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SEMIGROUPS WITH DIMINISHING ORBITAL DIAMETERS

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In this paper, fixed point theorems for semigroups of self-mappings on a metric space (X, d) subject to conditions on the size of the orbits are considered.

The concepts of diminishing orbital diameters (d.o.d.) for semigroups of mappings on a metric space and that of convex diminishing orbital diameters (c.d.o.d.) for semigroups of mappings on a convex subset of a normed linear space are introduced. Also discussed are the concepts of linearly ordered semigroups and in particular those that are Archimedean at some of its members. Certain results of Belluce and Kirk concerning a single mapping satisfying d.o.d. are generalized. Also included are results on semigroups of self-mappings on a weakly compact, convex subset of a Banach space.

1. The concept of "diminishing orbital diameters" of a single self-mapping f on a metric space (X, d) was first introduced by Belluce and Kirk in their paper [1]. For any point $x \in X$, let $O(x) = \{x, f(x), f^2(x), \cdots\}$ and $\delta[O(x)]$ denote the diameters of $O(x)$. It is clear that the sequence $\{\delta[O(f^n(x))]: n = 1, 2, \cdots\}$ is nonincreasing. Let $r(x) = \inf \{\delta[O(f^n(x))]: n = 1, 2, \cdots\}$, then $r(x) \geqq 0$ for every $x \in X$. The mapping f is said to have *diminishing orbital diameters* (*d.o.d.*) on X if and only if for every $x \in X$, the condition $r(x) < \delta[0(x)]$ holds whenever $\delta[0(x)] > 0$.

In this paper, we consider semigroups $\mathscr{F}$ of self-mappings with identity. Let $\mathscr{F}(x) = \{f(x): f \in \mathscr{F}\}$ and $\mathscr{F} f(x) = \{gf(x): g \in \mathscr{F}\}$. Suppose $\delta[\mathscr{F}(x)]$ denotes the diameter of $\mathscr{F}(x)$, and $r(x) = \inf \{\delta[\mathscr{F} f(x)]: f \in \mathscr{F}\}$ for every $x \in X$. $\mathscr{F}$ is said to have *diminishing orbital diameters* (*d.o.d.*) on X, if and only if for every $x \in X$, we have $\delta[\mathscr{F}(x)] < \infty$ and the conditions $r(x) < \delta[\mathscr{F}(x)]$ holds whenever $\delta[\mathscr{F}(x)] > 0$. It is clear that if $\mathscr{F}$ is generated by a single mapping f, then $\mathscr{F}$ has d.o.d. implies that f has d.o.d. and vice versa. When $\mathscr{F}$ is a group, it clearly fails to have d.o.d.

If $\mathscr{F}$ satisfies d.o.d., then for every $x \in X$, $\delta[\mathscr{F}(x)] > 0$ implies that there exists $g \in \mathscr{F}$ such that $\delta[\mathscr{F} g(x)] < \delta[\mathscr{F}(x)]$, i.e. there exists $p \in \mathscr{F}(x)$ such that $\delta[\mathscr{F}(p)] < \delta[\mathscr{F}(x)]$. A requirement on $\mathscr{F}$ weaker than d.o.d., when X is a convex subset of a normed linear space, is motivated by the above observation and the next example.

EXAMPLE. Let $X = E^2$ with the sup norm metric, and $f: X \rightarrow X$

defined by $f(a, b) = (|b|, -b)$ for every $(a, b) \in X$.

The mapping f can be easily shown to be nonexpansive. Let $\mathscr{F}$ be the semigroup generated by f and the identity. For $z = (1, 1)$, since $\delta[\mathscr{F}(z)] > 0$ and $\mathscr{F} \mid \mathscr{F}(z)$ is a family of isometries, $\mathscr{F}$ fails to have d.o.d. However, the point $p = (1, 1/2)$ in $\overline{\mathrm{co}}\,\mathscr{F}(z)$ satisfies $\delta[\mathscr{F}(p)] = 1 < 2 = \delta[\mathscr{F}(z)]$. In fact, every point $p \in \overline{\mathrm{co}}\,\mathscr{F}(z)$ satisfies $\delta[\mathscr{F}(p)] < \delta[\mathscr{F}(z)]$.

