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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $x'' + a(t)f(x) = e(t)$

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In this paper sufficient conditions are given which insure that all solutions of

$$x'' + a(t)f(x) = e(t)$$

tend to zero as $t \to \infty$. Results obtained are comparable to those obtained for the linear equations via two Liouville transformations. Also, related results concerning stability and boundedness of solutions and, when $e(t) = 0$, necessary and sufficient conditions for the uniqueness of the zero solution on an interval where $a(t)$ is negative are given.

1. We consider the equation:

$$x'' + (p(t) + d(t))f(x) = e(t), \; (' = d/dt) \, , \tag{1}$$

where p, d, e: $[0, \infty) \to R$, f: $R \to R$, $p(t) > 0$, $xf(x) > 0$ for $x \neq 0$, and $p''(t)$, $d(t)$, $e(t)$ and $f(x)$ are continuous.

In this paper the basic problem is to give conditions which guarantee that all solutions of (1) tend to zero as $t \to \infty$. Also, conditions are given which insure that all solutions of (1) are bounded and, in the case $e(t) = 0$, we examine the Liapunov stability of the zero solution of

$$\begin{aligned} x' &= w \\ w' &= -(p(t) + d(t))f(x) \, . \end{aligned} \tag{2}$$

The problem of insuring that all solutions of (1) tend to zero as t tends to infinity has been examined by a number of authors (cf. [3], [5], [7], [8], [9], [11], [12], [13] and the references therein). The problem of boundedness and stability of the solutions of (1) and (2) has also been studied quite extensively (cf. [2], [3], [5], [9], [12]). Our results however, differ from most of the results found in the above references in that it will not usually be assumed that $p(t) + d(t)$ is positive or even differentiable. Even if $a(t) = p(t) + d(t)$ is positive and differentiable our results will differ from much previous work in that $a'(t)$ or $a'(t)/a^r(t)$ where r is any fixed positive number may behave in an extremely wild manner (cf. [8], [9]).

We shall first examine the case where (1) is linear and then consider the nonlinear case. In the last section we compare the results obtained with some well known results for the linear equation.

2. In the linear case (1) has the form

$$x'' + (p(t) + d(t))x = e(t) \tag{3}$$

where p, d, and e are as above. The Liouville transformation

$$y(s) = x(t), \qquad s = \int_0^t \sqrt{p(u)}\,du \tag{4}$$

maps (3) into

$$\frac{d^2y}{ds^2} + [p'(t)/2p^{3/2}(t)]\frac{dy}{ds} + (1 + d(t)/p(t))y = e(t)/p(t)$$

which we write as

$$\ddot{y} + 2\mu(t)\dot{y} + (1 + d(t)/p(t))y = e(t)/p(t), \qquad (^{\cdot} = d/ds)\ , \tag{5}$$

where $\mu(t) = p'(t)/4p^{3/2}(t)$ and consider the equivalent system

$$\begin{aligned} \dot{y} &= z - \mu(t)y \\ \dot{z} &= -\mu(t)z + (\mu^2(t) + \dot{\mu}(t) - 1 - d(t)/p(t))y + e(t)/p(t)\ . \end{aligned} \tag{6}$$

THEOREM 1. *All solutions of* (3) *tend to zero as t tends to infinity if*:

(i) $\int^{\infty} |e(t)|/\sqrt{p(t)}\,dt < \infty$,

(ii) $\int^{\infty} (-2\mu(t) + |\mu^2(t) + \dot{\mu}(t) - d(t)/p(t)|)ds = -\infty$

and either

(iii) *There exists $M > 0$ so that for any r and s with $0 < r < s$,*

$$\int_r^s (-2\mu(t) + |\mu^2(t) + \dot{\mu}(t) - d(t)/p(t)|)ds \leqq M$$

or

(iv) $e(t) \equiv 0$.

