since $x, y \in \Delta(\alpha_a) = Sa^{-1}$, $xaa^{-1} = x$ and $yaa^{-1} = y$. Thus $x \leqq y$ and α_a is an order isomorphism of $\Delta(\alpha_a)$ onto $\nabla(\alpha_a)$. Thus

PROPOSITION 1.4. *Let S be an inverse semigroup. Then the embedding $a \to \alpha_a$ of S into I_S, of Theorem 1.3, also embeds S in J_S where S is considered as a partially ordered set with respect to the natural partial odering.*

Let X be a partially ordered set and $S \subseteqq J_X$ (we shall sometimes just write $S \subseteqq J_X$ for "S is an inverse subsemigroup of J_X"). We shall be interested in certain kinds of equivalence relations on X. Consider the following conditions on an equivalence ρ on X:

(i) $x \leqq y \leqq z$, $(x, z) \in \rho$ implies that $(x, y) \in \rho$;

(ii) $(x, y) \in \rho$, $x, y \in \Delta(a)$, $a \in S$, implies that $(xa, ya) \in \rho$.

If ρ satisfies these conditions then we shall call ρ a *convex congruence*, or just a *c-congruence* on X.

If X is actually a semilattice and we denote by $x \wedge y$ the greatest lower bound of any two elements x, y of X, then we can also consider the conditions:

(iii) $(x, y) \in \rho$ implies that $(x, x \wedge y) \in \rho$;

(iv) $(x, y) \in \rho$, $z \in X$ implies that $(x \wedge z, y \wedge z) \in \rho$.

If ρ satisfies conditions (i), (ii) and (iii) we shall call ρ an *s'-congruence*, while if ρ satisfies (ii) and (iv) then we shall call ρ a *semilattice congruence* or just an *s-congruence*. Although these definitions depend on S, S will generally be held fixed and so the terminology should not lead to any confusion. If X is a semilattice and ρ satisfies condition (iv), then clearly ρ satisfies conditions (i) and (iii). Thus an s-congruence is an s'-congruence and an s'-congurence is a c-congruence.

If X is totally ordered then the three types of congruence coincide.

By a *complete sublattice* A of a lattice B we mean a sublattice such that for any nonempty subset C of A the least upper bound (greatest lower bound) of C in A exists and is the least upper bound (greatest lower bound) of C in B.

PROPOSITION 1.5. *Let X be a partially ordered set and $S \subseteqq J_X$. Then the set $\Gamma(X)$ of c-congruences on X, partially ordered by set inclusion (as subsets of $X \times X$) is a complete lattice.*

If X is a semilattice then the set $\Gamma_1(X)$ of s'-congruences on X is a complete lattice (but not necessarily a sublattice of $\Gamma(X)$) and the set $\Gamma_2(X)$ of s-congruences is a complete sublattice of $\Gamma(X)$.

Proof. Let $\{\rho_i : i \in I\}$ be a family of c-congruences (s'-congruences, s-congruences). Then clearly $\bigcap_{i \in I} \rho_i$ is also a c-congruence (s'-congruence, s-congruence). Since $\Gamma(X)$ ($\Gamma_1(X)$, $\Gamma_2(X)$) has a largest element, the universal congruence $\rho = X \times X$, it follows from purely lattice theoretic considerations that $\Gamma(X)$ ($\Gamma_1(X)$, $\Gamma_2(X)$) is a complete lattice.

Now let C be a nonempty subset of $\Gamma_2(X)$. Clearly the greatest lower bound of C in $\Gamma(X)$ and $\Gamma_2(X)$ is just $\bigcap_{\rho \in C} \rho$. Now define a relation η on X by

$$(x, y) \in \eta \Leftrightarrow \text{for some } x = x_0, x_1, \cdots, x_n = y \in X ,$$
$$(x_{i-1}, x_i) \in \rho_i,\ i = 1, \cdots, n, \text{ for some } \rho_i \in C .$$

Then, from (1) Chapter 2, Theorem 4, η is an equivalence relation on X such that, if $(x, y) \in \eta$ and $z \in X$ then $(x \wedge z, y \wedge z) \in \eta$. Hence, to show that $\eta \in \Gamma_2(X)$, it only remains to be shown that if $(x, y) \in \eta$ and $(x, y) \in \Delta(a)$ then $(xa, ya) \in \eta$. Let $x = x_0, x_1, \cdots, x_n = y \in X$ and $\rho_1, \cdots, \rho_n \in C$ be such that $(x_{i-1}, x_i) \in \rho_i$, for $i = 1, \cdots, n$. Then $(x_0 \wedge x_{i-1}, x_0 \wedge x_i) \in \rho_i$, $i = 1, \cdots, n$ and, since $x_0 \wedge x_i \leqq x_0$, $x_0 \wedge x_i \in \Delta(a)$, for $i = 1, \cdots n$. Therefore, $((x_0 \wedge x_{i-1})a, (x_0 \wedge x_i)a) \in \rho_i$, for $i = 1, , \cdots, n$ and so $(xa, (x \wedge y)a) = ((x_0 \wedge x_0)a, (x_0 \wedge x_n)a) \in \eta$. Similarly, $(ya, (x \wedge y)a) \in \eta$. Hence $(xa, ya) \in \eta$ and $\eta \in \Gamma_2(X)$.

But η is the least upper bound of C in the lattice of equivalence relations on X and hence is the least upper bound of C in $\Gamma(X)$. Thus $\Gamma_2(X)$ is a complete sublattice of $\Gamma(X)$; in fact, we proved that $\Gamma_2(X)$ is a complete sublattice of the lattice of equivalence relations on X.

We now give an example to illustrate some of the points that have arisen.

EXAMPLE. Let X be the semilattice of Figure 1 and $S = E_{J_X}$.

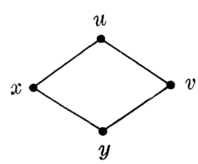

FIGURE 1.

Let ρ_1 be the equivalence relation on X which partitions X as $X = \{u\} \cup \{y\} \cup \{x, v\}$; let ρ_2 be the equivalence relation partitioning X as $X = \{x, u\} \cup \{v\} \cup \{y\}$ and let ρ_3 be the equivalence relation partitioning X as $X = \{x\} \cup \{y\} \cup \{u, v\}$.

Now ρ_1 is a c-congruence but not an s'-congruence since $(x, x \wedge v) = (x, y) \notin \rho_1$. Also ρ_2 is an s'-congruence but not an s-congruence since $(x, u) \in \rho_2$ but $(x \wedge v, u \wedge v) = (y, v) \notin \rho_2$. Similarly ρ_3 is an s'-congruence, but not an s-congruence. Finally, the least upper bound of ρ_2 and ρ_3 in $\Gamma(X)$ partitions X as $X = \{x, u, v\} \cup \{y\}$ which is not an s'-congruence.

2. From normal equivalences to congruences. Throughout this section, let X be a partially ordered set and S be an inverse subsemigroup of J_X. We now begin to relate the θ-classes of S and the congruences on X.

If A is a subset of S then we shall denote by $A\omega$ the set $\{s \in S: a \leqq s$, for some $a \in A\}$.