Let X be a convex subset of a normed linear space and $\mathscr{F}: X \to X$ a semigroup of self-mappings. $\mathscr{F}$ is said to have *convex diminishing orbital diameters* (*c.d.o.d.*) on X if and only if for every $x \in X$, $\delta[\mathscr{F}(x)] < \infty$ and the condition $\delta[\mathscr{F}(x)] > 0$ implies that there exists $p \in \overline{\mathrm{co}}\,\mathscr{F}(x)$ such that $\delta[\mathscr{F}(p)] < \delta[\mathscr{F}(x)]$.

We introduce next the concept of a linearly ordered semigroup. This is motivated by the observation of certain properties possessed by a flow (see [4]). Let $\{f_s: s \in S\}$ be a set of continuous self-mappings of a subset X of a Banach space B, where S (written additively) is a commutative topological semigroup with identity element $\bar{o}$ such that $f_{\bar{o}}(x) = x$ for all $x \in B$, and $f_s(f_t(x)) = f_t(f_s(x)) = f_{t+s}(x)$ for all $x \in B$ and $s, t \in S$, and satisfying the continuity condition that for each $t \in S$, $\sup\{\|f_t(x) - f_s(x)\| : x \in B\} \to 0$, as $s \to t$. Then $\{f_s: s \in S\}$ is called a *S-semigroup of operators* on X. In the case when $S = \boldsymbol{R}^+$, the non-negative real numbers with the usual topology, the semigroup $\{f_s: s \in \boldsymbol{R}^+\}$ is called a *flow*.

The following two properties are satisfied by a flow:

(1) Let $\mathscr{F}_t = \mathscr{F} f_t = \{f_a: a \in \boldsymbol{R}^+ + t\}$. For $s < t$, we have $\mathscr{F}_t \subseteqq \mathscr{F}_s$. Hence, the linear ordering of $\boldsymbol{R}^+$ induces a linear ordering in $\mathscr{F}$ in an obvious fashion.

(2) Let $f_s, f_t \in \mathscr{F}$, with $s \leqq t$. Suppose $s \neq 0$, then there is an integer n such that $n \cdot s \geqq t$. Hence, we have $\mathscr{F}(f_s)^n \subseteqq \mathscr{F} f_t$.

In general, let (X, d) be a metric space and $\mathscr{F}$ a semigroup of self-mappings on X. $\mathscr{F}$ will be called *linearly ordered* if it satisfies the condition that for every $f, g \in \mathscr{F}$, either $\mathscr{F} f \subseteqq \mathscr{F} g$ or $\mathscr{F} g \subseteqq \mathscr{F} f$. In the case when $\mathscr{F} f \subseteqq \mathscr{F} g$, we say that f follows g and denote this fact by $f \geqq g$. Let $\mathscr{F}$ be a linearly ordered semigroup. $\mathscr{F}$ is said to be Archimedean at $g \in \mathscr{F}$ with $g \neq Id$ (the identity mapping) if for every $f \in \mathscr{F}$ with $g \leqq f$, there exists a positive integer n such that $g^n \geqq f$. The semigroup $\mathscr{F}$ is said to be linearly ordered, Archimedean, if $\mathscr{F}$ is Archimedean at each $g \in \mathscr{F}$ where $g \neq Id$.

Clearly a flow and also any semigroup with a single generator are linearly ordered, Archimedean semigroups.

2. In this section, results of Belluce and Kirk [1], [5], concern-

ing mappings satisfying diminishing orbital diameters (d.o.d.) are generalized.

For a single mapping f on a metric space, Belluce and Kirk [1] showed that the condition that the mapping f satisfies d.o.d. is sufficient for it to have a fixed point if f is furthermore nonexpansive and possesses a f-closure point. It was shown later by Kirk [5] that for a compact metric space, this condition guarantees a fixed point when nonexpansiveness of f is replaced by continuity.