Proof. Define a Liapunov function V by

$$V(y, z, s) = z^2/2 + y^2/2\ . \tag{7}$$

Differentiating V along a solution of (6), we see that

$$\begin{aligned} \dot{V} &\leqq -2\mu(t)V + |\mu^2(t) + \dot{\mu}(t) - d(t)/p(t)||yz| + (|e(t)|/p(t))|z| \\ &\leqq (-2\mu(t) + |\mu^2(t) + \dot{\mu}(t) - d(t)/p(t)| + |e(t)|/2p(t))V + |e(t)|/2p(t) \end{aligned}$$

which we write as

$$\dot{V} \leqq \lambda_1(s)V + \lambda_2(s)\ . \tag{8}$$

Assume first that (iii) holds. Then, for $s \geqq s_0 \geqq 0$, V as a function of s along a solution of (6) satisfies

$$V(s) \leqq \exp \int_{s_0}^{s} \lambda_1(u)du[V(s_0) + \int_{s_0}^{s} (\exp - \int_{s_0}^{u} \lambda_1(v)dv)\lambda_2(u)du] .$$

Now, if $V(s)$ is eventually bounded away from zero, say $V(s) \geqq k > 0$ for $s \geqq s_0$, we have

$$V(s) \exp - \int_{s_0}^{s} \lambda_1(u)du \leqq V(s_0) + \int_{s_0}^{s} V(u) (\exp - \int_{s_0}^{u} \lambda_1(v)dv)(\lambda_2(u)/k)du$$

and by Gronwall's inequality,

$$V(s) \exp - \int_{s_0}^{s} \lambda_1(u)\, du \leqq V(s_0) \exp \int_{s_0}^{s} (\lambda_2(u)/k)\, du .$$

From (i), (ii), and (4), however, it follows that

$$\int^{\infty} \lambda_2(u)\, du < \infty \text{ and } \int^{\infty} \lambda_1(u)\, du = -\infty$$

and so it must be that $V(s)$ tends to zero as $s \to \infty$. This contradiction implies that there must be a sequence $\{s_n\}$ tending to infinity with n with the property that $V(s_n) \to 0$ as $n \to \infty$.
Now, for $s \geqq s_n$, we have,

$$\begin{aligned} V(s) &\leqq V(s_n) \exp \int_{s_n}^{s} \lambda_1(u)\, du + \int_{s_n}^{s} \lambda_2(u) \exp \int_{u}^{s} \lambda_1(v)\, dv\, du \\ &\leqq V(s_n) \exp M + \int_{s_n}^{s} \lambda_2(u) \exp M\, du \end{aligned}$$

and thus, $V(s)$ tends to zero as s tends to infinity.

From (7), we see that $y(s)$ must have limit zero as s goes to infinity. As $y(s) = x(t)$, the proof is now complete if (iii) holds.

If (iv) is valid, from (ii) and the fact that $\lambda_2(s)$ is identically zero, it follows that $V(s)$ tends to zero as s goes to infinity and the proof may be completed as above.

As an example, we see that all solutions of

$$x'' + (t^3 + t^{1/2} \sin t^{\alpha})\, x = t^{1/3} ,$$

with α being any positive number, tend to zero as $t \to \infty$. Here, $p(t) = t^3$, $d(t) = t^{1/2} \sin t^{\alpha}$. Also, we should note that (ii) in Theorem 1 implies $p(t) \to \infty$ as $t \to \infty$ as

$$\int^{\infty} \mu(t)\, ds = (1/4) \int^{\infty} (p'(t)/p(t))\, dt .$$

3. For the purposes of this section we write

$$p(t) = c(t)\, b(t) \tag{9}$$

where $c(t)$ and $b(t)$ are defined by

$$\begin{aligned} c(t) &= p(0) \exp \int_0^t [p'(s)_+/p(s)]\, ds \\ b(t) &= \exp - \int_0^t [p'(s)_-/p(s)]\, ds \end{aligned} \tag{10}$$

with $p'(t)_+ = \max\{p'(t), 0\}$ and $p'(t)_- = \max\{-p'(t), 0\}$ so that $p'(t) = p'(t)_+ - p'(t)_-$. We shall require throughout this section that

$$\int_0^\infty [p'(s)_-/p(s)]\, ds < \infty$$

so that $b(t)$ is bounded away from zero. With this decomposition of $p(t)$, we see that $c(t)$ is nondecreasing and $b(t)$ is nonincreasing and that $c'(t)$ and $b'(t)$ exist and are continuous. Actually, it is easy to see, as $p''(t)$ is continuous, that $c'(t)$ is of bounded variation and $c''(t)$ exists almost everywhere. As $c''(t)$ will not in general be continuous some of the systems which we will examine will be of Caratheodory type. This will present no difficulties, however, as the Liapunov functions which we use will always be continuously differentiable (see for example [15: pp. 10–11]) and, hence, in the future we will take the second derivative of $c(t)$ when convenient and without further comment.