Let τ be a normal equivalence on E_S and $x \in X$. Let $V(x) = \{e \in E_S: x \in \Delta(e)\}$ and $V_\tau(x) = \{\bigcup_{e \in V(x)} e\tau\}\omega$.
Then we have

LEMMA 2.1. *$V(x) \subseteqq V_\tau(y)$ implies that $V_\tau(x) \subseteqq V_\tau(y)$.*

Proof. Let $f, f_1 \in E_S$, $(f, f_1) \in \tau$ and $f_1 \in V(x)$. Then $f_1 \in V_\tau(y)$ and so $f_1 \geqq f_2$, $(f_2, f_3) \in \tau$ and $f_3 \in V(y)$, for some $f_2, f_3 \in E_S$. Hence $f \geqq ff_2$, $(ff_2, f_1f_2) \in \tau$, $f_1f_2 = f_2$, $(f_2, f_3) \in \tau$ and $f_3 \in V(y)$; that is, $f \geqq ff_2$, $(ff_2, f_3) \in \tau$ and $f_3 \in V(y)$. Hence $f \in V_\tau(y)$. Thus $\bigcup_{e \in V(x)} e\tau \subseteqq V_\tau(y)$ and so $V_\tau(x) \subseteqq V_\tau(y)$.

THEOREM 2.2. *Let X be a partially ordered set and $S \subseteqq J_X$. Let τ be a normal equivalence on E_S. Define the relation $\rho = \rho_\tau$ on X by*

$$(x, y) \in \rho \quad \text{if and only if} \quad V_\tau(x) = V_\tau(y)\ .$$

Then ρ is a c-congruence on X. Moreover, if σ is another normal equivalence on E_S and $\tau \subseteqq \sigma$, then $\rho_\tau \subseteqq \rho_\sigma$.

Proof. (i) Suppose that $x \leqq y \leqq z$ and $(x, z) \in \rho$. Then $V(z) \subseteqq V(y) \subseteqq V(x)$ and so $V_\tau(z) \subseteqq V_\tau(y) \subseteqq V_\tau(x) = V_\tau(z)$, by Lemma 2.1. Hence $V_\tau(x) = V_\tau(y)$ and so $(x, y) \in \rho$.

(ii) Suppose that $(x, y) \in \rho$, $a \in S$ and $x, y \in \Delta(a)$. Let $f \in V(xa)$. Then $xa \in \Delta(fa^{-1})$ and so $x \in \Delta(afa^{-1})$. Hence $afa^{-1} \in V(x) \subseteqq V_\tau(y)$. Therefore, for some $f_1, f_2 \in E_S$, we have $afa^{-1} \geqq f_1$, $(f_1, f_2) \in \tau$ and $f_2 \in V(y)$. Hence $ya = yf_2a \in \Delta(a^{-1}f_2) = \Delta(a^{-1}f_2a)$ where $(a^{-1}f_2a, a^{-1}f_1a) \in \tau$, $a^{-1}f_1a \leqq a^{-1}afa^{-1}a \leqq f$. Thus $f \in V_\tau(ya)$ and, by Lemma 2.1, $V_\tau(xa) \subseteqq V_\tau(ya)$. Similarly we have the converse inclusion and so $V_\tau(xa) = V_\tau(ya)$ and $(xa, ya) \in \rho$. Hence ρ is a c-congruence. Now $\tau \subseteqq \sigma$ implies that $V_\tau(x) \subseteqq V_\sigma(x)$, for all $x \in X$, and so $(x, y) \in \rho_\tau$ implies that $V(x) \subseteqq V_\tau(y) \subseteqq V_\sigma(y)$. Therefore $V_\sigma(x) \subseteqq V_\sigma(y)$, by Lemma 2.1, and similarly the converse inclusion holds. Thus $(x, y) \in \rho_\sigma$ and $\rho_\tau \subseteqq \rho_\sigma$.

In general, of course, this mapping from normal equivalences to c-congruences is not one-to-one. However, in some circumstances, as we now show, it will be.

For any sets A and B let $A\backslash B = \{x: x \in A,\ x \notin B\}$. For $e \in E_S$, let $\delta(e) = \Delta(e)\backslash\bigcup_{f<e} \Delta(f) = \{x: x \in \Delta(e),\ x \notin \Delta(f)$ for any $f \in E_S$ such that $f < e\}$.

By an *order isomorphism* α of one partially ordered set X into another Y, we mean a one-to-one mapping α of X into Y such that, for $x, y \in X$, $x \leqq y$ if and only if $x\alpha \leqq y\alpha$.

PROPOSITION 2.3. *Let X be a partially ordered set and $S \subseteqq J_X$. Let the normal equivalence τ on E_S induce the c-congruence $\rho = \rho_\tau$ on X as in Theorem* 2.2. *Let $e, f \in E_S$, $x \in \delta(e)$ and $y \in \delta(f)$. Then*

$$(x, y) \in \rho \quad \textit{if and only if} \quad (e, f) \in \tau\,. \tag{2.1}$$

Thus, if $X = \bigcup_{e \in E_S} \delta(e)$, then the definition of ρ in Theorem 2.2 *may be replaced by the statement* (2.1).

Finally, if $\delta(e) \neq \varnothing$, for all $e \in E_S$, then the mapping $\tau \to \rho_\tau$ defines an order isomorphism of the lattice $\Theta(S)$ into $\Gamma(X)$.

Proof. Let $e, f \in E_S$, $x \in \delta(e)$, $y \in \delta(f)$. First suppose that $(e, f) \in \tau$. Then, for $g \in V(x)$ we have that, $g \geqq e$, $(e, f) \in \tau$ and $f \in V(y)$. Thus $V(x) \subseteqq V_\tau(y)$, $V_\tau(x) \subseteqq V_\tau(y)$ and, by similarity, $V_\tau(x) = V_\tau(y)$; that is, $(x, y) \in \rho$.

Now suppose that $(x, y) \in \rho$. Then $V_\tau(x) = V_\tau(y)$. Hence $e \in V(x) \subseteqq V_\tau(y)$. Thus, for some $e_1, e_2 \in E_S$, $e \geqq e_1$, $(e_1, e_2) \in \tau$, $e_2 \geqq f$. Similarly, for some $f_1, f_2 \in E_S$, $f \geqq f_1$, $(f_1, f_2) \in \tau$ and $f_2 \geqq e$. Then

$$e \geqq e_1 f,\ (e_1 f, f) = (e_1 f, e_2 f) \in \tau$$

and

$$f \geqq e f_1,\ (e f_1, e) = (e f_1, e f_2) \in \tau\,.$$

Hence

$$(e_1 f, e f) = (e \cdot \alpha_1 f, e f) \in \tau$$

and

$$(e f_1, e f) = (e f_1 \cdot f, e f) \in \tau\,.$$

Therefore $(e_1 f, e f_1) \in \tau$ and so $(e, f) \in \tau$.

The remainder of the theorem then follows easily.

A congruence ρ on an inverse semigroup S is called *idempotent separating* if no two distinct idempotents of S lie in the same ρ-class. There exists a unique maximal idempotent separating congruence μ on S which can be characterized as follows (Howie [4]):

$$(a, b) \in \mu \Leftrightarrow a^{-1} e a = b^{-1} e b \quad \text{for all} \quad e \in E_S\,.$$

If μ is the identity congruence, then we shall call S *fundamental*.

Although, for $S \subseteqq J_X$ and X a semilattice, we shall be considering the general problem of defining a normal equivalence on E_S from an s'-congruence on X in the next section and althought it appears essential in general to assume that X is a semilattice and that the congruence on X is an s'-congruence, we can, at least, establish the following theorem without these assumptions.

THEOREM 2.4. *Let X be a partially ordered set and $S \subseteqq J_X$. Define the relation ν on X by*:

$$(x, y) \in \nu \Leftrightarrow V(x) = V(y) .$$

Then ν is c-congruence on X. Define the relation ξ on S by

$$\begin{aligned}(a, b) \in \xi \Leftrightarrow \text{(i)}\quad & \{x\nu : x\nu \cap \Delta(a) \neq \emptyset\} = \{x\nu : x\nu \cap \Delta(b) \neq \emptyset\} ;\\ \text{(ii)}\quad & x \in \Delta(a),\ y \in \Delta(b),\ (x, y) \in \nu \\ & \textit{implies that } (xa, yb) \in \nu .\end{aligned}$$

Then $\xi = \mu$, the maximum idenpotent separating congruence on S.

