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CANONICAL EXTENSIONS OF MEASURES AND THE EXTENSION OF REGULARITY OF CONDITIONAL PROBABILITIES*

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Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B}$ a sub σ-field of $\mathfrak{A}$. Let $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$, the σ-field generated by $\mathfrak{A}$ and H, where H is a subset of Ω not in $\mathfrak{A}$. P_e will be called a simple extension of P to $\mathfrak{A}'$ if P_e is a probability measure on $\mathfrak{A}'$ which agrees with P on $\mathfrak{A}$.

The purpose of this paper is to use a particular type of simple extension called a canonical extension, denoted as P_c to examine under what conditions the regularity of the conditional probability $P^{\mathfrak{B}}$ will extend to the regularity of $P_c^{\mathfrak{B}}$. Also, if $\mathfrak{A}$ is countably generated and $P_c^{\mathfrak{B}}$ is regular, a characterization of $P_c^{\mathfrak{B}}$ in terms of $P^{\mathfrak{B}}$ will be given.

The terminology in the following definitions will be used throughout this paper.

DEFINITION. The conditional probability of a set $A \in \mathfrak{A}$ given the σ-field $\mathfrak{B}$ is a $\mathfrak{B}$-measurable function denoted by $P^{\mathfrak{B}}(\cdot, A)$ such that for every $B \in \mathfrak{B}$

$$\int_B P^{\mathfrak{B}}(\cdot, A)dP_{\mathfrak{B}} = P(AB).$$

DEFINITION. The conditional probability (given $\mathfrak{B}$) is the collection of functions

$$\{P^{\mathfrak{B}}(\cdot, A) \mid A \in \mathfrak{A}\} .$$

This collection is denoted by $P^{\mathfrak{B}}$.

DEFINITION. For $A \in \mathfrak{A}$, a version of $P^{\mathfrak{B}}(\cdot, A)$ is a selection from the equivalence class of $P^{\mathfrak{B}}(\cdot, A)$ which will be denoted by $p(\cdot, A \mid \mathfrak{B})$.

DEFINITION. A version of the conditional probability $P^{\mathfrak{B}}$ is a function $p(\cdot, \cdot \mid \mathfrak{B})$ on $X \times \mathfrak{A}$ such that for each $A \in \mathfrak{A}$ $p(\cdot, A \mid \mathfrak{B})$ is a version of $P^{\mathfrak{B}}(\cdot, A)$. Also $p(w, \cdot \mid \mathfrak{B})$ will denote a section of $p(\cdot, \cdot \mid \mathfrak{B})$ at $w \in X$.

DEFINITION. A conditional probability $P^{\mathfrak{B}}$ is called regular if there exists a version, $p(\cdot, \cdot \mid \mathfrak{B})$, such that $p(w, \cdot \mid \mathfrak{B})$ is a measure on $\mathfrak{A}$ $P_{\mathfrak{B}}$ a.e.

Before the main body of the paper is presented, it should be observed that the regularity of $P^{\mathfrak{B}}$ itself is not in general sufficient to insure the regularity of $P_c^{\mathfrak{B}}$; for example, see [2], p. 210.

Finally, the scope of this paper is limited to results on canonical extensions. A forthcoming paper will deal with the preservation of regularity for simple extensions.

The main results. Observe that the σ-field

$$\mathfrak{A}' = \{A_1H + A_2H^c \,|\, A_1, A_2 \in \mathfrak{A}\}\ ,$$

and make

DEFINITION 1. Let A' be any element of $\mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$. A simple extension will be called a canonical extension, P_c, if there exists a number α between zero and one with $\beta = 1 - \alpha$ and $K \in \mathfrak{A}$ so that

$$\text{(1.1)} \qquad \begin{aligned} &\text{(a)} \quad A'K^c \in \mathfrak{A} \\ &\text{(b)} \quad P_c(A') = P(A'K^c) + \alpha P(A_1K) + \beta P(A_2K) \end{aligned}$$

with P_c a well defined probability measure on $\mathfrak{A}'$.