The following definition and lemma will be helpful in our analysis of the nonlinear case.

Suppose f is defined and continuous on $[0, \infty) \times N \to R^n$ where N is a neighborhood of the origin in R^n and consider the equation

$$x' = f(t, x)\ . \tag{11}$$

DEFINITION. The set $\{0\}$ is eventually uniformly stable with respect to (11) if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $\tau = \tau(\varepsilon) > 0$ so that $\|x(t, t_0, x_0)\| < \varepsilon$, $t \geqq t_0 \geqq \tau(\varepsilon)$ $t_0 \geqq \tau(\varepsilon)$, provided $\|x_0\| \leqq \delta$, where $x(t, t_0, x_0)$ is any solution of (11) with $x(t_0, t_0, x_0) = x_0$.

LEMMA 1. *If γ is a nonnegative constant and λ_i: $[0, \infty) \to [0, \infty)$ $i = 1, 2$, are continuous and satisfy $\int^\infty \lambda_i(t)\, dt < \infty$, $i = 1, 2$, then $\{0\}$ is eventually uniformly stable with respect to the scalar equation*

$$r' = \lambda_1(t) r^\gamma \operatorname{sgn} r + \lambda_2(t)\ . \tag{12}$$

Further, if $\gamma \leqq 1$, then all solutions of (12) are bounded as $t \to \infty$.

Proof. We consider only the case where $r_0 > 0$, the case where $r_0 \leqq 0$ being similar follows from the fact that $r' \geqq \lambda_1(t) r^\gamma \operatorname{sgn} r$. Let $\varepsilon > 0$ be given, let $\delta = \varepsilon/2$, and let $\tau = \tau(\varepsilon) > 0$ be chosen so that

$$\varepsilon^\gamma \int_\tau^\infty \lambda_1(t)\, dt + \int_\tau^\infty \lambda_2(t)\, dt < \varepsilon/2 .$$

We see from (12) that if $0 < r_0 < \delta$ and $t_0 \geqq \tau(\varepsilon)$, then $r(t, t_0, r_0) < \varepsilon$ for $t \geqq t_0 \geqq \tau(\varepsilon)$.

If $\gamma \leqq 1$, $r' \leqq \lambda_1(t) r + \lambda_2(t)$ for $r \geqq 1$ and the result follows.

We now consider equation (1) which we write as

$$x'' + (c(t)\, b(t) + d(t))\, f(x) = e(t) \tag{13}$$

where $c(t)$ and $b(t)$ are defined by (10).

For the nonlinear case, the Liouville transformation defined by

$$y(s) = x(t), \qquad s = \int_0^t \sqrt{c(u)}\, du \tag{14$'$}$$

maps (13) into

$$\ddot{y} + 2\mu(t)\, \dot{y} + (b(t) + d(t)/c(t))\, f(y) = e(t)/c(t) . \tag{14}$$

where $\mu(t) = c'(t)/4c^{3/2}(t)$.

We first consider the equivalent system,

$$\dot{y} = z$$

$$\dot{z} = -\, 2\mu(t) z - (b(t) + d(t)/c(t))\, f(y) + e(t)/c(t) \tag{15}$$

where we shall assume

$$\int^\infty (|d(t)| + |e(t)|)/c^{1/2}(t)\, dt < \infty . \tag{I}$$

Also, it frequently will be useful to assume that there exist nonnegative constants γ, k_1, and k_2 so that

$$f^2(x) \leqq k_1 F^\gamma(x) + k_2 \tag{II}$$

where $F(x)$ is defined by

$$F(x) = \int_0^x f(u)\, du .$$

We note that if $f(x) = x^n \operatorname{sgn} x$, $n > 0$, we may choose $\gamma = 2n/n + 1$ and $k_2 = 0$. Also, (II) is clearly satisfied in a bounded neighborhood of $x = 0$ by any $f(x)$ and will certainly be satisfied if $f(x)$ is bounded.