Proof. Let $(x, z) \in \nu$ and $x \leqq y \leqq z$. Then $V(x) \supseteqq V(y) \supseteqq V(z) = V(x)$. Thus $V(x) = V(y)$ and $(x, y) \in \nu$.

Now let $(x, y) \in \nu$ and $x, y \in \Delta(a)$. Let $e \in V(xa)$. Then $aea^{-1} \in V(x) = V(y)$. Thus $e \in V(ya)$ and $V(xa) \subseteqq V(ya)$. Similarly $V(ya) \subseteqq V(xa)$ and so $V(xa) = V(ya)$. Thus $(xa, ya) \in \nu$ and ν is a c-congruence.

It is straightforward to see that ξ is an equivalence relation. To show that $\xi = \mu$, we first show that $\tau = \xi|_{E_S} = \iota$. Let $(e, f) \in \tau$ and $x \in \Delta(e)$. Then $x\nu \cap \Delta(f) \neq \emptyset$ and so $y \in x\nu \cap \Delta(f)$, for some y. Then $f \in V(y) = V(x)$. Thus $x \in \Delta(f)$ and $\Delta(e) \subseteqq \Delta(f)$. Conversely, $\Delta(f) \subseteqq \Delta(e)$ and so $\Delta(e) = \Delta(f)$ and $e = f$.

Let $(a, b) \in \xi$. Then, for any $x \in X$, $x\nu \cap \Delta(a) \neq \emptyset$ if and only if $x\nu \cap \Delta(b) \neq \emptyset$. But $\Delta(a) = \Delta(aa^{-1})$ and $\Delta(b) = \Delta(bb^{-1})$. Hence $x\nu \cap \Delta(aa^{-1}) \neq \emptyset$ if and only if $x\nu \cap \Delta(bb^{-1}) \neq \emptyset$. Moreover, for $(x, y) \in \nu$, $x \in \Delta(aa^{-1})$, $y \in \Delta(bb^{-1})$, $(xaa^{-1}, ybb^{-1}) = (x, y) \in \nu$. Hence $(a, b) \in \xi$ implies that $(aa^{-1}, bb^{-1}) \in \xi$ and so $aa^{-1} = bb^{-1}$ and $\Delta(a) = \Delta(b)$.

Now we show that ξ is a congruence on S. Let $(a, b) \in \xi$ and $c \in S$. If $x \in \Delta(ac)$ then $x \in \Delta(a) = \Delta(b)$ and $xa \in \Delta(c)$. However, $(xa, xb) \in \nu$ and so $cc^{-1} \in V(xa) = V(xb)$. Thus $x \in \Delta(bc)$ and $\Delta(ac) \subseteqq \Delta(bc)$. By similarity, $\Delta(ac) = \Delta(bc)$ and condition (i) is satisfied by ac and bc. If $x \in \Delta(ac) = \Delta(bc)$, then $(xa, xb) \in \nu$, since $(a, b) \in \xi$, and so $(xac, xbc) \in \nu$, since ν is a c-congruence. Thus $(ac, bc) \in \xi$.

Now $x \in \Delta(ca)$ if and only if $x \in \Delta(c)$ and $xc \in \Delta(a) = \Delta(b)$. Thus $\Delta(ca) = \Delta(cb)$ and condition (i) is satisfied by ca and cb. Clearly ca and cb then satisfy condition (ii). Thus $(ca, cb) \in \xi$ and ξ is a congruence.

Since $\xi|_{E_S} = \iota$ we have that $\xi \subseteqq \mu$ and to complete the theorem we need only show that $\mu \subseteqq \xi$. Suppose that $(a, b) \in \mu$. Then $aa^{-1} =$

bb^{-1}, $\Delta(aa^{-1}) = \Delta(bb^{-1})$ and condition (i) is satisfied. Now let $x \in \Delta(a)$, $y \in \Delta(b)$ and $(x, y) \in \nu$. Let $f \in V(xa)$. Then $xa \in \Delta(f)$ and so $x \in \Delta(afa^{-1})$. But, since $(a, b) \in \mu$, $afa^{-1} = bfb^{-1}$. Thus $x \in \Delta(bfb^{-1})$. Now $V(x) = V(y)$ and so $y \in \Delta(bfb^{-1})$. Hence $yb \in \Delta(f)$ and $V(xa) \subseteqq V(yb)$. By similarity, we have that $V(xa) = V(yb)$ and $(xa, yb) \in \nu$. Thus condition (ii) is also satisfied by a and b and so $(a, b) \in \xi$. Hence $\xi = \mu$.

If, in Theorem 2.4, ν is the identity relation on X, then clearly $(a, b) \in \xi$ if and only if $a = b$. Thus we have immediately:

COROLLARY 2.5. *Let X be a partially ordered set and $S \subseteqq J_X$. If ν is the identity relation, then S is fundamental.*

Let X be a partially ordered set and $x \in X$. Then we shall denote by e_x the idempotent of J_X with domain equal to the principal ideal $< x >$. Let $S \subseteqq J_X$, then we say that S is *full* in J_X or (if X is a semilattice and $S \subseteqq T_X$) that S is *full* in T_X if $\{e_x : x \in X\} \subseteqq E_S$, where T_X is as defined in §3.

COROLLARY 2.6. *Let S be full inverse subsemigroup of J_X, then S is fundamental.*

Proof. If S is full then ν must be the identity relation and then so must ξ.

Corollary 2.6 is a slight generalization of a theorem ([6] Theorem 2.6) of Munn's and could be established directly along the same lines as Munn's proof. Corollary 2.5 is a little stronger, however, as the following example shows:

EXAMPLE. Let X be the set of real numbers under their natural ordering. Let $S = \{\alpha \in J_X : \Delta(\alpha)$ is not principal$\}$. Then S is an inverse subsemigroup of J_X. Clearly ν is the identity relation and hence S is fundamental. However, S is not a full inverse subsemigroup of J_X.

3. X a semilattice. Let X be a semilattice, then we can define another subsemigroup of I_X as follows. Let T_X denote the set of $\alpha \in I_X$ such that

(i) $\Delta(\alpha)$ and $\nabla(\alpha)$ are principal ideals;

(ii) α is an order isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$.

It is straightforward to verify that T_X is an inverse subsemigroup of I_X and J_X. For a discussion of T_X and its importance in connection with bisimple inverse semigroups see Munn [7].

PROPOSITION 3.1. *Let X be a partially ordered set and let $\bar{X}$*

denote the set of all ideals of X, *partially ordered by set inclusion. Then* $\bar{X}$ *is a semilattice and there exists an embedding* $\kappa: J_X \rightarrow T_{\bar{X}}$.

Proof. Clearly $\bar{X}$ is a semilattice. For $\alpha \in J_X$ define $\kappa_\alpha \in T_{\bar{X}}$ by:
(i) $\Delta(\kappa_\alpha) = \{I \in \bar{X}: I \subseteqq \Delta(\alpha)\}$;
(ii) for $I \in \Delta(\kappa_\alpha)$, $I\kappa_\alpha = \{x\alpha: x \in I\}$.
Then $\kappa: \alpha \rightarrow \kappa_\alpha$ is an isomophism of J_X into $T_{\bar{X}}$.