We generalize the above results on a single mapping on a metric space to the case of a commutative semigroup of self-mappings having d.o.d. While in the nonexpansive case a result of Edelstein [2] is used by Belluce and Kirk to prove their result for a single mapping, the following proposition uses a result of Holmes and Narayanaswami [3] concerning commutative asymptotically-nonexpansive semigroups of self-mappings.

A semigroup $\mathscr{F}: (X, d) \to (X, d)$ is called asymptotically-nonexpansive iff for every $x, y \in X$, there exists $g \in \mathscr{F}$ such that $d[fg(x), fg(y)] \leqq d(x, y)$ for all $f \in \mathscr{F}$.

The set $\{z \in X$: there exists $x \in X$ such that for every $f \in \mathscr{F}$, $\varepsilon > 0$, there exists $g \in \mathscr{F}$ with $d[fg(x), z] < \varepsilon\}$ is called the $\mathscr{F}$-closure of X and is denoted by $X^{\mathscr{F}}$.

LEMMA 2.1. (Proposition 2 in [3]). *Let (X, d) be a commutative semigroup of continuous asymptotically-nonexpansive mappings on X. If $z \in X^{\mathscr{F}}$, then $\mathscr{F} \mid \mathscr{F}(z)$ is a family of isometries.*

PROPOSITION 1. *Let X be a metric space, $\mathscr{F}$ a commutative semigroup of continuous asymptotically-nonexpansive self-mappings on X. If $\mathscr{F}$ has diminishing orbital diameters (d.o.d.) and there exists $z \in X^{\mathscr{F}}$, then z is a common fixed points of $\mathscr{F}$.*

Proof. By the preceding lemma, $\mathscr{F} \mid \mathscr{F}(z)$ is a family of isometries. Hence, we have

$$\begin{aligned}\delta[\mathscr{F} f(z)] &= \sup\{d[gf(z), g'f(z)]: g, g' \in \mathscr{F}\}\\ &= \sup\{d[fg(z), fg'(z)]: g, g' \in \mathscr{F}\}\\ &= \sup\{d[g(z), g'(z)]: g, g' \in \mathscr{F}\}\\ &= \delta[\mathscr{F}(z)]\ .\end{aligned}$$

Since the above is true for all $f \in \mathscr{F}$, we have

$$r(z) = \inf\{\delta[\mathscr{F} f(z)]: f \in \mathscr{F}\} = \delta[\mathscr{F}(z)]\ .$$

As $\mathscr{F}$ has d.o.d., we have $\delta[\mathscr{F}(z)] = 0$, showing that z is a common fixed point of $\mathscr{F}$.

The following corollary is immediate.

COROLLARY. *Let X and $\mathscr{F}$ be as in Proposition* 1 *and replace the condition that $\mathscr{F}$ has d.o.d. by the condition that $\mathscr{F}$ is nonisometric on the orbit of some point $z \in X^{\mathscr{F}}$* (i.e. there exists $g \in \mathscr{F}$ and points x, y in the orbit of some point z in $X^{\mathscr{F}}$ such that $d[g(x), g(y)] \neq d(x, y)$). *Then z is a common fixed point of $\mathscr{F}$.*

THEOREM 2.2. *Let X be a compact metric space, $\mathscr{F}: X \to X$ a commutative semigroup of continuous mappings with* d.o.d. *Then for every $x \in X$, there exists $z \in \bigcap_{f \in \mathscr{F}} \overline{\mathscr{F} f(x)}$ such that z is a common fixed point of $\mathscr{F}$.*

Proof. For every $x \in X$, by the commutativity of $\mathscr{F}$ and the compactness of X, $\bigcap_{f \in \mathscr{F}} \overline{\mathscr{F} f(x)} \neq \varnothing$. Let $A = \bigcap_{f \in \mathscr{F}} \overline{\mathscr{F} f(x)}$. Then A is a nonempty closed (and hence compact) subset which is invariant under $\mathscr{F}$.