If we define the Liapunov function $W(y, z, s)$ by

$$W(y, z, s) = (z^2/2 + b(t)\,F(y))\,E(s)\,, \tag{16}$$

where

$$E(s) = \exp - \int_0^s [(|d(t)| + |e(t)|)/c(t)]\,du$$

with $t = t(u)$, and differentiate W along trajectories of (15), we have,

$$\begin{aligned}\dot{W} &\leqq [(|d(t)| + |e(t)|)/c(t)]\,W + [(|d(t)||f(y)z| + |e(t)||z|)/c(t)]\,E(s)\\ &\leqq [(|d(t)|\,f^2(y) + |e(t)|)/2c(t)]\,E(s)\\ &\leqq (k_1\,|d(t)|/2c(t))\,F^{\gamma}(y)\,E(s) + [(k_2|d(t)| + |e(t)|)/2c(t)]\,E(s)\end{aligned}$$

and hence,

$$\dot{W} \leqq \lambda_1(s)\,W^{\gamma} + \lambda_2(s) \tag{17}$$

where $\lambda_i(s) \geqq 0$, $i = 1, 2$, and

$$\int^{\infty} \lambda_i(s)\,ds < \infty\,, \qquad i = 1, 2\,.$$

With the aid of (17) we are now prepared to prove the following theorem.

THEOREM 2. *If* (I) *and* (II) *are valid, then* $\{(0, 0)\}$ *is eventually uniformly stable with respect to* (15). *If in* (II) γ *can be chosen so that* $\gamma \leqq 1$ *and*

$$\int_0^{\pm\infty} f(u)\,du = \infty \tag{18}$$

then all solutions of (15), *and hence, all solutions of* (13) *are bounded in the future. If* $c(t)$ *is bounded and* $e(t) = 0$, *the zero solution of*

$$\begin{aligned} x' &= w\\ w' &= -\,(c(t)\,b(t) + d(t))\,f(x)\end{aligned} \tag{19}$$

is uniformly stable if either $\gamma \geqq 1$ *and* $k_2 = 0$ *or* $c(t)\,b(t) + d(t)$ *is positive and continuously differentiable.*

Proof. With the functions $\lambda_i(s)$, $i = 1, 2$, in (17) we see from Lemma 1 that $\{0\}$ is eventually uniformly stable with respect to (12). It follows now from [10; Theorem 3.14.1, p. 223], (16) and (17) that $\{(0, 0)\}$ is eventually uniformly stable with respect to (15).

If $\gamma \leqq 1$, we see that all solutions of (12) are bounded in the future. Hence, it follows from (17) and a standard comparison theorem that W is bounded along trajectories of (15). From (16) we see that if (18) is valid then the solutions of (15) are bounded in the future and as $x(t) = y(s)$, all the solutions of (13) are bounded in the future.

If $c(t)$ is bounded, the stability properties of (19) are the same as those of (15) as $x'(t) = \dot{y}(s)c^{1/2}(t)$. Also, if $k_2 = 0$ and $e(t) = 0$, $\lambda_2(s)$ is identically zero and if $\gamma \geqq 1$, the zero solution of (12) is unique and thus, the zero solution is uniformly stable ([10; p. 222]) and so the zero solution of (19) is uniformly stable ([10; Theorem 3.3.4, p. 141]). If $c(t)b(t) + d(t)$ is positive and continuously differentiable, the zero solution of (19) is unique, ([2; Theorem 7]) and the result follows.

REMARK 1. As (II) can be satisfied in a bounded neighborhood of $x = 0$ by any function f, we see that $\{(0, 0)\}$ is eventually uniformly stable with respect to (15) if (I) holds. Similarly, (II) is not required to hold for all x to obtain the stability of the zero solution of (19). Finally, we note that Theorem 2 is an extension to the nonlinear case of the theorem which states all solutions of

$$x'' + (1 + \phi(t) + \psi(t))x = 0$$

are bounded if $\int^{\infty} |\phi(t)|\,dt < \infty$, $\int^{\infty} |\psi'(t)|\,dt < \infty$, and $\psi(t) \to 0$ as $t \to \infty$ (cf [1; p. 112]).