We now give several ways in which inverse semigroups might be considered as subsemigroups of T_X for some semilattice X. First, from [7] Lemma 3.1,

PROPOSITION 3.2. *Let* S *be an inverse semigroup and* $E_S = E$. *Define a mapping* $\theta: S \rightarrow T_E$ *by the rule that* $a\theta = \theta_a$ *where*
(i) $\Delta(\theta_a) = Eaa^{-1}$;
(ii) *for* $e \in \Delta(\theta_a)$, $e\theta_a = a^{-1}ea$.
Then θ *is a homomorphism of* S *into* T_E *inducing the maximum idempotent separating congruence on* S *and hence is an isomorphism if* S *is fundamental.*

Combining either Theorem 1.3 (considering S as a trivially ordered set) or Proposition 1.4 with Proposition 3.1 we have:

PROPOSITION 3.3 *Let* S *be an inverse semigroup then there exists a semilattice* X *and an isomorphism* $\kappa: S \rightarrow T_X$.

Presently we shall be considering inverse subsemigroups S of J_X, where X is a semilattice, such that $X = \bigcup_{e \in E_S} \delta(e)$ or such that $\delta(e) \neq \varnothing$, for all $e \in E_S$. In this connection, we have

PROPOSITION 3.4. *Let* S *be an inverse semigroup then there exists a semilattice* X *and an isomorphism* $\kappa: S \rightarrow J_X$ *such that*
(i) $\delta(e\kappa) \neq \varnothing$ *for all* $e \in E_S$:
(ii) $X = \bigcup_{e \in E_S} \delta(e\kappa)$.

Proof. Let $\theta: S \rightarrow J_S$ be the embedding of Proposition 1.4. Let X denote the set of all subsets of S which are inversely well ordered with respect to the natural partial ordering of S, together with the empty set. Partially order X by set inclusion. Then X is clearly a semilattice. Define $\phi: J_S \rightarrow J_X$ as follows: for $\alpha \in J_S$,
(i) $\Delta(\alpha\phi) = \{A \in X: A \subseteqq \Delta(\alpha)\}$;
(ii) for $A \in \Delta(\alpha\phi)$, $A(\alpha\phi) = \{a\alpha: a \in A\}$.
Then ϕ is an isomorphism and so $\kappa = \theta \circ \phi$ is an isomorphism of S into J_X.

For $e \in E_S$, $e \in \Delta(e\theta)$ and so $\{e\} \in \Delta(e\kappa)$. Clearly $\{e\} \in \Delta(f\kappa)$, for $f \in$

E_S if and only if $e \leqq f$ in the natural partial order on S. Thus $\{e\} \in \delta(e\kappa)$ and $\delta(e\kappa) \neq \varnothing$ for all $e \in E_S$.

Let $A \in X$ have greatest element a, in the natural partial order on S. Then $a \in \delta((a^{-1}a)\kappa)$. Thus $X = \bigcup_{e \in E_S} \delta(e\kappa)$.

Finally, we give a representation of slightly less general applicability which is interesting on account of the relationship that the set X bears to the semigroup.

Before doing so, we need the following special case of Lemma 1.2. due to Munn [5]:

LEMMA 3.5. *Let S be an inverse semigroup and let a relation σ be defined on S by the rule that $x\sigma y$ if and only if there is an idempotent e in S such that $ex = ey$ (or, equivalently, $xe = ye$). Then σ is a congruence on S and S/σ is a group. Further, if τ is any congruence on S with the property that S/τ is a group, then $\sigma \subseteqq \tau$ and so S/τ is isomorphic with some quotient group of S/σ.*

Then σ is called the *minimum group congruence* on S.

PROPOSITION 3.6. *Let S be an inverse semigroup, let σ be the minimum group congruence on S, let μ be the maximum idempotent separating congruence on S and let $\sigma \cap \mu = \iota$, the identity congruence on S. Let $X = E_S \cup S/\sigma \cup \{0\}$, where for $x, y \in X$, we have $x \leqq y$ if and only if*

either (i) *$x, y \in E_S$ and $x \leqq y$ in the natural partial ordering of E_S;*

or (ii) *$y \in E_S$ and $x \in S/\sigma$;*

or (iii) *$x = 0$.*

Then X is a semilattice and there exists an embedding $\kappa: S \to T_X$, such that $\delta(e\kappa) \neq \varnothing$ for all $e \in E_S$.

Proof. Let $\theta: a \to \theta_a$ be the Munn representation of S of Proposition 3.2. Then, for $a \in S$, define $a\kappa \in T_X$ as follows:

(i) $\Delta(a\kappa) = E_S aa^{-1} \cup S/\sigma \cup \{0\}$;

(ii) $x(a\kappa) = x\theta_a$ if $x \in E_S \cap \Delta(a\kappa)$;

(iii) $x(a\kappa) = x(a\sigma)$ if $x \in S/\sigma$;

(iv) $x(a\kappa) = x$ if $x = 0$.

Then it is clear that κ is a homomorphism of S into T_X inducing the congruence $\sigma \cap \mu$, that is, the identity congruence. Thus κ is an isomorphism.

We now turn to the problem of relating, for $S \subseteqq J_X$ and X a semilattice, s'-congruences on X to normal equivalences or θ-classes of S. For ρ an s'-congruence on X and $a \in S$ we shall denote by $U(a)$

the set $\{x\rho : x\rho \cap \Delta(a) \neq \emptyset\}$. We suppress any indication of the dependence of $U(a)$ on ρ since this will not lead to any confusion.

THEOREM 3.7. *Let X be a semilattice, S be an inverse subsemigroup of J_X and ρ be an s'-congruence. For $a \in S$, define $\alpha_a \in J_{X/\rho}$, as follows:*

(i) $\Delta(\alpha_a) = U(a)$

(ii) *for $x\rho \in \Delta(\alpha_a)$, $(x\rho)\alpha_a = (x_1 a)\rho$ where x_1 is any element in $x\rho \cap \Delta(a)$.*

Then α: $a \rightarrow \alpha_a$ is a homomorphism of S into $I_{X/\rho}$. If ρ is an s-congruence then a partial ordering of X/ρ can be defined as follows:

$$x\rho \leqq y\rho \Leftrightarrow x_1 \leqq y_1 \quad \textit{for some} \quad x_1 \in x\rho,\ y_1 \in y\rho\ .$$

With respect to this partial ordering X/ρ is a semilattice and $S\alpha \subseteqq J_{X/\rho}$.

Proof. Since ρ is a c-congruence, α_a is clearly well defined and it is straight forward to show that $\alpha_a \in I_{X/\rho}$, that is, that α_a is one-to-one. Let $a, b \in S$ and $x\rho \in \Delta(\alpha_{ab})$. Then there exists an $x_1 \in x\rho \cap \Delta(ab)$. Hence $x_1 \in x\rho \cap \Delta(a)$ and $x_1 a \in \Delta(b)$. Thus $x\rho \in \Delta(\alpha_a)$ and $x_1 a \in (x\rho)\alpha_a \cap \Delta(b)$. Thus $(x\rho)\alpha_a \in \Delta(\alpha_b)$ and $x\rho \in \Delta(\alpha_a\alpha_b)$. Conversely, let $x\rho \in \Delta(\alpha_a\alpha_b)$. Then there exists an $x_1 \in x\rho \cap \Delta(a)$ and an $x_2 \in (x\rho)\alpha_a \cap \Delta(b) = (x_1 a)\rho \cap \Delta(b)$. With $x_3 = x_2 \wedge x_1 a$, we have $x_3 \in x_2\rho = (x\rho)\alpha_a$ and $x_3 \in \Delta(a^{-1}) \cap \Delta(b)$, since $x_1 a \in \Delta(a^{-1})$ and $x_2 \in \Delta(b)$. Thus $x_3 a^{-1} \in x\rho$, $x_3 a^{-1} \in \Delta(a)$ and $(x_3 a^{-1})a = x_3 \in \Delta(b)$. Thus $x_3 a^{-1} \in x\rho \cap \Delta(ab)$. Hence $x\rho \in \Delta(\alpha_{ab})$. Thus $\Delta(\alpha_{ab}) = \Delta(\alpha_a\alpha_b)$. Now let $x\rho \in \Delta(\alpha_{ab}) = \Delta(\alpha_a\alpha_b)$, and $x_1 \in x\rho \cap \Delta(ab)$. Then

$$(x\rho)\alpha_{ab} = (x_1 ab)\rho$$

and

$$(x\rho)\alpha_a\alpha_b = (x_1 a)\rho\alpha_b = (x_1 ab)\rho\ .$$

Hence $\alpha_a\alpha_b = \alpha_{ab}$ and α is a homomorphism.