By Zorn's lemma, there is a nonempty, minimal closed subset K of A which is invariant under $\mathscr{F}$. Suppose $\delta[K] > 0$. Let $z \in K$, then $\overline{\mathscr{F}(z)}$ is a closed subset of A which is invariant under $\mathscr{F}$. Hence $K = \overline{\mathscr{F}(z)}$.

Since $\mathscr{F}$ has d.o.d., we have $r(z) = \inf\{\delta[\mathscr{F} f(z)]: f \in \mathscr{F}\} < \delta[\mathscr{F}(z)] = \delta[K]$. Hence, there exists $h \in \mathscr{F}$ such that $\delta[\mathscr{F} h(z)] < \delta[\mathscr{F}(z)]$ which shows that $\delta[\overline{\mathscr{F} h(z)}] < \delta[\overline{\mathscr{F}(z)}]$. Consequently, $\overline{\mathscr{F} h(z)}$ is a proper closed subset of K which is invariant under $\mathscr{F}$, which is impossible. This contradiction shows that $\delta[K] = \delta[\mathscr{F}(z)] = 0$. Hence, z is a common fixed point of $\mathscr{F}$.

3. In the case when X is a weakly compact subset of a Banach space, the following is known (cf. Corollary 2 to Theorem 2 in [1]):–

Let X be a nonempty, convex, weakly compact subset of a Banach space and f a nonexpansive self-mapping on X. Suppose f has d.o.d., then f has a fixed point in X.

The assumption of convexity of X was removed later by Kirk in [6] where he proved the following:—

PROPOSITION 2 (Theorem 5 in [6]). *Suppose X is a nonempty, weakly compact subset of a Banach space. If $f: X \to X$ is nonexpansive and has d.o.d., then f has a fixed point in X.*

We proceed to prove a related result for semigroups of self-mappings.

LEMMA 3.1. *Let (X, d) be a metric space and $\mathscr{F}: X \to X$ a line-*

arly ordered semigroup of mappings. Suppose $\mathscr{F}$ has d.o.d. and there exists a $g \in \mathscr{F}$ with $g \neq Id$ such that

(i) *g has a fixed point*

(ii) *$\mathscr{F}$ is Archimedean at g.*

Then $\mathscr{F}$ has a common fixed point.

Proof. Let $z \in X$ be a fixed point of g. If $\mathscr{F}(z)$ is a singleton, then z is a common fixed point of $\mathscr{F}$. Suppose $\delta[\mathscr{F}(z)] > 0$. Since $\mathscr{F}$ has d.o.d., $\inf\{\delta[\mathscr{F} f(z)]: f \in \mathscr{F}\} < \delta[\mathscr{F}(z)]$. This implies that there exists a mapping $h \in \mathscr{F}$ such that $\delta[\mathscr{F} h(z)] < \delta[\mathscr{F}(z)]$. However, by (ii), there exists an integer n such that $g^n \geqq h$, i.e. $\mathscr{F} g^n \subseteqq \mathscr{F} h$. Hence, we obtain $\mathscr{F} g^n(z) \subseteqq \mathscr{F} h(z) \subseteqq \mathscr{F}(z) = \mathscr{F} g^n(z)$, showing that $\mathscr{F}(z) = \mathscr{F} h(z)$, which is a contradiction.

THEOREM 3. *Let X be a nonempty weakly compact subset of a Banach space and $\mathscr{F}: X \to X$ be a linearly ordered semigroup of mappings. Suppose $\mathscr{F}$ has d.o.d. and there exists a $g \in \mathscr{F}$ with $g \neq Id$ such that*

(i) *g is a nonexpansive mapping with d.o.d.*

(ii) *$\mathscr{F}$ is Archimedean at g.*

Then $\mathscr{F}$ has a common fixed point.

Proof. By Proposition 2, the mapping g has a fixed point in X. By the preceding lemma, $\mathscr{F}$ has a common fixed point.

COROLLARY. *Let X be as in Theorem 3. Suppose $\mathscr{F}: X \to X$ is a flow having d.o.d. If there exists a mapping $g \in \mathscr{F}$ with $g \neq Id$ such that g has a fixed point, then $\mathscr{F}$ has a common fixed point.*

Proof. Since a flow is a linearly ordered, Archimedean semigroup, the result is immediate from the lemma.