Write (14) now in the equivalent form

$$\begin{aligned} \dot{y} &= z - \mu(t)y \\ \dot{z} &= -\mu(t)z + (\dot{\mu}(t) + \mu^2(t))y - (b(t) + d(t)/c(t))\,f(y) + e(t)/c(t)\,. \end{aligned} \tag{20}$$

If $\mu(t)$ is bounded as $t \to \infty$, then as $z = \dot{\mu} + \mu y$, we see that all solutions of (15) are bounded in the future if and only if all solutions of (20) are bounded in the future. Also, we see that $\{(0, 0)\}$ is eventually uniformly stable with respect to (15) if and only if it is eventually uniformly stable with respect to (20). Thus, if $\mu(t)$ is bounded we may apply the results of Theorem 2 to (20).

THEOREM 3. *Assume* (I), (II), *and* (18) *hold and that* γ *in* (II) *can be chosen so that* $\gamma \leqq 1$. *If* $c(t) \to \infty$ *as* $t \to \infty$, $\mu(t)$ *is bounded, and if*

$$\int^{\infty} |\dot{\mu}(t) + \mu^2(t)|\,ds < \infty \tag{21}$$

then every solution of (13) *tends to zero as* $t \to \infty$.

Proof. We shall show that every solution of (20) tends to zero as $s \to \infty$ and, hence, as the solutions of (20) and (13) are related by $y(s) = x(t)$, we will have every solution of (13) tending to zero as $t \to \infty$.

Let $(y(s), z(s))$ be a solution of (20). If

$$\liminf_{s \to \infty} ||(y(s), z(s))|| = 0\,,$$

Theorem 2 yields that $\{(0, 0)\}$ is eventually uniformly stable with respect to (20) as $\mu(t)$ is bounded and, thus, we must have $(y(s), z(s)) \to (0, 0)$ as $s \to \infty$ and we are done.

From Theorem 2 we also see that $(y(s), z(s))$ is bounded in the future so if

$$\liminf_{s\to\infty} \|(y(s), z(s))\| > 0 ,$$

there is an annulus A with center at the origin and $s_0 \geqq 0$ with the property that $(y(s), z(s)) \in A$ for $s \geqq s_0$. If we define the Liapunov function $J(y, z, s)$ by

$$J = z^2/2 + b(t)\, F(y) ,$$

along trajectories of solutions of (20) we have,

$$\begin{aligned} \dot{J} \leqq -\mu(t)(z^2 + b(t)f(y)y) &+ |\dot{\mu}(t) + u^2(t)|\,|yz| \\ &+ (|d(t)|/c(t))\,|f(y)z| + (|e(t)|/c(t))\,|z| \,. \end{aligned}$$

For $(y, z) \in A$, there exist positive constants M_1, M_2 so that $|f(y)z| \leqq M_1$, $|yz| \leqq M_1$, $|z| \leqq M_1$ and $(z^2 + b(t)\, f(y)y) \geqq M_2$. Thus, along $(y(s), z(s))$ for $s \geqq s_0$ we have,

$$\dot{J} \leqq -\mu(t)M_2 + (|\dot{\mu}(t) + \mu^2(t)| + (|d(t)| + |e(t)|)/c(t))M_1 \,.$$

Since $c(t) \to \infty$ with t, we see that

$$\int^\infty \mu(t)\, ds = \infty$$

and so $J(y(s), z(s), s) \to -\infty$ as $s \to \infty$ which is a contradiction and the proof is complete.

REMARK 2. Theorems 2 and 3 are "best possible" as $c(t)b(t) + d(t) = a(t)$ need not be smooth nor even eventually positive. In particular, if $a(t)$ is negative on some interval $[t_1, t_2]$ and

$$\int^\infty [1 + F(x)]^{-1/2}\, dx < \infty$$

then the equation

$$x'' + a(t)\, f(x) = 0 \tag{22}$$

has solutions which go to infinity monotonically in finite time [4: Theorem 2]. Thus, if $\gamma > 1$ in (II) we cannot in general have all solutions of (13) bounded or tending to zero as $t \to \infty$ since $f(x) = x^3$ satisfies (II) with $\gamma > 1$. It $a(t)$ is positive, (22) may still have solutions with finite escape time, [6], if $a(t)$ is not locally of bounded variation.