If ρ is an s-congruence then X/ρ is clearly a semilattice and it only remains to be shown that $S\alpha \subseteqq J_{X/\rho}$.

So suppose that $x\rho \leqq y\rho$ and $y\rho \in \Delta(\alpha_a)$. Then there exists $x_1 \in x\rho$, $y_1, y_2 \in y\rho$ such that $x_1 \leqq y_1$ and $y_2 \in \Delta(a)$. Hence $(x_1, x_1 \wedge y_2) = (x_1 \wedge y_1, x_1 \wedge y_2) \in \rho$ and so $(x, x_1 \wedge y_2) \in \rho$ where $x_1 \wedge y_2 \leqq y_2 \in \Delta(a)$. Thus $x_1 \wedge y_2 \in \Delta(a)$ and $x\rho \in \Delta(\alpha_a)$. Therefore $\Delta(\alpha_a)$ is an ideal and it is routine to verify that α_a is order preserving. Thus $S\alpha \subseteqq J_{X/\rho}$.

To see the difficulty that arises if ρ is just a c-congruence, consider the semilattice X of Figure 2.

Let S be the inverse subsemigroup of J_X consisting of the idem-

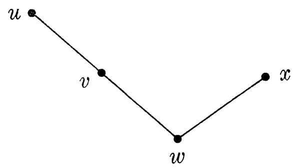

FIGURE 2.

potents e_1, e_2, e_3 where $\Delta(e_1) = \{x, w\}$, $\Delta(e_2) = \{u, v, w\}$ and $\Delta(e_3) = \{w\}$. Let ρ be the c-congruence on X determined by the partition $X = \{x, v\} \cup \{u\} \cup \{w\}$. Then there is no natural homomorphism of S into $J_{X/\rho}$.

From Theorem 3.7, we have

COROLLARY 3.8. *Let X be a semilattice and S be an inverse subsemigroup of J_X. Let ρ be an s'-congruence on X and define the relation $\tau = \tau_\rho$ on E_S as follows: for $e, f \in E_S$,*

$$(e, f) \in \tau \Leftrightarrow U(e) = U(f) .$$

Then τ is a normal equivalence on E_S. If $\rho \subseteqq \rho'$ then $\tau \subseteqq \tau'$.

In certain circumstances we can give a more direct difinition of the normal equivalence induced by an s-congruence.

LEMMA 3.9. *Let X be a semilattice and S be an inverse subsemigroup of J_X. Let ρ be an s-congruence on X and let ρ induce the normal equivalence τ on E_S. If $e_x, e_y \in E_S$ then*

$$(e_x, e_y) \in \tau \Leftrightarrow (x, y) \in \rho .$$

In particular, if $S \subseteqq T_X$ then this defines τ.

Proof. Let $(x, y) \in \rho$ and $z\rho \cap \Delta(e_x) \neq \varnothing$. Without loss of generality, let $z \in \Delta(e_x)$. Then $z \leqq x$, $(z, z \wedge y) = (z \wedge x, z \wedge y) \in \rho$ and $z \wedge y \in \Delta(e_y)$. Thus $z\rho \cap \Delta(e_y) \neq \varnothing$ and $U(e_x) \subseteqq U(e_y)$. By similarity, we have the converse inclusion and so $(e_x, e_y) \in \tau$.

Now suppose that $(e_x, e_y) \in \tau$. Then $x \in x\rho \cap \Delta(e_x)$ and so there exists an x_1 such that $(x, x_1) \in \rho$ and $x_1 \in \Delta(e_y)$, that is, $x_1 \leqq y$. Similarly, there exists a y_1 such that $(y, y_1) \in \rho$ and $y_1 \in \Delta(e_x)$, that is, $y_1 \leqq x$. Then $(x \wedge y, x_1) = (x \wedge y, x_1 \wedge y) \in \rho$ and $(x \wedge y, y_1) = (x \wedge y, x \wedge y_1) \in \rho$. Hence $(x_1, y_1) \in \rho$ and so $(x, y) \in \rho$ as required.

We conclude this section with an instance where the mapping $\rho \to \tau$ is one-to-one.

THEOREM 3.10. *Let X be a semilattice and S be a full inverse subsemigroup of J_X. If τ is a normal equivalence on E_S then τ induces* *an s-congruence on X. On the other hand, if ρ is an s-congruence on X, if ρ induces the normal equivalence τ on E_S and τ, in turn, induces the s-congruence ρ' on X, then $\rho = \rho'$. In particular, the mapping $\beta: \rho \rightarrow \tau$ defines an order isomorphism of $\Gamma_2(X)$ into $\Theta(S)$, and the mapping $\tau \rightarrow \rho$ into $\Gamma_2(X)$ is into $\Gamma_2(X)$. Thus, if S is full in T_X then, by Proposition 2.3, the mapping $\tau \rightarrow \rho$ defines an order isomorphism of $\Theta(S)$ onto $\Gamma_2(X)$.*

Proof. Let the normal equivalence τ on E_S induce the c-congruence ρ on X. For any $x, y \in X$, we clearly have

$$\begin{aligned} \Delta(e_x e_y) &= \Delta(e_x) \cap \Delta(e_y) \\ &= \{z: z \leqq x\} \cap \{z: z \leqq y\} \\ &= \{z: z \leqq x \wedge y\} \\ &= \Delta(e_{x \wedge y}). \end{aligned}$$

Hence $e_x e_y = e_{x \wedge y}$. Also, from Proposition 2.3, we have that $(x, y) \in \rho$ if and only if $(e_x, e_y) \in \tau$. So now suppose that $(x, y) \in \rho$ and $z \in X$. Then $(e_x, e_y) \in \tau$ and so $(e_{x \wedge z}, e_{y \wedge z}) = (e_x e_z, e_y e_z) \in \tau$. Hence $(x \wedge z, y \wedge z) \in \rho$ and ρ is an s-congruence.

Now suppose that ρ is an s-congruence, that ρ induces the normal equivalence τ and τ, in turn, induce ρ'. Let $(x, y) \in \rho$. Then, by Lemma 3.9, $(e_x, e_y) \in \tau$. Hence, for $e \in V(x)$, $e \geqq e_x$, $(e_x, e_y) \in \tau$ and $e_y \in V(y)$. Thus $e \in V_\tau(y)$ and $V(x) \subseteqq V_\tau(y)$. Similarly, $V(y) \subseteqq V_\tau(x)$ and so $V_\tau(x) = V_\tau(y)$ and $(x, y) \in \rho'$. Thus $\rho \subseteqq \rho'$.