3.2. The weaker hypothesis of c.d.o.d. is used in the next result. We proceed by first proving a lemma.

LEMMA. *Let X be a nonempty, convex, weakly compact subset of a Banach space. Suppose $\mathscr{F}: X \to X$ is a commutative lineary ordered semigroup of nonexpansive mappings with $\mathscr{F}$ having c.d.o.d. If for every weakly closed $\mathscr{F}$-invariant subset K_0 of X there exists a member $g_0 \in \mathscr{F}$ with $g_0 \neq Id$ such that*

(i) *g_0 has a fixed point in K_0*

(ii) *$\mathscr{F}$ is Archimedean at g_0, then $\mathscr{F}$ has a common fixed point.*

Proof. Let K be a nonempty convex subset which is minimal with respect to being weakly closed and invariant under $\mathscr{F}$. If $\delta[K] = 0$, we immediately obtain a common fixed point of $\mathscr{F}$.

Suppose $\delta[K] > 0$. By the hypotheses of the lemma, there exists a mapping $g_0 \in \mathscr{F}$ with $g_0 \neq Id$, such that $\mathscr{F}$ is Archimedean at g_0, and a point $z \in K$ such that $g_0(z) = z$. Suppose z is not a common fixed point of $\mathscr{F}$, then $\delta[\mathscr{F}(z)] > 0$. Since $\mathscr{F}$ has c.d.o.d., there exists a point $p \in \overline{\text{co}}\, \mathscr{F}(z) \subseteqq K$ such that $\delta[\mathscr{F}(p)] < \delta[\mathscr{F}(z)]$. Hence there exists $r > 0$ such that $\delta[\mathscr{F}(p)] < r < \delta[\mathscr{F}(z)]$.

Let $C_g(p) = \bigcap_{f \in \mathscr{F}} \bar{B}(fg(p), r)$ and $U = \bigcup_{g \in \mathscr{F}} C_g(p)$. Since for each $g \in \mathscr{F}$, the set $C_g(p)$ contains $\mathscr{F}(p)$, we have $U \neq \varnothing$. We proceed to show that U is convex and invariant under $\mathscr{F}$.

(1) Each $C_g(p)$ is clearly convex. Since $\mathscr{F}$ is linearly ordered, the collection $\{C_g(p) : g \in \mathscr{F}\}$ is nested. Hence, U is convex as the union of a nested family of convex subsets.

(2) Let $x \in U$, then $x \in C_g(p)$, for some $g \in \mathscr{F}$, i.e. $\|x - fg(p)\| \leqq r$, for all $f \in \mathscr{F}$. Since each $t \in \mathscr{F}$ is nonexpansive, we have

$$\|t(x) - tfg(p)\| \leqq \|x - fg(p)\| \leqq r \,,$$

for all $f \in \mathscr{F}$. Hence,

$$t(x) \in \bigcap_{f \in \mathscr{F}} \bar{B}(tfg(p), r) = C_{tg}(p) \subseteqq U \,,$$

showing that U is invariant under $\mathscr{F}$.

Consequently, (1) and (2) imply that $\bar{U}$ is closed, convex and invariant under $\mathscr{F}$.

Since $\bar{U} \cap K \supseteqq \mathscr{F}(p)$, and K is invariant under $\mathscr{F}$. $\bar{U} \cap K$ is a nonempty, closed, convex, weakly compact subset in K which is invariant under $\mathscr{F}$. By the minimality of K we have $K \cap \bar{U} = K$, which implies that $K \subseteqq \bar{U}$.