Also, if the zero solution of (19) is not unique to the right at $t = t_0$ it clearly cannot be stable at t_0. The next theorem gives criteria for the uniqueness of the zero solution of (19) from which we will see that if there is γ, $0 < \gamma < 1$, for which $f^2(x) \geqq k_1 F^{\gamma}(x)$ with $k_1 > 0$, then the zero solution is not unique to the right when $a(t) < 0$.

THEOREM 4. *Suppose* $a(t) < 0$ *on some interval* $[t_1, t_2]$. *The zero solution of* (19) *is unique on* $[t_1, t_2]$ *if and only if*

(a)
$$\int_{0+}^{1} [F(x)]^{-1/2}\, dx = \infty$$

and

(b)
$$\int_{0-}^{-1} [F(x)]^{-1/2}\, dx = -\infty$$

Proof. Suppose (a) doesn't hold and consider solutions of (19) with initial conditions $(0, w(t_1))$, $w(t_1) > 0$. As $a(t)$ and $f(x)$ are continuous, if $0 \leqq w(t_1) \leqq 1$, there exists $t_3 > t_1$ so that these solutions exist on $[t_1, t_3]$ and it follows, [15; Theorem 3.7], that these solutions are uniformly bounded on this interval. Without loss of generality we may assume $t_3 = t_2$.

From (19) we obtain $ww' = -a(t)f(x)x'$ which yields $w^2(t) - w^2(t_1) = -2\int_{t1}^{t} a(s)f(x(s))x'\,ds$. For $t_1 < t \leqq t_2$, as x and w are increasing,

$$-2m\int_{t_1}^{t} f(x(s))x'(s)\,ds > w^2(t) - w^2(t_1) > -2M\int_{t_1}^{t} f(x(s))x'(s)\,ds$$

where m and M are constants so that $m < a(t) < M < 0$ for $t_1 \leqq t \leqq t_2$. Hence,

$$w^2(t_1) - 2mF(x(t)) > w^2(t) > w^2(t_1) - 2MF(x(t))$$

or

$$[w^2(t_1) - 2mF(x(t))]^{1/2} > w(t) > [w^2(t_1) - 2MF(x(t))]^{1/2}\,.$$

As $x' = w$,

$$\int_{0}^{x(t)} [w^2(t_1) - 2MF(s)]^{-1/2}\,ds = \int_{t_1}^{t} [w^2(t_1) - 2MF(x(s))]^{-1/2}\,x'(s)\,ds \geqq t - t_1 \tag{23}$$

for $t_1 \leqq t \leqq t_2$. Now as $\int_{0+}^{1} [F(x)]^{-1/2}\,dx < \infty$, there is $\bar{x} > 0$ so that $\int_{0+}^{\bar{x}} [-2MF(x)]^{-1/2}\,dx < t_2 - t_1$. If $(x(t), w(t))$ is any solution of (19) with $0 < w(t_1) \leqq 1$, $x(t_1) = 0$, we have from (23)

$$\int_0^{x(t)} [-2MF(s)]^{-1/2}\, ds > \int_0^{x(t)} [w^2(t_1) - 2MF(s)]^{-1/2}\, ds \geqq t - t_1 .$$

In particular, we obtain

$$\int_0^{x(t_2)} [-2MF(s)]^{-1/2}\, ds \geqq t_2 - t_1$$

and so $x(t_2) > \bar{x} > 0$.

Choose a sequence of solutions of (19), $\{(x_n(t), w_n(t))\}$ defined on $[t_1, t_2]$ with $(x_n(t_1), w_n(t_1)) = (0, 1/n)$. This sequence of solutions is uniformly bounded and equicontinuous and by Ascoli's Theorem there is a subsequence which converges uniformly to a solution $(x(t), w(t))$ of (19) on $[t_1, t_2]$. Now, as $x_n(t_1) = 0$ and $w_n(t_1) = 1/n$, $x(t_1) = w(t_1) = 0$. Also, $x_n(t_2) > \bar{x} > 0$ for every n and we obtain $x(t_2) \geqq \bar{x} > 0$. This solution contradicts the uniqueness of the zero solution to the right. In a similar fashion one can show the existence of a solution $(x(t), w(t))$ of (19) in the fourth quadrant with the property that $(x(t_2), w(t_2)) = (0, 0)$ and $(x(t_3), w(t_3)) \neq (0, 0)$ for some t_3, $t_1 \leqq t_3 < t_2$. If (b) is not valid we proceed as above in the second or third quadrant.