Marczewski and Loś have shown, [4], that for any subset of X not in $\mathfrak{A}$, say H, there always exists a canonical extension P_c on $\mathfrak{A}'$. (It has been shown by the author in [1] that there exist many simple extensions which are not canonical.)

REMARK 2. One way of obtaining the set K of Definition 1 is by letting K_1 be an element of $\mathfrak{A}$ such that $(PK_1) = P_*(H)$ and K_2 be an element of $\mathfrak{A}$ such that $P(K_2) = P^*(H)$ with $K_1 \subset H \subset K_2$. Then, simply define $K = K_2 \backslash K_1$. (See [2], P. 71). Observe that there exists another $K' \in \mathfrak{A}$ which will extend P canonically to $\mathfrak{A}'$ as in Definition 1 if and only if $P(K \Delta K') = 0$.

LEMMA 3. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ be given. Let $p(\cdot, \cdot|\mathfrak{B})$ be a version of $P^{\mathfrak{B}}$ which makes $P^{\mathfrak{B}}$ regular. Let P_c be a canonical extension of P to $\mathfrak{A}'$ with α, β and K as in Definition* 1. *Suppose for w, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot|\mathfrak{B})$ is a canonical extension of $p(w, \cdot|\mathfrak{B})$ to $\mathfrak{A}'$ with the same α and β and K as P_c. Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. It will suffice to produce a version of $P_c^{\mathfrak{B}}$ which makes $P_c^{\mathfrak{B}}$ regular.

Let $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$. For w, $P_{\mathfrak{B}}$ a.e.,

$$\text{(3.1)} \qquad \begin{aligned} P_c(w, A'|\mathfrak{B}) = p(w, A'K^c|\mathfrak{B}) &+ \alpha p(w, A_1K\,|\,\mathfrak{B}) \\ &+ \beta p(w, A_2K|\mathfrak{B})\ . \end{aligned}$$

Thus it is immediate from (3.1) that $p_c(\cdot, A'|\mathfrak{B})$ is a $\mathfrak{B}$-measurable function for all $A' \in \mathfrak{A}'$ and for w, $P_{\mathfrak{B}}$ a.e., $p_c(w, \cdot|\mathfrak{B})$ is a measure on $\mathfrak{A}'$. It is also clear that for $A' \in \mathfrak{A}'$ and $B \in \mathfrak{B}$

$$\int_B P_c(\cdot, A'|\mathfrak{B})dP_c = P_c(A'B) \, . \tag{3.2}$$

For, integrating the right side of (3.1) with respect to P gives

$$P(A'K^cB) + \alpha P(A_1KB) + \beta P(A_2KB) = P_c(A'B) \, .$$

But $P_c = P$ on $\mathfrak{B}$ and so the integral of the right side of (3.1) is exactly the left side of (3.2) .

Hence, $p_c(\cdot, \cdot|\mathfrak{B})$ is the desired version.

THEOREM 4. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$, and $\mathfrak{A}'$ be as in Lemma 3. Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot|\mathfrak{B})$ is a version such that*

(4.1) $p(w, \cdot|\mathfrak{B})$ *is a measure* $P_{\mathfrak{B}}$ *a.e.*

(4.2) $p(w, \cdot|\mathfrak{B}) \ll Q(P_{\mathfrak{B}}$ *a.e.) where Q is a probability measure on* $\mathfrak{A}$.