Next, we proceed to show that $\bigcap_{q \in \mathscr{F}(z)} \bar{B}(q, r) \neq \varnothing$. Let $q \in \mathscr{F}(z)$, then $q \in K \subseteqq \bar{U}$. For any $\varepsilon > 0$, there is $q' \in U$ such that $\|q - q'\| < \varepsilon$. Now $q' \in U$ implies that $q' \in C_{h_\varepsilon}(p)$, for some $h_\varepsilon \in \mathscr{F}$. This shows that $\|q' - fh_\varepsilon(p)\| \leqq r$ for all $f \in \mathscr{F}$. Hence, $\|q - fh_\varepsilon(p)\| < r + \varepsilon$, for all $f \in \mathscr{F}$, which implies that $\bar{B}(z, r + \varepsilon) \supseteqq \mathscr{F} h_\varepsilon(p)$. This shows that $\bar{B}(z, r + \varepsilon) \supseteqq \overline{\text{co}}\, \mathscr{F} h_\varepsilon(p)$. Now, for each $g \in \mathscr{F}$, the set $\overline{\text{co}}\, \mathscr{F} g(p)$ is a closed, convex subset of the weakly compact set K and is therefore itself weakly compact. By the commutativity of $\mathscr{F}$, the collection $\{\overline{\text{co}}\, \mathscr{F} g(p) : g \in \mathscr{F}\}$ has f.i.p. Hence, $\bigcap_{g \in \mathscr{F}} \overline{\text{co}}\, \mathscr{F} g(p) \neq \varnothing$. Consequently, we obtain $\bar{B}(q, r + \varepsilon) \supseteqq \overline{\text{co}}\, \mathscr{F} h_\varepsilon(p) \supseteqq \bigcap_{g \in \mathscr{F}} \overline{\text{co}}\, \mathscr{F} g(p)$. Since the above result holds for every $\varepsilon > 0$, we have $\bar{B}(q, r) \supseteqq \bigcap_{g \in \mathscr{F}} \overline{\text{co}}\, \mathscr{F} g(p)$. As q is arbitrarily chosen from $\mathscr{F}(z)$ we obtain

$$\bigcap_{q \in \mathscr{F}(z)} \bar{B}(q, r) \supseteqq \bigcap_{g \in \mathscr{F}} \overline{\text{co}}\, \mathscr{F} g(p) \,.$$

As $\bigcap_{g \in \mathscr{F}} \overline{\text{co}}\, \mathscr{F} g(p) \subseteqq K$, the set $A = \bigcap_{q \in \mathscr{F}(z)} [\bar{B}(q, r) \cap K]$ is nonempty.

That A is a proper, convex, weakly closed, $\mathscr{F}$-invariant subset of K can be shown as follows:—

(i) Since $r < \delta[\mathscr{F}(z)]$, A is a proper subset of K.

(ii) Since each $\bar{B}(q, r) \cap K$ (where $q \in \mathscr{F}(z)$) is weakly closed, A is weakly closed.

(iii) To show that A is invariant under $\mathscr{F}$, it suffices to show that $\bigcap_{q \in \mathscr{F}(z)} \bar{B}(q, r)$ is invariant under $\mathscr{F}$.

Now, for any $x \in \bigcap_{q \in \mathscr{F}(z)} \bar{B}(q, r)$, we have $\|x - q\| \leqq r$ for all $q \in \mathscr{F}(z)$. Let $h \in \mathscr{F}$. For any $q \in \mathscr{F}(z)$, we have $q = f(z)$ for some $f \in \mathscr{F}$.

(a) For the case when $\mathscr{F} f \subseteqq \mathscr{F} : h$—

Since $f = f'h$ for some $f' \in \mathscr{F}$ we have

$$\|h(x) - q\| = \|h(x) - f(z)\| = \|h(x) - hf'(z)\| \leqq \|x - f'(z)\| \leqq r .$$

(b) For the case when $\mathscr{F} h \subseteqq \mathscr{F} f$:—

Since z is a fixed point of g_0, we have $q = f(z) = fg_0^n(z)$ for any $n \in \boldsymbol{N}$.