Now suppose (a) and (b) are valid but the zero solution of (19) is not unique to the right on $[t_1, t_2]$. Without loss of generality, we may assume the existence of a solution $(x(t), w(t))$ of (19) with $(x(t_1), w(t_1)) = (0, 0)$ and $(x(t), w(t)) \neq (0, 0)$ for $t_1 < t \leqq t_2$.

From (19) it is easy to see that $(x(t), w(t))$ must be in the first or third quadrant for $t_1 < t \leqq t_2$. We assume it is in the first quadrant, the argument in the other case being similar.

Arguing as above with $m < a(t) < M < 0$ for $t_1 \leqq t \leqq t_2$ yields

$$\int_{x(t)}^{x(t_2)} [w^2(t) - 2mF(s)]^{-1/2}\, ds \leqq t_2 - t \leqq t_2 - t_1 . \tag{24}$$

From (a), given $k > 0$, there is a value of x, say $\bar{x}$, $0 < \bar{x} < x(t_2)$, so that

$$\int_{\bar{x}}^{x(t_2)} [-2mF(s)]^{-1/2}\, ds > k .$$

Also, as $F(s) > 0$ on $[\bar{x}, x(t_2)]$, there exists $\delta > 0$ so that if $w^2(t) < \delta$, $w^2(t) - 2mF(s) \leqq -8mF(s)$. As $x(t_1) = w(t_1) = 0$, there exists $\bar{t}$, $t_1 < \bar{t} < t_2$, so that $0 < x(\bar{t}) < \bar{x}$ and $w^2(\bar{t}) < \delta$. From (24), we obtain

$$\begin{aligned} t_2 - t_1 &\geqq \int_{x(\bar{t})}^{x(t_2)} [w^2(\bar{t}) - 2mF(s)]^{-1/2}\, ds \\ &\geqq (1/2) \int_{\bar{x}}^{x(t_2)} [-2mF(s)]^{-1/2}\, ds \\ &\geqq k/2. \end{aligned}$$

The constant k is arbitrarily large, however, and we have a contradiction.

If the zero solution of (19) is not unique to the left the argument is similar being carried out in the second or fourth quadrant and the theorem is proved.

With regard to condition (II), suppose $f^2(x) \geqq kF^{\gamma}(x)$ with $0 < \gamma < 1$ and $k > 0$. Then

$$\begin{aligned} \infty &> \int_{0+}^{f(1)} u^{-(\gamma+1)/2}\, du \\ &= \int_{0+}^{1} f(s)\, [F(s)]^{-(\gamma+1)/2}\, ds \\ &\geqq k \int_{0+}^{1} [F(s)]^{-1/2}\, ds \end{aligned}$$

and (a) does not hold and hence, if $a(t) < 0$ on any interval we see that the zero solution of (19) is not unique to the right and cannot be stable where $a(t) < 0$. In a similar fashion it can be shown that if $f^2(x) \geqq kF^{\gamma}(x)$ with $1 < \gamma$ and $k > 0$ then (19) has solutions which have finite escape time whenever $a(t) < 0$.

If in (II), γ must be chosen greater than one it follows from Theorem 2 and the proof of Theorem 3 that solutions of (13) with $x(t_0)$ and $x'(t_0)/\sqrt{c(t_0)}$ small where t_0 is large will tend to zero $t \to \infty$ if the hypotheses of Theorem 3, with the exception of $\gamma \leqq 1$, are valid. However, if (II) does not hold with $\gamma \leqq 1$, further restrictions must be made on $cb + d$ to guarantee that all solutions of (13) tend to zero. The final two theorems of this paper consider this case.

The next theorem should be considered in conjunction with results on the boundedness of solutions of (13) (cf. [2], [3], [9] and the references contained therein).