Conversely, let $(x, y) \in \rho'$. Then $V_\tau(x) = V_\tau(y)$. Hence $e_x \in V_\tau(y)$ and $e_y \in V_\tau(x)$. Thus there exist $e_1, e_2, f_1, f_2 \in E_S$ such that

$$e_x \geqq e_1,\ (e_1, e_2) \in \tau \quad \text{and} \quad e_2 \geqq e_y \tag{3.1}$$

and

$$e_y \geqq f_1,\ (f_1, f_2) \in \tau \quad \text{and} \quad f_2 \geqq e_x\ . \tag{3.2}$$

Therefore

$$e_x \geqq e_1 e_y,\ (e_1 e_y, e_y) = (e_1 e_y, e_2 e_y) \in \tau\ ,$$

and

$$e_y \geqq f_1 e_x,\ (f_1 e_x, e_x) = (f_1 e_x, f_2 e_x) \in \tau\ .$$

Hence

$$(e_1 e_y, e_x e_y) = (e_x e_1 e_y, e_x e_y) \in \tau$$

and

$$(f_1e_x, e_xe_y) = (e_yf_1e_x, e_ye_x) \in \tau .$$

Thus $(e_1e_y, f_1e_x) \in \tau$ and $(e_x, e_y) \in \tau$. Hence, by Lemma 3.9, $(x, y) \in \rho'$ and $\rho' \subseteqq \rho$. Thus $\rho = \rho'$.

Let the s-congruences ρ and ρ' induce the normal equivalences τ and τ'. If $\rho \subseteqq \rho'$ then $\tau \subseteqq \tau'$, by Corollary 3.8. Let $\tau \subseteqq \tau'$. Since, by the above τ and τ' induce, in turn, ρ and ρ' it follows from Theorem 2.2 that $\rho \subseteqq \rho'$. Hence β is an order isomorphism of $\Gamma_2(X)$ into $\Theta(S)$.

4. The case $\delta(e) \neq \varnothing$. Throughout this section we assume that X is a semilattice, that $S \subseteqq J_X$ and that $\delta(e) \neq \varnothing$ for all $e \in E_S$. The representations of Propositions 3.2, 3.3, 3.4 and 3.6 all satisfy this condition. However, for the main result of this section we shall require further hypotheses.

LEMMA 4.1. *Let X be a semilattice, $S \subseteqq J_X$ and $\delta(e) \neq \varnothing$, for all $e \in E_S$. Let τ be a normal equivalence on E_S and suppose that τ induces an s'-congruence ρ on X. Let ρ, in turn, induce the normal equivalence τ' on E_S. Then $\tau' \subseteqq \tau$.*

Proof. Let $(e, f) \in \tau'$. Then $U(e) = U(f)$. Let $x \in \delta(e)$. Then $x\rho \cap \Delta(f) \neq \varnothing$ and so there exists a $y \in x\rho$ such that $y \in \Delta(f)$ or $f \in V(y)$. Thus $f \in V(y) \subseteqq V_\tau(y) = V_\tau(x)$ and so there exist $f_1, f_2 \in E_S$ such that

$$f \geqq f_1,\ (f_1, f_2) \in \tau \quad \text{and} \quad f_2 \geqq e , \tag{4.1}$$

since $f_2 \in V(x)$ if and only if $f_2 \geqq e$. Similarly, there exist $e_1, e_2 \in E_S$ such that

$$e \geqq e_1,\ (e_1, e_2) \in \tau \quad \text{and} \quad e_2 \geqq f . \tag{4.2}$$

Now (4.1) and (4.2) are just the statements (3.1) and (3.2) with e and f replacing e_x and e_y. Hence, as in Theorem 3.10, we can deduce that $(e, f) \in \tau$.

In the absence of the assumption that $\delta(e) \neq \varnothing$, for all $e \in E_S$, Lemma 4.1 need not hold.

EXAMPLE. Let $I = [0, 1]$, the interval of real numbers from 0 to 1 under the natural ordering. Let I' denote the half open interval $[0, 1)$. Let S be the subsemigroup $\{e_i: i \in I\}$ of idempotents of $J_{I'}$ where

$$\Delta(e_i) = \begin{cases} \{r \in I: r \leqq i\} & \text{if} \quad i \neq 1 , \\ \{r \in I: r < 1\} & \text{if} \quad i = 1 . \end{cases}$$

Let τ be the normal equivalence on $S = E_S$ determined by the partition $S = \{e_i : i < 1\} \cup \{e_1\}$ of S. Then τ induces the s-congruence $\rho = I' \times I'$ on I' and ρ, in turn, induces the normal equivalence $\tau' = S \times S$ on S. Thus $\tau \subset \tau'$.

Even in the presence of the assumption that $\delta(e) \neq \emptyset$, for all $e \in E_S$, we may not have $\tau = \tau'$.

EXAMPLE. Let X be the semilattice of Figure 2.

Let S be the subsemigroup of J_X consisting of the idempotents f, g, h where $\Delta(f) = \{u, v, w, x\}$, $\Delta(g) = \{v, w\}$, $\Delta(h) = \{w\}$. If τ is the normal equivalence partitioning S as $S = \{f, g\} \cup \{h\}$ then ρ_τ has classes $\{u, v\}$, $\{w\}$, $\{x\}$ and ρ_τ is an s-congruence.

However, if ρ_τ induces the normal equivalence τ' then τ' is the identity equivalence and so $\tau' \subset \tau$.

THEOREM 4.2. *Let X be a semilattice, S be an inverse subsemigroup of J_X and $\delta(e) \neq \emptyset$, for all $e \in S$. Let a normal equivalence τ on E_S induce an s'-congruence ρ on X. Let ρ, in turn, induce the normal equivalence τ' on E_S. If any of the following conditions hold then $\tau = \tau'$:*

(1) *X is totally ordered;*

(2) *ρ is an s'-congruence and $X = \bigcup_{e \in E_S} \delta(e)$; in particular, if S is full in T_X;*

(3) *ρ is an s-congruence and $S \subseteqq T_X$.*

Note. If X is totally ordered or, by Theorem 3.10, if S is full in T_X, then every normal equivalence induces an s-congruence.

Proof. We have from Lemma 4.1, that $\tau' \subseteqq \tau$ in each case.

(1) Let $(e, f) \in \tau$ and suppose that $x\rho \cap \Delta(e) \neq \emptyset$. Without loss of generality let $x \in \Delta(e)$. Since X is totally ordered so also must E_S be totally ordered. If $f \geqq e$ then $\Delta(f) \supseteqq \Delta(e)$ and $x\rho \cap \Delta(f) \neq \emptyset$. So suppose that $f < e$ and that $y \in \delta(f)$. If $y \geqq x$ then $x \in \Delta(f)$ and again $x\rho \cap \Delta(f) \neq \emptyset$. Suppose that $x > y$. Then $V(x) \subseteqq V(y)$ and so $V_\tau(x) \subseteqq V_\tau(y)$. Now let $g \in V(y)$. Then $g \geqq f$, $(f, e) \in \tau$ and $e \in V(x)$. Hence $g \in V_\tau(x)$. Thus $V(y) \subseteqq V_\tau(x)$, $V_\tau(y) = V_\tau(x)$ and $(x, y) \in \rho$. Thus we again have $x\rho \cap \Delta(f) \neq \emptyset$. Thus $U(e) \subseteqq U(f)$ and conversely, by similarity. Thus $(e, f) \in \tau'$ and so $\tau = \tau'$.

(2) Let $(e, f) \in \tau$ and $x\rho \cap \Delta(e) \neq \emptyset$. Let $x \in \Delta(e)$ and $x \in \delta(k)$. Then $k \leqq e$ and $(k, kf) = (ke, kf) \in \tau$. Let $y \in \delta(kf)$. Then, by Proposition 2.3, $(x, y) \in \rho$ and $y \in \Delta(kf) \subseteqq \Delta(f)$. Thus $U(e) \subseteqq U(f)$ and conversely, by similarity. Hence $(e, f) \in \tau'$ and $\tau = \tau'$.