Let P_c be a canonical extension of P to $\mathfrak{A}'$ with respect to α, β and K as in (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. Suppose $K' = K_2 \backslash K_1$, where $K_1 \subset H \subset K_2$, $Q_*(H) = Q(K_1)$ and $Q^*(H) = Q(K_2)$. Consider any set $A \subset K_2 \backslash H$ where $A \in \mathfrak{A}$. $Q(A) = 0$. By (4.2) $p(w, A|\mathfrak{B}) = 0$ ($P_{\mathfrak{B}}$ a.e.) and so therefore $P(A) = 0$ also. Similarly, if $B \subset H \backslash K_1$, where $B \in \mathfrak{A}$, then $Q(B) = 0$ and hence $p(w, B|\mathfrak{B}) = 0$ and so $P(B) = 0$ also. Thus $p^*(w, H|\mathfrak{B}) = p(w, K_2|\mathfrak{B})$ ($P^{\mathfrak{B}}$ a.e.) and $p(w, K_1|\mathfrak{B}) = p_*(w, H|\mathfrak{B})$ ($P_{\mathfrak{B}}$ a.e.). Also, $P(K_1) = P_*(H)$ and $P^*(H) = P(K_2)$. According to Remark 2, $p(w, \cdot|\mathfrak{B})$ can be extended canonically to $\mathfrak{A}'$ with respect to α, β and K' and by Lemma 3 the proof is complete.

The following result is a consequence of Theorem 4.

THEOREM 5. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B}$ and $\mathfrak{A}'$ be as in Lemma 3. Suppose $P^{\mathfrak{B}}$ is regular and $p(\cdot, \cdot|\mathfrak{B})$ is a version such that*

(5.1) $p(w, \cdot|\mathfrak{B})$ *is a measue* $P_{\mathfrak{B}}$ *a.e.*

(5.2) *there exists a sequence $\{w_n\}_{n=1}$ such that for every $\varepsilon > 0$ and any $w(P_{\mathfrak{B}}$ a.e.) there is an w_n with*

$$\sup_{A \in \mathfrak{A}} |p(w, A|\mathfrak{B}) - p(w_n, A|\mathfrak{B})| < \varepsilon \, .$$

Let P_c be a canonical extension of P to $\mathfrak{A}'$ with α, β and K as in (1.1). *Then, $P_c^{\mathfrak{B}}$ is regular.*

Proof. Let Q be a probability measure defined as

$$\sum_{n=1}^{\infty} \frac{1}{2^n} p(w_n, \cdot|\mathfrak{B}) .$$

Condition (5.2) insures that $p(w, \cdot|\mathfrak{B}) \ll Q \ P_{\mathfrak{B}}$ a.e. and the result follows from Theorem 4.

The following proposition is presented for the sake of completeness.

Let $(X, \mathfrak{A}, P)$ be a probability space with $(X, \bar{\mathfrak{A}}, \bar{P})$ denoting the completion. Suppose H is in $\bar{\mathfrak{A}}$ but not in $\mathfrak{A}$. Let $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$.

PROPOSITION 6. *Let $(X, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, and $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ with $H \in \bar{\mathfrak{A}} \backslash \mathfrak{A}$ be given. Let P_1 denote the restriction of $\bar{P}$ to $\mathfrak{A}'$. If $P^{\mathfrak{B}}$ is regular then so is $P_1^{\mathfrak{B}}$.*

The proof can be viewed as an easy consequence of Lemma 3 and is therefore omitted.

The remainder of this paper is devoted to the single

THEOREM 7. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}$ generated by a countable field, $\mathscr{A}$. Let $\mathscr{A}'$ be the field generated by $\mathscr{A}$ and H and $\mathfrak{A}' = \sigma(\mathscr{A}')$. Let P_c be a canonical extension of P to $\mathfrak{A}'$ with respect to α, β and K and suppose $P_c^{\mathfrak{B}}$ is regular where $\mathfrak{B} \subset \mathfrak{A}$. Then, there exists a version $p'(\cdot, \cdot|\mathfrak{B})$ of $P_c^{\mathfrak{B}}$ such that $P_{\mathfrak{B}}$ a.e. $p'(w, \cdot|\mathfrak{B})$ is a probability measure which is a canonical extension of $p'(w, \cdot|\mathfrak{B})|\mathfrak{A}$ with respect to the same α, β and K that are associated with P_c.*

The following lemmas are introduced before presenting the main body of the proof.