THEOREM 5. *Suppose that* $d(t)$ *is identically zero and that all solutions* $x(t)$ *of* (13) *are bounded. If* μ *is bounded,* $c(t) \to \infty$ *as* $t \to \infty$, *and if* (I) *and* (21) *are valid, then every solution of* (13) *tends to zero as* $t \to \infty$.

Proof. Define $W_1(y, z, s)$ by

$$W_1(y, z, s) = (z^2/2 + b(t)\, F(y) + 1/2)\, E_1(s)$$

where

$$E_1(s) = \exp - \int_0^s |e(t)|/c(t)\, du \text{ with } t = t(u)\ .$$

Along trajectories of (15), we have

$$\dot{W}_1(y, z, s) \leqq (e(t)/c(t))E_1(s)z - (|e(t)|/c(t))W_1 \leqq 0 .$$

Hence, $z(s)$ is bounded and, as $x(t) = y(s)$, the solutions of (15) are bounded.

As μ is bounded, the solutions of (20) are bounded. Also, from Remark 1 we see that the origin is eventually uniformly stable with respect to (20). The remainder of the proof is parallel to the proof of Theorem 3 and so is omitted.

THEOREM 6. *Suppose $d(t)$ is identically zero, $c(t) \to \infty$ as $t \to \infty$, and* (I), (18), (21) *are valid. Further, suppose $yf(y) \geqq 2\nu F(y)$ for some constant $\nu > 0$ and given $R > 0$, there is $k(R) > 0$ with the property that $|yz| \leqq k(R)(z^2/2 + F(y))$ for $y^2 + z^2 \geqq R^2$. Then all solutions of* (13) *tend to zero as $t \to \infty$.*

Proof. Defining $J(y, z, s)$ as in the proof of Theorem 3 and differentiating along the trajectories of solutions of (20) yields for $y^2 + z^2 \geqq R^2$

$$\begin{aligned}\dot{J} &\leqq -\mu(t)(z^2 + b(t)f(y)y) + |\dot{\mu}(t) + \mu^2(t)||yz| + (|e(t)|/c(t))|z| \\ &\leqq (-k_1\mu(t) + k_2k(R)|\dot{\mu}(t) + \mu^2(t)| + (|e(t)|/2c(t)))J \\ &\quad + (|e(t)|2c(t))\end{aligned}$$

where $k_1 = \min\{2\nu, 2\}$ and $k_2 = 1/(\inf b(t))$. It follows now from (I), (18), and (21) that the solutions of (20) are ultimately bounded for bound $2R$; that is, every solution of (20) must eventually enter and remain in the open $2R$ ball about the origin. The result now follows as R is arbitrary and, hence, $x(t) = y(s) \to 0$ as $t \to \infty$.

4. It is interesting to note that Theorem 6 is directly comparable to a result for linear equations obtained via two Liouville transformations. Consider the linear equation

$$x'' + a(t)x = 0 \tag{25}$$

where $a(t)$ has a continuous second derivative on $[0, \infty)$, $a'(t) \geqq 0$, $a(t) > 0$, and $a(t) \to \infty$ as $t \to \infty$. The transformation (4) with $a(t) = c(t) = p(t)$, maps (25) into

$$\ddot{y} + 2\mu(t)\dot{y} + y = 0$$

where $\mu(t)$ is as above. If we define $v(s)$ by

$$y(s) = v(s)\exp - \int_0^s \mu(t(u))\, du$$

we obtain

$$\ddot{v} + (1 - \mu^2(t) - \dot{\mu}(t))v = 0 . \tag{26}$$

Thus, if (21) is valid, all solutions of (26) are bounded and all solutions of (25) tend to zero as $t \to \infty$. Unfortunately, for the equation

$$x'' + a(t)x^\alpha = 0 \tag{27}$$

with $a(t)$ as above and $\alpha > 1$ the quotient of odd integers, the second Liouville transformation does not yield as tractable an equation as (26). However, it follows as a corollary of Theorem 6 that (21) is also a sufficient condition to insure that all solutions of (27) tend to zero as $t \to \infty$.

Also, if $0 < \alpha < 1$, we see from Theorem 3 that (21) and $\mu(t)$ bounded suffice to guarantee that all solutions of (27) tend to zero as $t \to \infty$.

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