(3) Let $(e, f) \in \tau$. Let $\Delta(e) = \langle x_e \rangle$ and $\Delta(f) = \langle x_f \rangle$. By

Proposition 2.3, $(x_e, x_f) \in \rho$. Let $x\rho \cap \Delta(e) \neq \varnothing$ and suppose that $x \in \Delta(e)$. Then $x \leqq x_e$ and $(x, x \wedge x_f) = (x \wedge x_e, x \wedge x_f) \in \rho$, since ρ is an s-congruence. Also $x \wedge x_f \in \Delta(f)$ and so $x\rho \cap \Delta(f) \neq \varnothing$. Hence $U(e) \subseteqq U(f)$ and conversely. Thus $(e, f) \in \tau'$ and $\tau = \tau'$.

5. **Inducing congruences on S.** Let X be a semilattice, $S \subseteqq J_X$ and ρ be an s'-congruence on X. We have seen that ρ indhces a normal equivalence on E_S and in this section we show how to define two congruence relations on S in the corresponding θ-class directly. In certain circumstances these will be the smallest and largest congruences in that θ-classes.

PROPOSITION 5.1. *Let X be a semilattice, S be an inverse subsemigroup of J_X and let ρ be an s'-congruence on X. Define the relation $\xi = \xi_\rho$ on S by*

$$(a, b) \in \xi \Leftrightarrow \begin{cases} \text{(i)} & U(a) = U(b)\ ; \\ \text{(ii)} & x \in \Delta(a),\ y \in \Delta(b) \text{ and } (x, y) \in \rho \\ & \text{implies that } (xa, yb) \in \rho. \end{cases}$$

Then ξ is a congruence on S, in fact, the congruence induced on S by the homomorphism α of Theorem 3.7. If ρ is induced by some normal equivalence σ on E_S, as in Theorem 2.2, if $\tau = \xi|_{E_S}$ and $\delta(e) \neq \varnothing$, for all $e \in E_S$, then $\xi = \mu_\tau$, the maximum congruence in the θ-class containing ξ.

Proof. Since ξ is just the congruence on S induced by the homomorphism α of Theorem 3.7, the first part of the theorem requires no verification.

For the final assertion, since we must have $\xi \subseteqq \mu_\tau$, it suffices to show that $\mu_\tau \subseteqq \xi$.

Let $(a, b) \in \mu_\tau$. Then $(aa^{-1}, bb^{-1}) \in \tau$, while $\Delta(a) = \Delta(aa^{-1})$ and $\Delta(b) = \Delta(bb^{-1})$. Hence, by the definition of τ, a and b satisfy condition (i). Now let $(x, y) \in \rho$, $x \in \Delta(a)$ and $y \in \Delta(b)$. We want $(xa, yb) \in \rho$. Since ρ is induced from σ we wish to show that $V_\sigma(xa) = V_\sigma(yb)$.

Let $e \in V(xa)$. Then $xa \in \Delta(e)$ and $x \in \Delta(aea^{-1})$. Hence $aea^{-1} \in V(x) \subseteqq V_\sigma(y)$ and so, for some $f_1, f_2 \in E_S$, we have

$$aea^{-1} \geqq f_1,\ (f_1, f_2) \in \sigma \quad \text{and} \quad f_2 \in V(y)\ .$$

Hence $yb = yf_2b \in \Delta(b^{-1}f_2b)$, where $(b^{-1}f_1b, b^{-1}f_2b) \in \sigma$, since σ is a normal equivalence. Also $(b^{-1}f_1b, a^{-1}f_1a) \in \tau$, by Lemma 1.2, since $(a, b) \in \mu_\tau$. But, by Lemma 4.1, $\tau \subseteqq \sigma$. Hence $(a^{-1}f_1a, b^{-1}f_2b) \in \sigma$ and

$$e \geqq a^{-1}aea^{-1}a \geqq a^{-1}f_1a,\ (a^{-1}f_1a, b^{-1}f_2b) \in \sigma \quad \text{and} \quad b^{-1}f_2b \in V(yb)\ .$$

Thus $e \in V_\sigma(yb)$ and $V_\sigma(xa) \subseteqq V_\sigma(yb)$. By similarity, we have equality and so $(xa, yb) \in \rho$, as required. Hence $(a, b) \in \xi$, $\mu_\tau \subseteqq \xi$ and so $\mu_\tau = \xi$.

PROPOSITION 5.2. *Let X be a semilattice and S be an inverse subsemigroup J_X. Let ρ be an s'-congruence on X. Define the relation η on S by*

$(a, b) \in \eta \Leftrightarrow$ (i) $U(a) = U(b)$

(ii) *If $x\rho \in (a) = U(b)$ then there exists a $y \in x\rho$ such that $y \in \varDelta(a) \cap \varDelta(b)$ and $za = zb$, for all $z \leqq y$, $z \in X$.*

Then η is a congruence on S. If $\eta|_{E_S} = \tau$ and either of the following two conditions holds then $\eta = \sigma_\tau$, the minimum congruence in the θ-class containing η:

(1) $S \supseteqq E_{J_X}$;

(2) *ρ is an s-congruence and S is full in T_X.*

Proof. Let $(a, b) \in \eta$. We first show that $(a, b) \in \xi$, where ξ is as in Proposition 5.1. Then, for any $c \in S$, we shall have (ac, bc) and $(ca, cb) \in \xi$ and so, since ξ is a congruence, we shall have $U(ac) = U(bc)$ and $U(ca) = U(ca) = U(cb)$.

Since the conditions (i) are identical, we need only verify that a and b satisfy condition (ii) in Proposition 5.1. Let $x \in \varDelta(a)$, $y \in \varDelta(b)$ and $(x, y) \in \rho$. Then there exists a y_1 such that $(x, y_1) \in \rho$ and $za = zb$, for all $z \leqq y_1$. Hence $y_1 a = y_1 b$, $(xa, y_1 a) \in \rho$, $(yb, y_1 b) \in \rho$ and so $(xa, yb) \in \rho$. Thus $(a, b) \in \xi$, $U(ac) = U(bc)$ and $U(ca) = U(cb)$.

Now let $x\rho \in U(ac) = U(bc)$. Then $x\rho \cap \varDelta(a) \neq \varnothing$ and $x\rho \cap \varDelta(b) \neq \varnothing$. Hence there is a $y_1 \in x\rho$ such that $za = zb$ for all $z \leqq y_1$. Let $y_2 \in x\rho \cap \varDelta(ac)$, $y_3 \in x\rho \cap \varDelta(bc)$ and $y = y_1 \wedge y_2 \wedge y_3$.

Then $y \in x\rho \cap \varDelta(ac) \cap \varDelta(bc)$ and for all $z \leqq y$, $zac = zbc$. Thus $(ac, bc) \in \eta$.

The proof that $(ca, cb) \in \eta$ is similar and so η is a congruence.

To show that $\eta = \sigma_\tau$, we need, by Lemma 1.2, to show that, for any $(a, b) \in \eta$,

(1) $(aa^{-1}, bb^{-1}) \in \tau$;

(2) there exists an $e \in E_S$ such that $(e, aa^{-1}) \in \tau$ and $ea = eb$.

The first requirement is satisfied since η is a congruence and $\eta|_{E_S} = \tau$.

Now suppose that $S \supseteqq E_{J_X}$. Let $U(a) = U(b) = \{x_i\rho : i \in I\}$. For each $i \in I$, let $y_i \in x_i\rho$ be such that $za = zb$, for all $z \leqq y_i$. Let e be the idempotent S with domain $\bigcup_{i \in I} \langle y_i \rangle$. Then clearly, by the definition of e, $U(aa^{-1}) = U(a) \subseteqq U(e)$. On the other hand, we clearly have $e \leqq aa^{-1}$ and so $U(e) \subseteqq U(aa^{-1})$. Thus $U(e) = U(aa^{-1})$ and $(e, aa^{-1}) \in \tau$. Also $ea = eb$ and so $(a, b) \in \sigma_\tau$. Thus $\eta = \sigma_\tau$.