LEMMA 8. *Let $(X, \mathfrak{A}, P)$ be a probability space with $\mathfrak{A}' = \sigma(\mathfrak{A}, H)$ and P_e an arbitrary simple extension of P to $\mathfrak{A}'$. Let K be the set associated with a canonical extension of P to $\mathfrak{A}'$ as in Remark 2. Then, for each set $A \in \mathfrak{A}$ there exist constants α_A and β_A with $0 \leqq \alpha_A \leqq 1$ and $0 \leqq \beta_A \leqq 1$ and such that $P_e(AHK) = \alpha_A P(AK)$ and $P_e(AH^cK) = \beta_A P(AK)$.*

Proof. For $A \in \mathfrak{A}, AK \supset AHK$. If $P(AK) \neq 0$, then $\alpha_A = P_e(AHK)/P(AK)$; otherwise, let α_A be arbitrary between zero and one. β_A is obtained similarly.

LEMMA 9. *Assume the hypothesis of Lemma 8. Let $\mathscr{A}$ be a field which generates $\mathfrak{A}$ and $\mathscr{A}'$ the field generated by $\mathscr{A}$ and H. Let $\alpha(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \alpha_A$ and $\beta(\mathscr{A}) \equiv \sup_{A \in \mathscr{A}} \beta_A$. Then, a necessary and sufficient condition that P_e be a canonical extension of P to $\mathfrak{A}'$ is that*

$\alpha(\mathscr{A}) = \alpha_X$ or $\beta(\mathscr{A}) = \beta_X$ for some $\mathscr{A}$ which generates $\mathfrak{A}$.

Proof. Necessity is obvious and only sufficiency is proved. Let $\mathscr{A}$ be some field which generates $\mathfrak{A}$ and $\alpha(\mathscr{A}) = \alpha_X$. (For simplicity, write $\alpha(\mathscr{A}) = \alpha$.) By hypothesis,

$$P_e(HK) = \alpha P(K) .$$

For $A \in \mathscr{A}$ it follows by Lemma 8 that

$$P_e(AHK) = \alpha_A P(AK) \tag{9.1}$$

and

$$P_e(A^cHK) = \alpha_{A^c} P(A^cK) . \tag{9.2}$$

The following equalities also hold

$$\alpha P(K) = \alpha P(AK) + \alpha P(A^cK) \tag{9.3}$$

$$P_e(HK) = P_e(AHK) + P_e(A^cHK) . \tag{9.4}$$

By (9.1) – (9.4) it follows that

$$0 = (\alpha - \alpha_A)P(AK) + (\alpha - \alpha_{A^c})P(A^cK) . \tag{9.5}$$

If $P(AK) = 0$, set $\alpha_A = \alpha$ or if $P(A^cK) = 0$, set $\alpha_{A^c} = \alpha$ (see Lemma 8). Otherwise, (9.5) forces $\alpha - \alpha_A = \alpha - \alpha_{A^c} = 0$ and hence for any $A \in \mathscr{A}$, $P_e(AHK) = \alpha P(AK)$.

Next, the fact that $P_e(AH^cK) = \beta P(AK)$, $\beta = 1 - \alpha$, is immediate from the following chain of equalities:

$$\begin{aligned} P(A) = P_e(AH + AH^c) &= P_e((AH + AH^c)K^c) + P_e(AHK) \\ &+ P_eAH^cK) = P(AK^c) + \alpha P(AK) + P_e(AH^cK) . \end{aligned}$$