(1) Suppose $\mathscr{F} fg_0 \subseteqq \mathscr{F} h$. As in (a), we obtain

$$\|h(x) - q\| = \|h(x) - fg_0(z)\| \leqq r .$$

(2) Suppose $\mathscr{F} fg_0 \supseteqq \mathscr{F} h$, then $\mathscr{F} g_0 \supseteqq \mathscr{F} h$, i.e. $g_0 \leqq h$. Since $g_0 \neq Id$ and $\mathscr{F}$ is Archimedean at g_0, there exists an integer j such that $g_0^j \geqq h$. This shows that $\mathscr{F} g_0^j \subseteqq \mathscr{F} h$, or $\mathscr{F} fg_0^j \subseteqq \mathscr{F} h$. Hence $fg_0^j = f''h$ for some $f'' \in \mathscr{F}$, which implies that

$$\|h(x) - q\| = \|h(x) - fg_0^j(z)\| = \|h(x) - f''h(z)\| \leqq r .$$

Hence, $\|h(x) - q\| \leqq r$ for all $q \in \mathscr{F}(z)$ in either case (a) or (b), i.e. $h(x) \in \bigcap_{q \in \mathscr{F}(z)} \bar{B}(q, r)$, showing that A is invariant under $\mathscr{F}$.

However, results (i)—(iii) contradict the minimality of K, which shows that K is necessarily a singleton, i.e. a common fixed point.

THEOREM 4. *Let X be a nonempty, convex, weakly compact subset of a Banach space. Let $\mathscr{F} : X \rightarrow X$ be a commutative, linearly ordered semigroup of nonexpansive mappings such that $\mathscr{F}$ has c.d.o.d. Suppose $\mathscr{F}$ is Archimedean at $g_0 \in \mathscr{F}$ with $g_0 \neq Id$ such that g_0 has d.o.d. Then $\mathscr{F}$ has a common fixed point in X.*

Proof. Let K be a nonempty convex subset in X which is minimal with respect to being weakly closed and invariant under $\mathscr{F}$. Since a mapping having d.o.d. on X also has d.o.d. on every $\mathscr{F}$-invariant subset of X, g_0 has d.o.d. on K. By Proposition 2, g_0 has a fixed point in K since K is weakly compact. By the preceding lemma, $\mathscr{F}$ has a common fixed point in X.

COROLLARY. *Let X be a nonempty, convex, weakly compact subset of a Banach space. Suppose $\mathscr{F}: X \to X$ is a flow of nonexpansive mappings with c.d.o.d. If there exists $g_0 \in \mathscr{F}$ with $g_0 \neq Id$ such that g_0 has d.o.d., then $\mathscr{F}$ has a common fixed point.*

Proof. Since a flow is a commutative, linearly ordered, Archimedean semigroup, the result is immediate from Theorem 4.

4. Examples of commutative semigroups with diminishing orbital diameters can be easily constructed. The following is an example of such a semigroup.

EXAMPLE. Let

$$X = \left\{\left(0, \frac{1}{2^n}\right): n = 0, 1, \cdots\right\} \cup \left\{\left(\frac{1}{2^n}, \frac{1}{2^n}\right): n = 0, 1, \cdots\right\} \cup (0, 0)\,.$$

Suppose $\{f_n: n = 0, 1, \cdots\}$ is a family of mappings defined by:

$$f_n(x, y) = \left(0, \frac{1}{2^n}\right), \text{ where } y > \frac{1}{2^n}$$

$$f_n\left(0, \frac{1}{2^n}\right) = \left(\frac{1}{2^n}, \frac{1}{2^n}\right),$$

$$f_n\left(\frac{1}{2^n}, \frac{1}{2^n}\right) = \left(0, \frac{1}{2^n}\right) \text{ and}$$

$$f_n(x, y) = (x, y) \text{ where } y < \frac{1}{2^n}\,.$$

Since $\{f_n: n = 0, 1, \cdots\}$ is a commutative family, the semigroup $\mathscr{F}$ generated by it and the identity mapping is commutative. Each f_n is nonexpansive and fails to have d.o.d. while $\mathscr{F}$ clearly satisfies d.o.d. Indeed, for each point $p = (x, y)$ with $\delta[\mathscr{F}(p)] > 0$, since $y = 1/2^n$ where $n \in \{0, 1, \cdots\}$ we have $\delta[\mathscr{F} f_m(p)] < \delta[\mathscr{F}(p)]$ for $m > n$.

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