Finally suppose that ρ is an s-congruence and that $S \subseteqq T_X$. Let $aa^{-1} = e_x$ and $bb^{-1} = e_y$. Since $(e_x, e_y) \in \tau$, by Lemma 3.9, $(x, y) \in \rho$ and so there exists a z such that $(x, z) \in \rho$ and $z_1 a = z_1 b$ for all $z_1 \leqq z$. Then, again by Lemma 3.9, $(e_x, e_z) \in \tau$ while clearly $e_z a = e_z b$. Thus

$(a, b) \in \sigma_\tau$ and $\eta = \sigma_\tau$.

COROLLARY 5.3. *Let S be a full inverse subsemigroup of T_X. Let τ be a normal equivalence on E_S and let τ induce the s-congruence ρ on X. Then the congruences ξ and η of Propositions* 5.1 *and* 5.2 *are respectively μ, the maximum congruence, and σ_τ, the minimum congruence in the θ-class determined by τ.*

Proof. That τ induces an s-congruence ρ and that ρ, in turn induces τ follows from Proposition 3.10. The result then follows from Propositions 5.1 and 5.2.

6. $\Theta(S)$ and $\Gamma_2(X)$. By a *lattice* (*semilattice*) *homomorphism* α of a lattice (semilattice) A into a lattice (semilattice) B we mean a mapping α of A into B such that $(x \wedge y)\alpha = x\alpha \wedge y\alpha$ and $(x \vee y)\alpha = x\alpha \vee y\alpha((x \wedge y)\alpha = x\alpha \wedge y\alpha)$ for all $x, y \in A$. A lattice (semilattice) *isomorphism* is then a one-to-one lattice (semilattice) homomorphism.

In the next two theorems we essentially summarize some of the previous results.

THEOREM 6.1. *Let X be a semilattice. If X is a full inverse subsemigroup of J_X, then the mapping $\alpha: \tau \rightarrow \rho_\tau$, of Theorem* 2.2, *from $\Theta(S)$ into $\Gamma(X)$ is a semilattice homomorphism onto $\Gamma_2(X)$.*

If S is a full inverse subsemigroup of T_X then α is a lattice isomorphism of $\Theta(S)$ onto $\Gamma_2(X)$.

If X is totally ordered and $\delta(e) \neq \varnothing$, for all $e \in E_S$, then α is an order isomorphism of $\Theta(S)$ into $\Gamma_2(X)$.

Proof. That α maps $\Theta(S)$ onto $\Gamma_2(X)$, when S is full in J_X, follows from Theorem 3.10. Let τ_1 and τ_2 be normal equivalences, let $\tau_3 = \tau_1 \cap \tau_2$ and $\rho_i = (\tau_i)\alpha$, $i = 1, 2, 3$. Then from Theorem 2.2, $\rho_3 \subseteqq \rho_1 \cap \rho_2$. Let $(x, y) \in \rho_1 \cap \rho_2$. Then by Proposition 2.3, $(e_x, e_y) \in \tau_1 \cap \tau_2 = \tau_3$. Hence, again by Proposition 2.3, $(x, y) \in \rho_3$. Thus $\rho_3 = \rho_1 \cap \rho_2$ and α is a semilattice homomorphism.

If S is full in T_X, then by Proposition 3.10, α is a one-to-one semilattice homomorphism of $\Theta(S)$ onto $\Gamma_2(X)$ and hence is a lattice isomorphism.

If X is totally ordered, then every c-congruence is an s-congruence and so, by Proposition 2.3, α is an o-isomorphism of $\Theta(S)$ into $\Gamma_2(X)$.

THEOREM 6.2. *Let X be a semilattice and S be an inverse subsemigroup of J_X. Let β denote the mapping $\rho \rightarrow \tau_\rho$ of Corollary* 3.8.

If S is full in J_X then β is an o-isomorphism of $\Gamma_2(X)$ into $\Theta(S)$. If S is full in T_X then $\beta = \alpha^{-1}$, where α is defined as in Theorem

6.1.

If X is totally ordered and $\delta(e) \neq \varnothing$, for all $e \in E_S$, then β is an order preserving mapping of $\Gamma_2(X)$ onto $\Theta(S)$.

Proof. If S is full in J_X then, from Theorem 3.10, β is an order isomorphism of $\Gamma_2(X)$ into $\Theta(S)$.

If S is full in T_X then, from Theorem 3.10, $\beta\alpha = \iota_{\Gamma_2(X)}$ and, from Theorem 4.2, $\alpha\beta = \iota_{\Theta(S)}$.

Hence $\beta = \alpha^{-1}$.

Finally, if X is totally ordered and $\delta(e) \neq \varnothing$, for all $e \in E_S$, then β is order preserving, by Corollary 3.8, and β maps $\Gamma_2(S)$ onto $\Theta(S)$ by Theorem 4.2.

If S is a full inverse subsemigroup of J_X, it is natural to ask to what extent the properties of S are determined by those of $S \cap T_X$. We shall denote by $S\Gamma_2(X)$ the lattice of s-congruences under S to distinguish it from the lattice of s-congruences $T\Gamma_2(X)$ under some other semigroup T.

PROPOSITION 6.3. *Let X be a semilattice and S be a full inverse subsemigroup J_X. Let $T = S \cap T_X$. Then $S\Gamma_2(X) = T\Gamma_2(X)$.*

Proof. Clearly $S\Gamma_2(X) \subseteq T\Gamma_2(X)$. Let $\rho \in T\Gamma_2(X)$, $(x, y) \in \rho$, $x, y \in \Delta(a)$, for some $x, y \in X$, $a \in S$. Let e_x denote the idempotent of T with domain $< x >$. Since $\rho \in T\Gamma_2(X)$, we have $(x, x \wedge y) \in \rho$ and $x, x \wedge y \in \Delta(a)$. Also $x, x \wedge y \in \Delta(e_x)$. Hence $x, x \wedge y \in \Delta(e_x a)$ and $e_x a \in T$. Hence $(xe_x a, (x \wedge y)e_x a) \in \rho$; that is, $(xa, (x \wedge y)a) \in \rho$. Similarly $(ya, (x \wedge y)a) \in \rho$ and so $(xa, ya) \in \rho$. Thus $\rho \in S\Gamma_2(X)$ and we have the result.

COROLLARY 6.4. *Under the hypothesis of Proposition 6.3, there exists a semilattice homomorphism of $\Theta(S)$ onto $\Theta(T)$.*

Proof. The result follows from Theorem 6.1 and Proposition 6.3.

REMARK. Let S be an inverse semigroup and μ be the maximum idempotent separating congruence on S. Since $\Theta(S) = \Theta(S/\mu)$ and since, by Proposition 3.2, S/μ is isomorphic to a full inverse subsemigroup of T_{E_S} one might question the need to study other kinds of inverse subsemigroups of J_X apart from those that are full subsemigroups of T_X. (If S is a full inverse subsemigroup of T_X then it is not difficult to see that the representation of S as a semigroup of partial transformations of X is isomorphic in a natural way to the repesentation of S given by Proposition 3.2. on E_S.) However, this assumes a prior knowledge of the semigroup sufficient to identify the representation of S on E_S. If the semigroup is known as a semigroup of partial transformations, it may be quite difficult to identify the representation on E_S while it might be relatively simple to work with the semigroup of partial transformations as given.

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