Hence, where $\mathscr{A}' = \{A_1H + A_2H^c \mid A_i \in \mathscr{A}\ i = 1, 2\}$, A' in $\mathscr{A}'$ can be written as $A' = A_1H + A_2H^c$ and it follows that

$$P_e(A') = P(A'K^c) + \alpha P(A_1K) + \beta P(A_2K) .$$

Finally, let

$$\begin{aligned} \phi_\alpha &= \{A \in \mathfrak{A} \mid P_e(AHK) = \alpha P(AK)\} \\ \phi_\beta &= \{A \in \mathfrak{A} \mid P_e(AH^cK) = \beta P(AK)\} . \end{aligned}$$

Both ϕ_α and ϕ_β are monotone classes containing $\mathscr{A}$; hence, the proof is complete by the monotone class theorem (see [3], p. 60).

Theorem 7 can now be proved.

Proof. For $w \in X$, $P_{\mathfrak{B}}$ a.e., and $A \in \mathscr{A}$, write

$$p'(w, AHK|\mathfrak{B}) = \alpha_{w,A}p(w, AK|\mathfrak{B})$$

where $0 \leqq \alpha_{w,A} \leqq 1$ as in Lemma 8 and $p(w, \cdot|\mathfrak{B})$ will be written for $p'(w, \cdot|\mathfrak{B})|_{\mathfrak{A}}$. For fixed $A \in \mathscr{A}$, $\alpha_{w,A}$ is a $\mathfrak{B}$-measurable function where

$$(7.1)\qquad \begin{aligned} &\alpha_{w,A} = p'(w, AHK|\mathfrak{B})/p(w, AK|\mathfrak{B}) \text{ for } p(w, AK|\mathfrak{B}) \neq 0 \\ &\alpha_{w,A} = \alpha \text{ if } p(w, AK|\mathfrak{B}) = 0\ . \end{aligned}$$

(In (7.1) α is associated with P_c and by Lemma 9, $\alpha = \sup_{A\in\mathscr{A}}\alpha_A$).

For $A \in \mathscr{A}$ let

$$(7.2)\qquad U_A \equiv \{w|\alpha_{w,A} > \alpha\}\ .$$

Observe that U_A is contained in the complement of the set of w's where $p(w, AK|\mathfrak{B}) = 0$.

Also, $U_A \in \mathfrak{B}$ (see (7.1)). Hence, since P_c is a canonical extension, it follows that

$$(7.3)\qquad \alpha P(AU_AK) = P_c(AU_AHK) = \int_{U_A} p'(w, AHK|\mathfrak{B})dP_c\ .$$

Also,

$$(7.4)\qquad \begin{aligned} &\int_{U_A} p'(w, AHK|\mathfrak{B})dP_c = \int_{U_A} \alpha_{w,A}p(w, AK|\mathfrak{B})dP \geqq \\ &\int_{U_A} \alpha p(w, AK|\mathfrak{B})dP = \alpha P(AU_AK)\ . \end{aligned}$$

Hence, the defining properties of U_A together with (7.3) and (7.4) say that $P(U_A) = 0$.

If $L_A \equiv \{w|\alpha_{w,A} < \alpha\}$, then an argument similar to the preceding one shows $P(L_A) = 0$.

Hence, for each set $A \in \mathscr{A}$, there exists a $P_{\mathfrak{B}}$ null set on the complement of which $\alpha_{w,A} = \alpha$. But where $\mathscr{A}$ is countable, it follows that there exists a $P_{\mathfrak{B}}$ null set, N, on the complement of which $\alpha_{w,A} = \alpha$ for all $A \in \mathscr{A}$. Thus,

$$(7.5)\qquad p'(w, AHK|\mathfrak{B}) = \alpha p(w, AK|\mathfrak{B})$$

for all $w \in N^c$ and $A \in \mathscr{A}$.

Finally, if $\alpha_w \equiv \sup_{A\in\mathscr{A}}\alpha_{w,A}$, then it is immediate from (7.5) that $P_{\mathfrak{B}}$ a.e. $\alpha_w = \alpha = \alpha_X$ and by Lemma 9 the theorem is proved.

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